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Applicability To Fixed Point Theorem On Tricomplex Valued *b*-Metric Space For Four Maps Through Control Functions

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Abstract: The aspiration of this paper is to we incorporate a fixed point and unique common fixed point in tricomplex valued b- Metric space for mixed type rational contractions of two weakly compatible of mappings with coefficients as point dependent control functions. We quote some example to validate and strengthen our results in the context of tricomplex valued b- Metric space.

Key Words - Tricomplex valued b-metric space, contractive type mapping, and common fixed point. Complete TCVbMS

I. Introduction

There are lot of exciting work has been done by several researchers on fixed point results and common fixed point results in different types of generalized metric spaces. In 1989, Bakhtin[12] introduced the notion of b-metric space as a generalization of metric space. After that many authors used the concept of b-metric space by different way and find some fixed point results. In recent year J. Choi et. al. [10] proposed the concept of bicomplex numbers and their elementary functions, which was generalization of complex numbers and proved some common fixed point theorems in connection with two weakly compatible mapping. Later on, In 2019, Jebril et. al.[13,14] demonstrated a variety of fixed point outcomes via mixed type contractions in bicomplex valued metric spaces . Datta et. al. [8, 9] demonstrated some common FPT in bicomplex valued b-metric spaces in 2020 and 2021. After that several researchers established fixed point and common fixed point results under rational contractions for pair of mapping in bicomplex valued metric spaces, refer [1, 4, 8, 9, 10, 11, 13, 14]. In 1991, Prince [5] gave the concept of tricomplex metric spaces. Most recently, G. Mani et. al. [20][22] studied the notion of tricomplex valued metric spaces and its properties and established some fixed point theorems and common fixed point theorems and its applications

In the present work, we generalize theorem (3.1) in tricomplex valued b-metric spaces using some analytical type contraction conditions for two pair of weakly compatible mappings using control functions. We quote some example to validate and strengthen our results.

II. Preliminaries:

We denote the symbol $€_0, €_1, €_2$ and $€_3$ as a set of real, complex and bicomplex and tricomplex numbers respectively. The set of bicomplex numbers defined as:

$€_2 = \{w: w = a_1 + a_2 i_1 + a_3 i_2 + a_4 i_1 i_2, a_1, a_2, a_3, a_4 \in €_0\}$

i.e $€_2 = \{w: w = z_1 + z_2 i_2 \ z_1, z_2 \in €_1\}$

Where $z_1 = a_1 + a_2 i_1$, $z_2 = a_3 + a_4 i_1$ and i_1, i_2 are independent imaginary units such that $i_1^2 = -1 = i_2^2$. In 1991, Price[5] defined the tricomplex numbers as follows:

$$\mu = a_1 + a_2 i_1 + a_3 i_2 + a_4 j_1 + a_5 i_3 + a_6 j_2 + a_7 j_3 + a_8 i_4$$

where $a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8 \in €_0$ and independent units $i_1, i_2, i_3, i_4, j_1, j_2$ *and* j_3 *are such that* $i_1^2 = i_2^2 = -1$, $i_4 = i_1 j_3 = i_1 i_2 i_3$, $j_2 = i_1 i_3 = i_3 i_1$, $j_2^2 = 1$, $j_1 = i_1 i_2 = i_2 i_1$ *and* $j_1^2 = 1$;

The set of tricomplex numbers defined as:

$€_3 = \{\mu: \mu = a_1 + a_2 i_1 + a_3 i_2 + a_4 j_1 + a_5 i_3 + a_6 j_2 + a_7 j_3 + a_8 i_4: , a_1, a_2, a_3, a_4, , a_5, a_6, a_7, a_8 \in €_0\}$

i.e $€_3 = \{\mu: \mu = \theta_1 + \theta_2 i_3 \ \theta_1, \theta_2 \in €_2\}$

Where $\theta_1 = z_1 + z_2 i_2$, $\theta_2 = z_3 + z_4 i_2$ If $\vartheta = \eta_1 + i_3 \eta_2$ and $\omega = ƙ_1 + i_3 ƙ_2$ be any two tricomplex numbers then their sum is

$$\vartheta \pm \omega = (\eta_1 + i_3\eta_2) \pm (ƙ_1 + i_3 ƙ_2) = (\eta_1 + ƙ_1) \pm i_3(\eta_2 + ƙ_2)$$

and product is

$\vartheta\omega = (\eta_1 + i_3\eta_2)(ƙ_1 + i_3ƙ_2) = (\eta_1\eta_2 - \eta_2ƙ_2) + i_3((\eta_1ƙ_2 + \eta_2ƙ_1)$. There are four idempotent elements in $€_3$: and they are $0, 1, e_1 = \frac{1+j_3}{2}$ *and* $e_2 = \frac{1-j_3}{2}$. Hence, e_1 *and* e_2 are nontrivial such that $e_1 + e_2 = 1$ *and* $e_1 e_2 = 0$. Every tricomplex numbers $\theta_1 + \theta_2 i_3$ can be uniquely expressed as a combination of e_1 *and* e_2.

$$\mu = \theta_1 + \theta_2 i_3 = (\theta_1 - \theta_2 i_2)e_1 + (\theta_1 - \theta_2 i_2)e_2.$$

The notation of μ represents the idempotent of the tricomplex numbers and the coefficients of the complex numbers $\mu_1 = (\theta_1 - \theta_2 i_2)$ *and* $\mu_2 = (\theta_1 + \theta_2 i_2)$ are called idempotent components of bicomplex numbers of μ. An element $\mu = \theta_1 + \theta_2 i_3 \in €_3$ is invertible if there exists a numbers ω in $€_3$ such that $\mu\omega = 1$. ω is called multiplicative inverse of ϑ. An element having an inverse in $€_3$ is called nonsingular and the element not having an inverse in $€_3$ is said to be singular element of $€_3$. An element $\mu = \theta_1 + \theta_2 i_3 \in €_3$ is nonsingular if $|\theta_1^2 + \theta_2^2| \neq 0$ and singular if $|\theta_1^2 + \theta_2^2| = 0$ .The inverse of ϑ is defined as $\mu^{-1} = \omega = \frac{\theta_1 - i_3\theta_2}{\theta_1^2 + \theta_2^2}$.

