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A Peer Search On Integer Solutions To Ternary Quadratic Equation

$$x^2 + d\,y^2 = z^2 + d^{2\alpha+1}$$

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Abstract: Various choices of non-zero integer solutions to non-homogeneous second degree polynomial equation with three unknowns given by $x^2 + d\,y^2 = z^2 + d^{2\alpha+1}$ are illustrated through different technical procedures.

Keywords : Mixed quadratic equation, Second degree equation with three unknowns ,Integer solutions, Substitution technique, Factorization method.

I. Introduction

It is quite obvious that Diophantine equations, one of the areas of number theory, are rich in variety [1, 2]. In particular, the ternary quadratic Diophantine equations in connection with geometrical figures occupy a pivotal role in the orbit of mathematics and have a wealth of historical significance. In this context, one may refer [3-10] for second degree Diophantine equations with three unknowns representing different geometrical figures.

In this paper, the ternary quadratic Diophantine equation representing hyperboloid of one sheet given by $x^2 + d\,y^2 = z^2 + d^{2\alpha+1}$ is studied for determining its integer solutions successfully through substitution strategy. The solutions in integers of the two dimensional geometrical figure, namely, Hyperbola represented by pell equations of the form $y^2 = D x^2 + N, D > 0$ and square-free , are employed in obtaining the solutions in integers for the Hyperboloid of one sheet given in title . A few interesting relations among the solutions are presented. A formula for generating sequence of integer solutions based on the given integer solution is obtained.

II. Method of analysis

The non-homogeneous second degree equation with three unknowns to be solved in integers is

$$x^2 + d\,y^2 = z^2 + d^{2\alpha+1}, d > 0 \tag{1}$$

Different procedures for solving (1) are as shown below :

Procedure 1

Note that (1) is equivalent to the pair of equations

$$\begin{aligned} x + z &= d(d^{\alpha} + y) \\ x - z &= (d^{\alpha} - y) \end{aligned} \tag{2}$$

Solving the above system of double equations ,we get

$$\begin{aligned} x &= \frac{(d+1)d^{\alpha} + (d-1)y}{2}, \\ z &= \frac{(d-1)d^{\alpha} + (d+1)y}{2} \end{aligned} \tag{3}$$

As the main thrust is to find integer solutions, the values of x, z in (3) are integers when

Choice (i) $d = odd = 2k+1, y = arbitrary = s$

Choice (ii) $d = even = 2k, y = even = 2s$

Considering Choice (i) ,the integer solutions to (1) are given by

$$\begin{aligned} x &= (k+1)(2k+1)^{\alpha} + ks, \\ y &= s, \\ z &= k(2k+1)^{\alpha} + (k+1)s. \end{aligned}$$

Considering Choice (ii) ,the integer solutions to (1) are given by

$$\begin{aligned} x &= (2k+1)2^{\alpha-1}k^{\alpha} + (2k-1)s, \\ y &= 2s, \\ z &= (2k-1)2^{\alpha-1}k^{\alpha} + (2k+1)s. \end{aligned}$$

Procedure 2

Write (1) as

$$x^2 + d\,y^2 = (z^2 + d^{2\alpha+1})*1 \tag{4}$$

Assume the integer 1 on the R.H.S. of (4) as

$$1 = \frac{(d - n^2 + i2n\sqrt{d})(d - n^2 - i2n\sqrt{d})}{(d+n^2)^2} \tag{5}$$

Substituting (5) in (4) and applying factorization method ,consider

$$(x + i\sqrt{d}\,y) = \frac{(d - n^2 + i2n\sqrt{d})}{(d+n^2)}(z + i\,d^{\alpha}\sqrt{d}) \tag{6}$$

On equating the coefficients of corresponding terms , we have

$$\begin{aligned} x &= \frac{[(d-n^2)z - 2nd^{\alpha+1}]}{(d+n^2)}, \\ y &= \frac{[2nz + (d-n^2)d^{\alpha}]}{(d+n^2)}. \end{aligned} \tag{7}$$

As our interest is to find integer solutions, the values of x, y in (7) are integers for suitable choices to d, n, z. In other words, for any given integer values of d, n; it is possible to choose z in integers so that the values of x, y are integers. For simplicity and brevity , a few illustrations are exhibited below in Table 1:

