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Sheffer's A – Type Generating Function On The Polynomial

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Abstract:- In this paper, some simple inequalities have been used to obtain some equation with summation T_k (x) where the value of k = 0 to k = rs + s – 1 , so T_{rs+s-1} (x) is of degree rs exactly and that T_k (x) is always of degree less than equal to rs which shows that $\{\emptyset_n(x)\}$ is of sheffer's A – type rs = m. This process the result to extend the theorem to the case when the generating function is of the form
A (t) ψ [x H (t) + g (t)] we develop the work of J. L. Goldbery [1]

Key Word:- Generating function, Degree, Polynomials , inequalities, Differential operator Introduction .

1. Introduction:-

If a set of Polynomial $\{\phi_n(x)\}$ is defined by

(1.1)

$$\begin{cases} \sum_{n=0}^{\infty} \phi_n(x) t^n = A(t)\psi[xH(t)] \text{ with} \\ \sum_{n=0}^{\infty} \psi_n t^n = \psi(t), \psi_n \neq 0, \sum_{n=0}^{\infty} a_n t^n = A(t), a_0 \neq 0 \\ \text{and } \sum_{n=0}^{\infty} h_n t^{n+1} = H(t), h_0 \neq 0, \end{cases}$$

then a necessary and sufficient condition for $\{\phi_n(x)\}$ to be Sheffer A-type m > o has been proved by J.L. Goldberg [1].

The object of this paper is to extend the theorem to the case when the generating Function is of the form A (t) ψ [xH (t) + g (t)].

2. Some results and theorems for a necessary and sufficient condition for $\{\phi_n(x)\}$ to be Sheffer A- type m > 0 has been proved by J. L. Goldberg [1].

Before providing the main theorem, let us first prove the following,

Theorem (1):- A necessary and sufficient condition that the set of polynomials $\{\phi_n(x)\}$ be of σ – type Zero with

2.1 $\sigma \equiv D \prod_{i=1}^{q} \left[\left(x + \frac{g(t)}{H(t)}\right) D + b_i - 1\right]$ is that $\phi_n(x)$ possesses the generating function in

2.2

$$A\,(t)\; {}_0F_q\left(-\,;b_1,b_2,\ldots.,b_q;xH\,(t)+g\,(t)\right)=\sum_{n=0}^{\infty}\phi_n(x)\,t^n,$$

in which

2.3

$$\begin{cases} H\,(t)\sum_{n=0}^{\infty} h_n t,^{n+1} h_0 \neq 0, A\,(t) = \sum_{n=0}^{\infty} a_n\, t^n, a_0\ \neq 0 \\ \text{and} \\ g\,(t) = \sum_{n=0}^{\infty} g_n\, t^{n+2} \end{cases}$$

Let us assume that (2.2) along with (2.3) is true. Then $\{\phi_n\ (x)\}$ is a simple set . The function H (t) has a formal inverse J (t) defined by J (H(t)) = H (J(t)) = t, so that.

2.4

$$J\,(t)=\sum_{n=0}^{\infty} c_n\, t^{n+1}, c_\circ\ \neq 0, c_n\ are\ constants.$$

The function y = ${}_0f_q$ (- ; b_1,…..,b_q ; z) is a solution of the differential equation,

2.5 $$\left[\theta_2 \prod_{i=1}^{q} (\theta_2 +\ b_{i-1}) - z\right]\ y = 0\ ;\ \theta_2 =\ z\frac{d}{dz}.$$

Taking z = [xH (t) + g (t) is is easy to show from (2.5) that

2.6

$$J\,(\sigma)\sum_{n=0}^{\propto} \phi_n\ (x)t^n = \sum_{n=1}^{\infty} \phi_{n-_1}(x)t^n.$$

Therefore J $(\sigma)\ \phi_\circ\ (x) = 0\ and\ J\ (\sigma)\ \phi_n\ (x) = \phi_{n-1}\ (x), n \leq 1$.

Hence $\{\phi_n\ (x)\}$ is of σ – type zero.

Next let us assume that $\{\phi_n\ (x)\}$ is of Sheffer σ – type zero and belongs to an operator J (σ) of the form (2.4).

Then the function J (t) has a formal inverse H (t) given by (2.3)

On the basis of the above assumptions the necessity part follows easily.

Now, let the set $\{\phi_n\ (x)\}$ be defined by

2.7

$$\begin{cases} \sum_{n=0}^{\infty} \phi_n\,(x) t^n = A\,(t)\psi\,[x\,H\,(t) + g\,(t)\,], with \\ \sum_{n=0}^{\infty} \psi_n\, t^n = \psi\,(t), \psi_n \neq 0, \sum_{n=0}^{\infty} a_n t^n = A\,(t), a_\circ \neq 0 \\ \sum_{n=0}^{\infty} h_n t^{n+1} = H\,(t), h_\circ \neq 0\ and\ \sum_{n=0}^{\infty} g_n t^{n+2} = g\,(t). \end{cases}$$

Theorem (2) :- If $\{\phi_n\,(x)\}$is defined by (2.7) then a necessary and sufficient condition for $\{\phi_n\,(x)\}$ to be sheffer A-type m (>0) is that there exist a positive integer r. which divides m, and numbers $b_1, b_2, \ldots, b_r$; neither zero nor negative integers, such that

2.8 ψ [x H (t) + g (t) = $_0 f_r$ [- ; $b_1, \ldots, b_r$; $\propto$ {x H (t) + g (t)}]

For some non-zero constant $\propto$, with H^{-1} (t) a polynomial of degree s = $\frac{m}{r}$ exactly.