The norm $\|.\|$ of $€_3$ is a positive real valued functions and $\|.\|: €_3 \to €_0^+$ is defined by

$$\|\mu\| = \|\theta_1 + i_3\theta_2\| = (|\theta_1|^2 + |\theta_2|^2)^{1/2} = \left(\frac{|\theta_1 - \theta_2 i_2|^2 + |\theta_1 + \theta_2 i_2|^2}{2}\right)^{1/2}.$$

$\|\mu\| = (a_1^2 + a_2^2 + a_3^2 + a_4^2 + a_5^2 + a_6^2 + a_7^2 + a_8^2)^{1/2}$

Where, $\mu = a_1 + a_2 i_1 + a_3 i_2 + a_4 j_1 + a_5 i_3 + a_6 j_2 + a_7 j_3 + a_8 = \theta_1 + \theta_2 i_3 \in €_3$. Clearly $€_3$ is a Banach space as the linear space $\mu\omega$ is complete.

If $\mu, \omega \in €_3$ then $\|\mu\omega\| \leq 2\|\mu\|\|\omega\|$ hold instead of $\|\mu\omega\| \leq \|\mu\|\|\omega\|$ and then $€_3$ is not Banach algebra. The partial order relation $\lesssim_{i_3}$ on $€_3$ was defined by G. Mani et. al. [10]. Let $€_3$ be the set of tricomplex numbers $\mu = \theta_1 + \theta_2 i_3$ *and* $\omega = ƙ_1 + i_3ƙ_2 \in €_3$ then $\mu \lesssim_{i_3} \omega$ if $\theta_1 \lesssim_{i_3} ƙ_1$ *and* $\theta_2 \lesssim_{i_3} \acute{ƙ}_2$, if one of the following condition is satisfied .

(i) $\theta_1 = ƙ_1, \theta_2 = ƙ_2$

(ii) $\theta_1 \prec_{i_2} ƙ_1, \theta_2 = ƙ_2$

(iii) $\theta_1 = ƙ_1, \theta_2 \prec_{i_2} ƙ_2$

(iv) $\theta_1 \prec_{i_2} ƙ_1, \theta_2 \prec_{i_2} ƙ_2$

In particular, we can write $\mu \lnsim_{i_3} \omega$. if $\mu \prec_{i_3} \omega$ and $\mu \neq \omega$. i.e , one of $(ii), (iii)$ *and* (iv) is holds and $\mu \prec_{i_3} \omega$ if only (iv) holds. For any two tricomplex numbers $\mu, \omega \in €_3$ .the following conditions holds:

(a) $\mu \prec_{i_3} \omega$ if $\|\mu\| \leq \|\omega\|$;

(b) $\|\mu + \omega\| \leq \|\mu\| + \|\omega\|$;

(c) $\|a\mu\| = |a|\|\mu\|$, where $a \in €_0^+$;

(d) $\|\mu\,\omega\| \leq 2|\mu|\|\omega\|$ and equality holds only when at least one of μ and ω is non-singular;

(e) $\|\mu^{-1}\| = \|\mu\|^{-1}$ if μ is non-singular;

(f) $\left\|\frac{\mu}{\omega}\right\| = \frac{\|\mu\|}{\|\omega\|}$ if ω is non-singular;

Now we recall some basic definition which will be utilized in our paper:

Definition 2.1[20]: Let $\mathfrak{K}$ be a nonempty set then the function $\Gamma: \mathfrak{K} \times \mathfrak{K} \to €_3$ satisfies the following conditions. if

(i) $0 \precsim_{i_3} \Gamma(z_1, z_2)$ $for\ all\ z_1, z_2 \in \mathfrak{K}, and\ \Gamma(z_1, z_2) = 0 \Leftrightarrow z_1 = z_2$;

(ii) $đ(z_1, z_2) = đ(z_2, z_1)$;

(iii)$\Gamma(z_1, z_2) \precsim_{i_3} \Gamma(z_1, z_3) + \Gamma(z_3, z_2)$ for all $z_1, z_2, z_3 \in \mathfrak{K}$;

Then the pair $(\mathfrak{K}, đ)$ is called $TCVMS$.

Definition 2.2[20]: Let $\mathfrak{K}$ be a nonempty set and $s \geq 1$ be a given real number then the function $\Gamma: \mathfrak{K} \times \mathfrak{K} \to €_3$ satisfies the following conditions. if

(i) $0 \precsim_{i_3} \Gamma(z_1, z_2)$ $for\ all\ z_1, z_2 \in \mathfrak{K}, and\ \Gamma(z_1, z_2) = 0 \Leftrightarrow z_1 = z_2$;

(ii) $\Gamma(z_1, z_2) = \Gamma(z_2, z_1)$;

(iii)$\Gamma(z_1, z_2) \precsim_{i_3} s[\Gamma(z_1, z_3) + \Gamma đ(z_3, z_2)]$ for all $z_1, z_2, z_3 \in \mathfrak{K}$;

Then the pair $(\mathfrak{K}, đ)$ is called $TCVbMS$.

Definition 2.3[22]: Let $(\mathfrak{K}, đ)$ be a $TCVbMS$ and $\{z_n\}$ be a sequence in $\mathfrak{K}$ then

(i) A sequence $\{z_n\}$ in $\mathfrak{K}$ is said to be convergent, $\{z_n\}$ convergence to z If for any $0 <_{i_3} c \in €_3$ then there exist $n_0 \in \mathbb{N}$ $such\ that\ đ(z_n, z) <_{i_3} c, for\ all\ n \geq n_0$ and we can denote this $\lim_{n\to\infty} z_n = z\ or\ z_n \to z\ as\ n \to \infty$.

(ii) A sequence $\{z_n\}$ in X is said to be Cauchy sequence If for any $0 <_{i_3} c \in €_3$, there exists $n_0 \in \mathbb{N}, such\ that\ d(z_n, z_{n+m}) <_{i_3} c$, where, $m, n \in \mathbb{N}$ $and\ n > n_0$.

(iii) If every Cauchy sequence is convergent in $(\mathfrak{K}, đ)$ then $(\mathfrak{K}, đ)$ is said to be a complete $TCVMS$.

Lemma 2.4[20]: Let $(\mathfrak{K}, \mathcal{S})$ be a $TCVSMS$ and let $\{X_n\}$ be a sequence in $\mathfrak{K}$ then $\{x_n\}$ is convergent and converges to x if and only if $\|\Gamma(x_n, x)\| \to 0$ as n→ ∞.