Table 1-Illustrations

n	d	z	y	x
1	3	$2N-3^{\alpha}$	N	$N-2*3^{\alpha}$
2	3	$7N-4*3^{\alpha+1}$	$4N-7*3^{\alpha}$	$-N$
3	3	$2N+3^{\alpha}$	N	$-(N+2*3^{\alpha})$
1	2	$3N+4*2^{\alpha}$	$(2N+3*2^{\alpha})$	N

Procedure 3

Introduction of the transformations

$$x=d^{\alpha}X, y=d^{\alpha}Y, z=d^{\alpha}W \tag{8}$$

in (1) leads to

$$X^2+dY^2=W^2+d \tag{9}$$

Note that (9) is equivalent to the pair of equations

$$W+X=d(Y+1)$$

$$W-X=(Y-1)$$

Solving the above system of double equations ,we get

$$W=\frac{(d+1)\,Y+(d-1)}{2}, \quad X=\frac{(d-1)\,Y+(d+1)}{2} \tag{10}$$

As the aim is to find integer solutions , the values of W, X in (10) are integers when

Choice (iii) $d=odd=2s+1, Y=arbitrary$

Choice (iv) $d=even=2s, Y=odd=2k+1$

Considering Choice (iii) ,the integer solutions to (1) are given by

$$x=d^{\alpha}(sY+s+1),$$

$$y=d^{\alpha}Y,$$

$$z=.d^{\alpha}(sY+Y+s)$$

Considering Choice (iv) ,the integer solutions to (1) are given by

$$x=d^{\alpha}[2s(k+1)-k],$$

$$y=,d^{\alpha}(2k+1),$$

$$z=.d^{\alpha}[2s(k+1)+k].$$

Procedure 4

Choosing

$$z=u+dv, x=u-dv \tag{11}$$

in (1) , it simplifies to

$$y^2=4uv+d^{2\alpha} \tag{12}$$

It is possible to choose u, v so that the R.H.S. of (12) is a perfect square from which the value of y is obtained. In view of (11) ,the corresponding values of z, x are found. For simplicity , a few examples are presented below :

Example 1

Take

$$u=d^{2\alpha}, v=d^{2\alpha}\pm d^{\alpha}$$

From (12) , one observes

$$y=2d^{2\alpha}\pm d^{\alpha}$$

From (11) ,one has

$$z = d^{2\alpha} + d(d^{2\alpha} \pm d^{\alpha}),\, x = d^{2\alpha} - d(d^{2\alpha} \pm d^{\alpha})$$

Example 2

Take

$$u = d^{3\alpha},\, v = d^{\alpha} \pm 1$$

From (12) , one observes

$$y = 2d^{2\alpha} \pm d^{\alpha}$$

From (11) ,one has

$$z = d^{3\alpha} + d(d^{\alpha} \pm 1),\, x = d^{3\alpha} - d(d^{\alpha} \pm 1)$$

Example 3

Take

$$u = (k^2 + k)\, d^{\alpha},\, v = (k^2 + k \pm 1)d^{\alpha}$$

From (12) , one observes

$$y = d^{\alpha} \pm (2k^2 + 2k)d^{\alpha}$$

From (11) ,one has

$$z = (k^2 + k)\, d^{\alpha} + d(k^2 + k \pm 1)d^{\alpha},\, x = (k^2 + k)\, d^{\alpha} - d(k^2 + k \pm 1)d^{\alpha}$$

Note 1

A similar analysis may be performed by considering the transformations

$$x = u + d\,v,\, z = u - d\,v$$

Remarkable observation

If the non-zero integer triple (x_0, y_0, z_0) is any solution to (1) , then the triples

$$((2d-1)x_0 + 2d\,y_0 + 2d\,z_0, 2\,x_0 + y_0 + 2\,z_0, (2d\,x_0 + 2d\,y_0 + (2d+1)z_0) \quad \text{and}$$

$$((8d^2+1)x_0 + 8d^2\,y_0 + (8d^2+4d)z_0, 8d\,x_0 + (8d+1)\,y_0 + (8d+4)z_0,$$

$$(8d^2+4d)x_0 + (8d^2+4d)\,y_0 + (8d^2+8d+1)z_0)$$

also satisfy (1).

Conclusion:

In this paper, we have presented varieties of integer solutions on the hyperboloid of one sheet represented by the non-homogeneous ternary quadratic Diophantine equation given in title. As the choices of ternary quadratic equations are rich in variety, the readers of this paper may search for other representations to hyperboloid of one sheet in obtaining patterns of integer solutions successfully.

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