Since $\psi_n \neq 0\ (n \geq 0)$, so $\{\phi_n\,(x)\}$ is a simple set; say ϕ_n (x) = $a_n x_n$ + O (x^{n-1}), $a_n\ \neq 0\ (n \geq 0)$.

Therefore there exist a unique differential operator J (x_1 D) such that

J (x, D) ϕ_n (x) = ϕ_{n-1} (x), n ≥ 1 where

$$J\,(x, D) = \sum_{n=0}^{\infty} T_n\,(x)\,D^{n+1}, D = \frac{d}{dx}\ and$$

T_n (x) = $t_n x^n$ + O (x^{n-1}), a polynomial of degree ≤ n. Since $a_\circ \neq 0$ *we have* $t_\circ \neq 0$.

Let H^{-1} (t) be the formal power series inverse of H (t).

Now, let us first assume that $\{\phi_n\,(x)\}$ is of sheffer A-type m >0. Then from.

J (x, D) ϕ_n (x) = $\phi_{n-1}(x)$ we have

2.9 $n \propto_n \{\, t_\circ + (n-1)\,t_1 + \ldots + (n-1)(n-2)\ldots(n-m)\,t_m\} = \propto_{n-1}, n = 1, 2 \ldots \ldots$

obtained by equating coefficients x^{n-1}. (2.9) can be written as

2.10 $n \propto_n C \prod_{k=1}^{r} (n + b_{k^{-1}}) = \propto_{n-1}, C \neq O\ and\ b_k\ \neq 0, 1, -2, \ldots$.. Solving (2.10) for $\propto_n$ we get

2.11 $\propto_n = \frac{\propto_\circ}{C^n\, n!\, \prod_{k=1}^{r} (b_k)_n}$, $(b_k)_n = b_k (b_k + 1)\ldots(b_k + n - 1)$.

Also we have .

$\propto_n = a_\circ\, h_\circ{}^n\, \psi_n$which yields together with (2.11) ,

2.12

$$\sum_{n=0}^{\infty} \phi_n(x) t^n = A\,(t)\ o^{F_r}\,[-\,; b_1, \ldots . b_r;\ \propto \{x\,H\,(t) + g\,(t)\}], \propto = (Ch_\circ)^{-1}\ \neq 0.$$

Now to show that H– 1 (t) is a polynomial in t of degree s $\left(\frac{m}{r} = s\right)$.

$\{\phi_n\,(x)\}$ is σ - type zero with $\sigma\ \equiv D \prod_{k=1}^{r} \left\{ \left(x + \frac{g}{H}\right) D + b_{k^{-1}} \right\}$. (follows from theorem 1).

So, there exist a unique differential operator J* (σ) such that

2.13

$$J^{*}(\sigma)\phi_n(x)=\sum_{k=0}^{\infty}\gamma k\left\{D\prod_{i=1}^{r}\left(x+\frac{g(t)}{h(t)}\right)D+b_{i^{-1}}\right\}^{k+1}\quad \phi_n(x)=\phi_{n-1\,(x)(n=1,2,\ldots..)}$$

Since J (x, D) is unique so (2.13) can be rearranged in terms of powers of D into J (x, D). Then T_k (x) are of highest degree m if and only it r s = m.

$$Thus\quad J^{*}(t)=\sum_{k=0}^{s-1}\gamma k\ t^{k+1}.$$

But H^{-1}(t) = J* (t) (follows from theorem 1) and so H^{-1} (t) is a polynomial in t of degree s = $\frac{m}{r}$ which proves the necessity.

Next let there exist a positive integer r which divides m and numbers $b_1,\ldots.b_r$ such that (2.8) holds for some non-zero constant σ with H^{-1} (t) a polynomial of degree $\frac{m}{r}$ = s exactly. Then we are to show that $\{\phi_n(x)\}$ is of sheffer A- type m>0. From the above hypothesis $\{\phi_n(x)\}$ is of σ −type zero with

$\sigma\ \equiv D\prod_{k=1}^{r}\left\{\left(x+\frac{g}{H}\right)D+\ b_{k^{-1}}\right\}$ Also Since J* (t) = H^{-1} (t), so we have

2.14

$$\sum_{k=0}^{s-1}\gamma k\left\{D\prod_{i=1}^{r}\left((x+\frac{g}{H})D+b_{i^{-1}}\right)\right\}\phi_n^{\ k+1}(x)=\sum_{k=0}^{rs+s-1}T_k(x).$$

$.D^{k+1}\phi_n(x)=\phi_{n-1}(x)\quad (s\geq 1)for\ n=1,\ 2,\ldots\ldots$

CONCLUSION:-

From (2.14) T_{rs+s-1} (x) is of degree rs exactly and that T_k (x) is always of degree ≤ r s,. This proves the necessary and sufficient condition for $\{\phi_n(x)\}$ to be of Sheffer A type m > 0, Now Sheffer A type rs = m. The choice g_n = 0 (n = 1, 2,……) reduces the theorem (2) to that of J. L. Goldberg's theorems and results.

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