Lemma 2.5[20]: Let $(\mathfrak{K}, \mathcal{S})$ be a tricomplex valued S-metric space and let $\{x_n\}$ be a sequence in $\mathfrak{K}$ then $\{x_n\}$ is a Cauchy sequence if and only if $\|\Gamma(x_n, x_{n+m})\| \to 0$ as n→ ∞ , $m \in \mathcal{N}$.

Main Theorem.

III. MAIN RESULTS

Recently R. Ramaswamy et. al.[6] proved the following fixed point theorem in a Complete tricomplex valued metric spaces for four maps as follows:

Theorem 3.1: Let $(\mathfrak{K}, \varsigma)$ be a Complete $TCVSMS$ and, $, S: \mathfrak{K} \to \mathfrak{K}$. if their exist mapping $f, \mathfrak{z}, \boldsymbol{\beta}, \boldsymbol{\gamma}: \mathfrak{K} \times \mathfrak{K} \times \mathfrak{K} \to \mathfrak{K}$ such that for all $\tau, \eta \in \mathfrak{K}$ satisfying the conditions:

(a) $f(QS\tau, \eta, a) \leq f(\tau, \eta, a)\ and\ \ f(\tau, QS\eta, a) \leq f(\tau, \eta, a)$,

$\mathfrak{z}(QS\tau, \eta, a) \leq \mathfrak{z}(\tau, \eta, a)\ \ and\ \ \mathfrak{z}(\tau, QS\eta, a) \leq \mathfrak{z}(\tau, \eta, a)$,

$\boldsymbol{\beta}(QS\tau, \eta, a) \leq \boldsymbol{\beta}(\tau, \eta, a)\ \ and\ \ \boldsymbol{\beta}(\tau, QS\eta, a) \leq \boldsymbol{\beta}(\tau, \eta, a)$,

$\boldsymbol{\gamma}(QS\tau, \eta, a) \leq \boldsymbol{\gamma}(\tau, \eta, a)\ \ and\ \ \boldsymbol{\gamma}(\tau, QS\eta, a) \leq \boldsymbol{\gamma}(\tau, \eta, a)$,

(b) $\varsigma(S\tau, Q\eta) \precsim_{i_3} f(\tau, \eta, a)\varsigma(\tau, \eta) + \mathfrak{z}(\tau, \eta, a)\frac{\varsigma(\tau, S\tau)\varsigma(\eta, S\eta)}{1+\varsigma(\tau, \eta)} + \boldsymbol{\beta}(\tau, \eta, a)\frac{\varsigma(\eta, S\tau)\varsigma(\tau, S\eta)}{1+\varsigma(\tau, \eta)} +$
$\boldsymbol{\gamma}(\tau, \eta, a)\left(\frac{\varsigma(\tau, S\tau)\varsigma(\tau, Q\eta) + \varsigma(\eta, Q\eta)\varsigma(\eta, S\tau)}{1+\varsigma(\tau, Q\eta) + \varsigma(\eta, S\tau)}\right)$

$for\ all\ \tau,\eta \in \mathfrak{K}\ and\ for\ a\ fixed\ element\ a \in X,$

(c) $f(\tau,\eta,a) + \mathfrak{z}(\tau,\eta,a) + \boldsymbol{\beta}(\tau,\eta,a) + \boldsymbol{\gamma}(\tau,\eta,a) < 1.$

Then S and Q have a unique common fixed point.

Inspired by above theorem, in this paper we generalize the theorem (3.1) in a tricomplex valued b-metric space for four mapping satisfying more rational conditions using control functions

Theorem 3.2: Let $(\mathfrak{K},\Gamma)$ be a complete $TCVbMS$ with the coefficient $s \geq 1$ and $F,G,P,U:\mathfrak{K} \to \mathfrak{K}$ satisfying the conditions:

(i) $G(\mathfrak{K}) \subseteq P(\mathfrak{K})\ and\ F(\mathfrak{K}) \subseteq U(\mathfrak{K}),$

(ii) The pair $(F,P)\ and\ (G,U)$ are weakly compatible,

(iii) $P(\mathfrak{K})\ or\ U(\mathfrak{K})$ is a complete subspace of $\mathfrak{K}$,

(iv) If there exist mapping $\alpha,\beta,\gamma,\delta,\Omega,\vartheta,\xi:\mathfrak{K}^3 \to \left[0,\frac{1}{s}\right)$ $such\ that\ for\ all\ \tau,\eta \in \mathfrak{K};$

$$\alpha(F\tau) \leq \alpha(\tau);\quad \beta(F\tau) \leq \beta(\tau),\quad \gamma(F\tau) \leq \gamma(\tau),$$
$$\delta(F\tau) \leq \delta(\tau),\ \Omega(F\tau) \leq \Omega(\tau),\quad \vartheta(F\tau) \leq \vartheta(\tau),\ and\quad \xi(F\tau) \leq \xi(\tau)$$
$$\alpha(G\tau) \leq \alpha(\tau);\quad \beta(G\tau) \leq \beta(\tau),\quad \gamma(G\tau) \leq \gamma(\tau),$$
$$\delta(G\tau) \leq \delta(\tau),\ \Omega(G\tau) \leq \Omega(\tau),\quad \vartheta(G\tau) \leq \vartheta(\tau),\ and\quad \xi(G\tau) \leq \xi(\tau)$$
$$\alpha(P\tau) \leq \alpha(\tau);\quad \beta(P\tau) \leq \beta(\tau),\quad \gamma(P\tau) \leq \gamma(\tau),$$
$$\delta(P\tau) \leq \delta(\tau),\Omega(P\tau) \leq \Omega(\tau),\quad \vartheta(P\tau) \leq \vartheta(\tau),\ and\quad \xi(P\tau) \leq \xi(\tau)$$
$$\alpha(U\tau) \leq \alpha(\tau);\quad \beta(U\tau) \leq \beta(\tau),\quad \gamma(U\tau) \leq \gamma(\tau),$$
$$\delta(U\tau) \leq \delta(\tau),\Omega(U\tau) \leq \Omega(\tau),\quad \vartheta(U\tau) \leq \vartheta(\tau),\ and\quad \xi(U\tau) \leq \xi(\tau)$$

(v) $\Gamma(F\tau,G\eta) \precsim_{i_3} \alpha(\tau)\Gamma(P\tau,,U\eta) + \beta(\tau)\Gamma(P\tau,F\tau)$

$$+\gamma(\tau)\Gamma(U\eta,G\eta) + \delta(\tau)[\Gamma(U\eta,F\tau) + \Gamma(P\tau,G\eta)] + \Omega(\tau)\left(\frac{\Gamma(P\tau,F\tau)\Gamma(U\eta,G\eta)}{1+\Gamma(P\tau,U\eta)}\right)$$
$$+\vartheta(\tau)\left(\frac{\Gamma(U\eta,F\tau)\Gamma(P\tau,G\eta)}{1+\Gamma(P\tau,U\eta)}\right)$$
$$+\xi(\tau)\,max\{\Gamma(P\tau,F\tau),\Gamma(P\tau,U\eta),\Gamma(U\eta,G\eta),\Gamma(U\eta,F\tau)\}$$

and $\alpha(\tau) + \beta(\tau) + \gamma(\tau) + 2s\delta(\tau) + \Omega(\tau) + \vartheta(\tau) + \xi(\tau) < 1$

Then $F,G,P,and\ U$ have a unique common fixed point.

Proof: let an arbitrary point $\boldsymbol{\tau_0} \in \mathfrak{K}$. *We define a sequence* $\{\eta_n\}$ *in* $\mathfrak{K}$.

Such that $\eta_{2n+1} = U\tau_{2n+1} = F\tau_{2n}$ *and* $\eta_{2n+2} = P\tau_{2n+2} = G\tau_{2n+1}, n = 0,1,2 \ldots$ *then from inequality* (v) *we have*

$$\Gamma(\eta_{2n+1},\eta_{2n+2}) = \Gamma(F\tau_{2n},G\tau_{2n+1})$$
$$\precsim_{i_3} \alpha(\tau_{2n})\Gamma(P\tau_{2n},U\tau_{2n+1}) + \beta(\tau_{2n})\Gamma(P\tau_{2n},F\tau_{2n}) + \gamma(\tau_{2n})\Gamma(U\tau_{2n+1},G\tau_{2n+1})$$
$$+\delta(\tau_{2n})[\Gamma(U\tau_{2n+1},F\tau_{2n}) + \Gamma(P\tau_{2n},G\tau_{2n+1})] + \Omega(\tau_{2n}a)\left(\frac{\Gamma(P\tau_{2n},F\tau_{2n})\Gamma(U\tau_{2n+1},G\tau_{2n+1})}{1+\Gamma(P\tau_{2n},U\tau_{2n+1})}\right)$$
$$+\vartheta(\tau_{2n})\left(\frac{\Gamma(U\tau_{2n+1},F\tau_{2n})\Gamma(P\tau_{2n},G\tau_{2n+1})}{1+\Gamma(P\tau_{2n},U\tau_{2n+1})}\right)$$
$$+\xi(\tau_{2n})max\{\Gamma(P\tau_{2n},F\tau_{2n}),\Gamma(P\tau_{2n},U\tau_{2n+1}),\Gamma(U\tau_{2n+1},G\tau_{2n+1}),\Gamma(U\tau_{2n+1},F\tau_{2n})\}$$

$$\Gamma(\eta_{2n+1},\eta_{2n+2}) \precsim_{i_3} \alpha(\eta_{2n})\Gamma(\eta_{2n},\eta_{2n+1}) + \beta(\eta_{2n})\Gamma(\eta_{2n},\eta_{2n+1}) + \gamma(\eta_{2n})\Gamma(\eta_{2n+1},\eta_{2n+2})$$
$$+\delta(\eta_{2n})[\Gamma(\eta_{2n+1},\eta_{2n+1}) + \Gamma(\eta_{2n},\eta_{2n+2})] + \Omega(\eta_{2n})\left(\frac{\Gamma(\eta_{2n},\eta_{2n+1})\Gamma(\eta_{2n+1},\eta_{2n+2})}{1+\Gamma(\eta_{2n},\eta_{2n+1})}\right)$$
$$+\vartheta(\eta_{2n})\left(\frac{\Gamma(\eta_{2n+1},\eta_{2n+1})\Gamma(\eta_{2n},\eta_{2n+2})}{1+\Gamma(\eta_{2n},\eta_{2n+1})}\right)$$
$$+\xi(\eta_{2n})max\{\Gamma(\eta_{2n},\eta_{2n+1}),\Gamma(\eta_{2n},\eta_{2n+1}),\Gamma(\eta_{2n+1},\eta_{2n+2}),0\}$$

Case (a): if $max\{\Gamma(\eta_{2n},\eta_{2n+1}),\Gamma(\eta_{2n},\eta_{2n+1}),\Gamma(\eta_{2n+1},\eta_{2n+2}),0\} = \Gamma(\eta_{2n},\eta_{2n+1})$ and

Put $\Lambda_{2n} = \Gamma(\eta_{2n},\eta_{2n+1})$ then

$$\|\Lambda_{2n+1}\| \precsim_{i_3} \alpha(\eta_{2n})\|\Lambda_{2n}\| + \beta(\eta_{2n})\|\Lambda_{2n}\| + \gamma(\eta_{2n})\|\Lambda_{2n+1}\| + \delta(\eta_{2n})[0 + \|\Lambda_{2n}\| + \|\Lambda_{2n+1}\|]$$
$$+\Omega(\eta_{2n})\left\|\frac{\Lambda_{2n}}{1+\Lambda_{2n}}\right\|\|\Lambda_{2n+1}\| + \xi(\eta_{2n})\|\Lambda_{2n}\|$$

$$\|\Lambda_{2n+1}\| \precsim_{i_3} [\alpha(\eta_{2n}) + \beta(\eta_{2n}) + \delta(\eta_{2n}) + \xi(\eta_{2n})]\|\Lambda_{2n}\|$$
$$+[\gamma(\eta_{2n}) + \delta(\eta_{2n}) + \Omega(\eta_{2n})]\|\Lambda_{2n+1}\|$$

$$\Rightarrow \|\Lambda_{2n+1}\| \precsim_{i_3} (\alpha+\beta+\delta+\xi)(\eta_{2n})\|\Lambda_{2n}\| + (\gamma+\delta+\Omega)(\eta_{2n})\|\Lambda_{2n+1}\|$$

$$\Rightarrow \{1-(\gamma+\delta+\Omega)\}(\eta_{2n})\|\Lambda_{2n+1}\| \lesssim_{i_3} (\alpha+\beta+\delta+\xi)(\eta_{2n})\|\Lambda_{2n}\|$$

$$\Rightarrow \|\Lambda_{2n+1}\| \lesssim_{i_3} \frac{(\alpha+\beta+\delta+\xi)(\eta_{2n})}{\{1-(\gamma+\delta+\Omega)\}(\eta_{2n})}\|\Lambda_{2n}\|$$

let $₤ = \frac{(\alpha+\beta+\delta+\xi)(\eta_{2n})}{\{1-(\gamma+\delta+\Omega)\}(\eta_{2n})} < 1$ *Thus*

$$\Rightarrow \|\Lambda_{2n+1}\| \lesssim_{i_3} ₤\|\Lambda_{2n}\|$$

Similarly it can be follow that

$$\Rightarrow \|\Lambda_{2n+2}\| \lesssim_{i_3} ₤'\|\Lambda_{2n}\|$$

Where $₤' = \frac{(\alpha+\gamma+\delta+\xi)(\eta_{2n})}{\{1-(\beta+\delta+\Omega)\}(\eta_{2n})}$

Let $\hbar = Max\{₤, ₤'\}$, *then* $0 \le ₤ < 1$, *since* $₤, ₤' \in [0,1)$, *then for every* $n \in \mathcal{N}$ *by above inequality we get* $\Rightarrow \|\Lambda_{2n+2}\| \lesssim_{i_3} \hbar\|\Lambda_{2n}\|$.

Hence, $\Lambda_n \le \hbar\Lambda_{n-1} \le \hbar^2\Lambda_{n-2} \le \cdots .. \le \hbar^n\Lambda_0$

That is $\Gamma(\eta_n, \eta_{n+1}) = \hbar^n\Gamma(\eta_0, \eta_1)$

Case (b): if $max\{\Gamma(\eta_{2n}, \eta_{2n+1}), \Gamma(\eta_{2n}, \eta_{2n+1}), \Gamma(\eta_{2n+1}, \eta_{2n+2}), 0\} = \Gamma(\eta_{2n+1}, \eta_{2n+2})$ and

Put $\Lambda_{2n} = \Gamma(\eta_{2n}, \eta_{2n+1})$ then

$$\|\Lambda_{2n+1}\| \lesssim_{i_3} \alpha(\eta_{2n})\|\Lambda_{2n}\| + \beta(\eta_{2n})\|\Lambda_{2n}\| + \gamma(\eta_{2n})\|\Lambda_{2n+1}\| + \delta(\eta_{2n})[0 + \|\Lambda_{2n}\| + \|\Lambda_{2n+1}\|] + \Omega(\eta_{2n})\left\|\frac{\Lambda_{2n}}{1+\Lambda_{2n}}\right\|\|\Lambda_{2n+1}\| + \xi(\eta_{2n})\|\Lambda_{2n+1}\|$$

$$\|\Lambda_{2n+1}\| \lesssim_{i_3} [\alpha(\eta_{2n}) + \beta(\eta_{2n}) + \delta(\eta_{2n})]\|\Lambda_{2n}\| + [\gamma(\eta_{2n}) + \delta(\eta_{2n}) + \Omega(\eta_{2n})]\|\Lambda_{2n+1}\|$$

$$\Rightarrow \|\Lambda_{2n+1}\| \lesssim_{i_3} (\alpha+\beta+\delta)(\eta_{2n})\|\Lambda_{2n}\| + (\gamma+\delta+\Omega+\xi)(\eta_{2n})\|\Lambda_{2n+1}\|$$

$$\Rightarrow \{1-(\gamma+\delta+\Omega+\xi)\}(\eta_{2n})\|\Lambda_{2n+1}\| \lesssim_{i_3} (\alpha+\beta+\delta)(\eta_{2n})\|\Lambda_{2n}\|$$

$$\Rightarrow \|\Lambda_{2n+1}\| \lesssim_{i_3} \frac{(\alpha+\beta+\delta)(\eta_{2n})}{\{1-(\gamma+\delta+\Omega+\xi)\}(\eta_{2n})}\|\Lambda_{2n}\|$$

let $₤ = \frac{(\alpha+\beta+\delta)(\eta_{2n})}{\{1-(\gamma+\delta+\Omega+\xi)\}(\eta_{2n})} < 1$ *Thus*

$$\Rightarrow \|\Lambda_{2n+1}\| \lesssim_{i_3} \mathfrak{z}\|\Lambda_{2n}\|$$

Similarly it can be follow that

$$\Rightarrow \|\Lambda_{2n+2}\| \lesssim_{i_3} \mathfrak{z}'\|\Lambda_{2n}\|$$

Where $\mathfrak{z}' = \frac{(\alpha+\gamma+\delta+\xi)(\eta_{2n})}{\{1-(\beta+\delta+\Omega)\}(\eta_{2n})}$

Let $\hbar = Max\{\mathfrak{z}, \mathfrak{z}'\}$, *then* $0 \le \mathfrak{z} < 1$, *since* $\mathfrak{z}, \mathfrak{z}' \in [0,1)$, *then for every* $n \in \mathcal{N}$ *by above inequality we get* $\Rightarrow \|\Lambda_{2n+2}\| \lesssim_{i_3} \hbar\|\Lambda_{2n}\|$.

Hence, $\Lambda_n \le \hbar\Lambda_{n-1} \le \hbar^2\Lambda_{n-2} \le \cdots .. \le \hbar^n\Lambda_0$

That is $\Gamma(\eta_n, \eta_{n+1}) = \hbar^n\Gamma(\eta_0, \eta_1)$

If $m > n$ *then*

$$\begin{aligned}\|\Gamma(\eta_n, \eta_m)\| &\lesssim_{i_3} s[\|\Gamma(\eta_n, \eta_{n+1})\| + \|\Gamma(\eta_{n+1}, \eta_m)\|] \\ &\lesssim_{i_3} s\|\Gamma(\eta_n, \eta_{n+1})\| + s^2[\|\Gamma(\eta_{n+1}, \eta_{n+2})\| + \|\Gamma(\eta_{n+2}, \eta_m)\|] \\ &\lesssim_{i_3} s\|\Gamma(\eta_n, \eta_{n+1})\| + s^2\|\Gamma(\eta_{n+1}, \eta_{n+2})\| + s^3\|\Gamma(\eta_{n+2}, \eta_{n+3})\| + \cdots \ldots\ldots .... \\ &\lesssim_{i_3} s\hbar^n\|\Gamma(\eta_0, \eta_1)\| + s^2\hbar^{n+1}\|\Gamma(\eta_0, \eta_1)\| + s^3\hbar^{n+2}\|\Gamma(\eta_0, \eta_1)\| + \cdots \ldots\ldots .... \\ &\lesssim_{i_3} s\hbar^n(1 + s\hbar + s^2\hbar^2 + s^3\hbar^3 + \cdots \ldots\ldots\ldots ..)\|\Gamma(\eta_0, \eta_1)\| \\ &\lesssim_{i_3} \frac{s\hbar^n}{1-s\hbar}\|\Gamma(\eta_0, \eta_1)\| \to 0 \ as\ m, n \to \infty.\end{aligned}$$

Hence $\{\eta_n\}$ *is a Cauchy sequence in* $\mathfrak{K}$.

$P(\mathfrak{K})$ is a complete subspace of $\mathfrak{K}$, Since $\eta_{2n+2} = P\,\tau_{2n+2} \in P(\mathfrak{K})$ and $\{\eta_n\}$ is a Cauchy sequence , there exists $\sigma \in P(\mathfrak{K})$ such that $\eta_{2n+2} \to \sigma$ $as\ n \to \infty$. Then there exists $u \in \mathfrak{K}$ such that $Pu = \sigma$. thus

$$\lim_{n\to\infty} F\,\tau_{2n} = \lim_{n\to\infty} U\,\tau_{2n+1} = \lim_{n\to\infty} G\,\tau_{2n+1} = \lim_{n\to\infty} P\,\tau_{2n+2} = \sigma.$$

Now consider by inequality (v)

$$\begin{aligned}\Gamma(Fu, G\,\tau_{2n+1}) \lesssim_{i_3} &\ \alpha(Pu)\Gamma(Pu, U\,\tau_{2n+1}) + \beta(Pu)\Gamma(Pu, Fu) + \gamma(Pu)\Gamma(U\,\tau_{2n+1}, G\,\tau_{2n+1}) \\ &+ \delta(Pu)[\Gamma(U\,\tau_{2n+1}, Fu) + \Gamma(Pu, G\,\tau_{2n+1})] \\ &+ \Omega(Pu)\left(\frac{\Gamma(Pu, Fu)\Gamma(U\,\tau_{2n+1}, G\,\tau_{2n+1})}{1 + \Gamma(Pu, U\,\tau_{2n+1})}\right) \\ &+ \vartheta(Pu)\left(\frac{\Gamma(U\,\tau_{2n+1}, Fu)\Gamma(Pu, G\,\tau_{2n+1})}{1 + \Gamma(Pu, U\,\tau_{2n+1})}\right) \\ &+ \xi(Pu)max\{\Gamma(Pu, Fu), \Gamma(Pu, U\,\tau_{2n+1}), \Gamma(U\,\tau_{2n+1}, G\,\tau_{2n+1}), \Gamma(U\,\tau_{2n+1}, Fu)\}\end{aligned}$$

$$\lim_{n\to\infty}\|\Gamma(Fu,G\,\tau_{2n+1})\| \precsim_{i_3} \lim_{n\to\infty}\Big\{\alpha(Pu)\|\Gamma(Pu,U\,\tau_{2n+1})\| + \beta(Pu)\|\Gamma(Pu,Fu)\| + \gamma(Pu)\|\Gamma(U\,\tau_{2n+1},G\,\tau_{2n+1})\| + \delta(Pu)[\|\Gamma(U\,\tau_{2n+1},Fu)\| + \|\Gamma(Pu,G\,\tau_{2n+1})\|] + \Omega(Pu)\left(\frac{\|\Gamma(Pu,Fu)\|\|\Gamma(U\,\tau_{2n+1},G\,\tau_{2n+1})\|}{\|1+\Gamma(Pu,U\,\tau_{2n+1})\|}\right) + \vartheta(Pu)\left(\frac{\|\Gamma(U\,\tau_{2n+1},Fu)\|\|\Gamma(Pu,G\,\tau_{2n+1})\|}{\|1+\Gamma(Pu,U\,\tau_{2n+1})\|}\right) + \xi(Pu)max\{\|\Gamma(Pu,Fu)\|,\|\Gamma(Pu,U\,\tau_{2n+1})\|,\|\Gamma(U\,\tau_{2n+1},G\,\tau_{2n+1})\|,\|\Gamma(U\,\tau_{2n+1},Fu)\|\}\Big\}$$

$$\|\Gamma(Fu,\sigma)\| \precsim_{i_3} \Big\{\alpha(\sigma)\|\Gamma(\sigma,\sigma)\| + \beta(\sigma)\|\Gamma(\sigma,Fu)\| + \gamma(\sigma)\|\Gamma(\sigma,\sigma)\| + \delta(\sigma)[\|\Gamma(\sigma,Fu)\| + \|\Gamma(\sigma,\sigma)\|] + \Omega(\sigma)\left(\frac{2\|\Gamma(\sigma,Fu)\|\|\Gamma(\sigma,\sigma)\|}{\|1+\Gamma(\sigma,\sigma)\|}\right) + \vartheta(\sigma)\left(\frac{2\|\Gamma(\sigma,Fu)\|\|\Gamma(\sigma,\sigma)\|}{\|1+\Gamma(\sigma,\sigma)\|}\right) + \xi(\sigma)max\{\|\Gamma(\sigma,Fu)\|,\|\Gamma(\sigma,\sigma)\|,\|\Gamma(\sigma,\sigma)\|,\|\Gamma(\sigma,Fu)\|\}\Big\}$$

$\Rightarrow \|\Gamma(Fu,\sigma)\| \precsim_{i_3} \beta(\sigma)\|\Gamma(\sigma,Fu)\| + \delta(\sigma)\|\Gamma(\sigma,Fu)\| + \xi(\sigma)\|\Gamma(\sigma,Fu)\|$

$\Rightarrow \|\Gamma(\sigma,Fu)\| \precsim_{i_3} (\beta+\delta+\xi)(\sigma)\|\Gamma(\sigma,Fu)\|$

$\Rightarrow ((1-(\beta+\delta+\xi))(\sigma)\|\Gamma(\sigma,Fu)\| \precsim_{i_3} 0$

$\Rightarrow \|\Gamma(\sigma,Fu)\| \precsim_{i_3} 0.$ *Therefore* $Fu=\sigma.$

Thus $Pu = Fu = \sigma$

Since $F(\mathfrak{K}) \subseteq U(\mathfrak{K})$ *then there exists* $v \in \mathfrak{K}$ *such that* $Fu = Uv.$

Thus $Pu = Fu = Uv = \sigma.$

Now by inequality (v)

$$\Gamma(Fu,Gv) \precsim_{i_3} \alpha(Pu)\Gamma(Pu,Uv) + \beta(Pu)\Gamma(Pu,Fu) + \gamma(Pu)\Gamma(Uv,Gv) + \delta(Pu)[\Gamma(Uv,Fu) + \Gamma(Pu,Gv)] + \Omega(Pu)\left(\frac{\Gamma(Pu,Fu)\Gamma(Uv,Gv)}{1+\Gamma(Pu,Uv)}\right) + \vartheta(Pu)\left(\frac{\Gamma(Uv,Fu)\Gamma(Pu,Gv)}{1+\Gamma(Pu,Uv)}\right) + \xi(Pu)max\{\Gamma(Pu,Fu),\Gamma(Pu,Uv),\Gamma(Uv,Gv),\Gamma(Uv,Fu)\}$$

$$\Rightarrow \|\Gamma(\sigma,Gv)\| \precsim_{i_3} \alpha(\sigma)\|\Gamma(\sigma,\sigma)\| + \beta(\sigma)\|\Gamma(\sigma,\sigma)\| + \gamma(\sigma)\|\Gamma(\sigma,Gv)\| + \delta(\sigma)[\|\Gamma(\sigma,\sigma)\| + \|\Gamma(\sigma,Gv)\|] + \Omega(\sigma)\left(\frac{2\|\Gamma(\sigma,\sigma)\|\|\Gamma(\sigma,Gv)\|}{\|1+\Gamma(\sigma,\sigma)\|}\right) + \vartheta(\sigma)\left(\frac{2\|\Gamma(\sigma,\sigma)\|\|\Gamma(\sigma,Gv)\|}{\|1+\Gamma(\sigma,\sigma)\|}\right) + \xi(\sigma)max\{\|\Gamma(\sigma,\sigma)\|,\|\Gamma(\sigma,\sigma)\|,\|\Gamma(\sigma,Gv)\|,\|\Gamma(\sigma,\sigma)\|\}$$

$\Rightarrow \|\Gamma(\sigma,Gv)\| \precsim_{i_3} \gamma(\sigma)\|\Gamma(\sigma,Gv)\| + \delta(\sigma)\|\Gamma(\sigma,Gv)\| + \xi(\sigma)\|\Gamma(\sigma,Gv)\|$

$\Rightarrow \|\Gamma(\sigma,Gv)\| \precsim_{i_3} (\gamma+\delta+\xi)(\sigma)\|\Gamma(\sigma,Gv)\|$

$\Rightarrow ((1-(\gamma+\delta+\xi))(\sigma)\|\Gamma(\sigma,Gv)\| \precsim_{i_3} 0$

$\Rightarrow \|\Gamma(\sigma,Gv)\| \precsim_{i_3} 0.$ *Therefore* $Gv=\sigma.$

Hence $Pu = Fu = Uv = Gv = \sigma.$

Since the pair (F,P) *and* (G,U) are weakly compatible, then

$P\sigma = PFu = FPu = F\sigma$ and $U\sigma = UGv = GUv = G\sigma$

Now by inequality (v)

$$\Gamma(F\sigma,Gv) \precsim_{i_3} \alpha(P\sigma)\Gamma(P\sigma,Uv) + \beta(P\sigma)\Gamma(P\sigma,F\sigma) + \gamma(P\sigma)\Gamma(Uv,Gv) + \delta(P\sigma)[\Gamma(Uv,F\sigma) + \Gamma(P\sigma,Gv)] + \Omega(P\sigma)\left(\frac{\Gamma(P\sigma,F\sigma)\Gamma(Uv,Gv)}{1+\Gamma(P\sigma,Uv)}\right) + \vartheta(P\sigma)\left(\frac{\Gamma(Uv,F\sigma)\Gamma(P\sigma,Gv)}{1+\Gamma(P\sigma,Uv)}\right) + \xi(P\sigma)max\{\Gamma(P\sigma,F\sigma),\Gamma(P\sigma,Uv),\Gamma(Uv,Gv),\Gamma(Uv,F\sigma)\}$$

$$\Rightarrow \|\Gamma(F\sigma,\sigma)\| \precsim_{i_3} \alpha(F\sigma)\|\Gamma(F\sigma,\sigma)\| + \beta(F\sigma)\|\Gamma(F\sigma,F\sigma)\| + \gamma(F\sigma)\|\Gamma(\sigma,\sigma)\| + \delta(F\sigma)[\|\Gamma(\sigma,F\sigma)\| + \|\Gamma(F\sigma,\sigma)\|] + \Omega(F\sigma)\left(\frac{2\Gamma\|F\sigma,F\sigma\|\|\Gamma(\sigma,\sigma)\|}{\|1+\Gamma(F\sigma,\sigma)\|}\right) + \vartheta(F\sigma)\left(\frac{2\|\Gamma(\sigma,F\sigma)\|\|\Gamma(F\sigma,\sigma)\|}{\|1+\Gamma(F\sigma,\sigma)\|}\right) + \xi(F\sigma)max\{\|\Gamma(F\sigma,F\sigma)\|,\|\Gamma(F\sigma,\sigma)\|,\|\Gamma(\sigma,\sigma)\|,\|\Gamma(\sigma,F\sigma)\|\}$$

$\Rightarrow \|\Gamma(F\sigma,\sigma)\| \precsim_{i_3} \alpha(F\sigma)\|\Gamma(F\sigma,\sigma)\|$

$$+\delta(F\sigma)[\|\Gamma(\sigma,F\sigma)\|+\|\Gamma(F\sigma,\sigma)\|]+\vartheta(F\sigma)\left(\frac{2\|\Gamma(\sigma,F\sigma)\|\|\Gamma(F\sigma,\sigma)\|}{\|1+\Gamma(F\sigma,\sigma)\|}\right)+\xi(F\sigma)\|\Gamma(F\sigma,\sigma)\|$$

$\Rightarrow \|\Gamma(F\sigma,\sigma)\| \precsim_{i_3} \alpha(F\sigma)\|\Gamma(F\sigma,\sigma)\|$

$$+2\delta(F\sigma)\|\Gamma(F\sigma,\sigma)\|+2\vartheta(F\sigma)\|\Gamma(F\sigma,\sigma)\|\left(\left\|\frac{\Gamma(F\sigma,\sigma)}{1+\Gamma(F\sigma,\sigma)}\right\|\right)++\xi(F\sigma)\|\Gamma(F\sigma,\sigma)\|$$

$\Rightarrow \|\Gamma(F\sigma,\sigma)\| \precsim_{i_3} (\alpha+2\delta+2\vartheta)(F\sigma)\|\Gamma(F\sigma,\sigma)\|$

$\Rightarrow ((1-(\alpha+2\delta+2\vartheta))(F\sigma)\|\Gamma(F\sigma,\sigma)\| \precsim_{i_3} 0$

$\Rightarrow \|\Gamma(F\sigma,\sigma)\| \precsim_{i_3} 0.$ *Therefore* $F\sigma=\sigma.$

Thus $F\sigma = P\sigma = \sigma.$ ……………………….(a)

and

$\|\Gamma(\sigma,G\sigma)\| \precsim_{i_3} \alpha(\sigma)\|\Gamma(\sigma,\sigma)\|+\beta(\sigma)\|\Gamma(\sigma,\sigma)\|$

$$+\gamma(\sigma)\|\Gamma(\sigma,G\sigma)\|+\delta(\sigma)[\|\Gamma(\sigma,\sigma)\|+\|\Gamma(\sigma,G\sigma)\|]+\Omega(\sigma)\left(\frac{2\|\Gamma(\sigma,\sigma)\|\|\Gamma(\sigma,G\sigma)\|}{\|1+\Gamma(\sigma,\sigma)\|}\right)$$

$$+\vartheta(\sigma)\left(\frac{2\|\Gamma(\sigma,\sigma)\|\|\Gamma(\sigma,G\sigma)\|}{\|1+\Gamma(\sigma,\sigma)\|}\right)$$

$$+\xi(\sigma)\|\Gamma(\sigma,\sigma)\|,\|\Gamma(\sigma,U\sigma)\|,\|\Gamma(\sigma,G\sigma)\|,\|\Gamma(\sigma,\sigma)\|$$

$\Rightarrow \|\Gamma(\sigma,G\sigma)\| \precsim_{i_3} \gamma(\sigma)\|\Gamma(\sigma,G\sigma)\|+\ \delta(\sigma)\|\Gamma(\sigma,G\sigma)\|+\xi(\sigma)\|\Gamma(\sigma,G\sigma)\|$

$\Rightarrow \|\Gamma(\sigma,G\sigma)\| \precsim_{i_3} (\gamma+\delta+\xi)(\sigma)\|\Gamma(\sigma,G\sigma)\|$

$\Rightarrow ((1-(\gamma+\delta+\xi))(\sigma)\|\Gamma(\sigma,G\sigma)\| \precsim_{i_3} 0$

$\Rightarrow \|\Gamma(\sigma,G\sigma)\| \precsim_{i_3} 0.$ *Therefore* $G\sigma=\sigma.$

Thus $G\sigma = U\sigma = \sigma.$ ……………………….(b)

From equation (a) and (b) we get

$F\sigma = P\sigma = G\sigma = U\sigma = \sigma.$

Hence $F, G, P,$ *and* U have a unique common fixed point.

Uniqueness:

Suppose that $\mathfrak{z}' \in \mathfrak{K}$ be another common fixed point such that
$F\mathfrak{z}' = P\mathfrak{z}' = \mathfrak{z}' = G\mathfrak{z}' = U\mathfrak{z}'$ then by inequality (v)

$\Gamma(F\sigma,G\mathfrak{z}') \precsim_{i_3} \alpha(P\sigma)\Gamma(P\sigma,U\mathfrak{z}')+\beta(P\sigma)\mathcal{S}(P\sigma,F\sigma)$

$$+\gamma(P\sigma)\Gamma(U\mathfrak{z}',G\mathfrak{z}')+\delta(P\sigma)[\mathcal{S}(U\mathfrak{z}',F\sigma)+\Gamma(P\sigma,G\mathfrak{z}')]+\Omega(P\sigma)\left(\frac{\Gamma(P\sigma,F\sigma)\Gamma(U\mathfrak{z}',G\mathfrak{z}')}{1+\Gamma(P\sigma,U\mathfrak{z}')}\right)$$

$$+\vartheta(P\sigma)\left(\frac{\Gamma(U\mathfrak{z}',F\sigma)\Gamma(P\sigma,G\mathfrak{z}')}{1+\Gamma(P\sigma,U\mathfrak{z}')}\right)$$

$$+\xi(P\sigma)max\{\Gamma(P\sigma,F\sigma),\mathcal{S}(P\sigma,U\mathfrak{z}'),\Gamma(U\mathfrak{z}',G\mathfrak{z}'),\Gamma(U\mathfrak{z}',F\sigma)\}$$

$\Rightarrow \Gamma(\sigma,\mathfrak{z}') \precsim_{i_3} \alpha(\sigma)\Gamma(\sigma,\mathfrak{z}')+\beta(\sigma)\Gamma(\sigma,\sigma)$

$$+\gamma(\sigma)\Gamma(\mathfrak{z}',\mathfrak{z}')+\delta(\sigma)[\mathcal{S}(\mathfrak{z}',z)+\Gamma(\sigma,\mathfrak{z}')]+\Omega(\sigma)\left(\frac{\Gamma(\sigma,\sigma)\Gamma(\mathfrak{z}',\mathfrak{z}')}{1+\Gamma(\sigma,\mathfrak{z}')}\right)$$

$$+\vartheta(\sigma)\left(\frac{\Gamma(\mathfrak{z}',\sigma)\Gamma(\sigma,\mathfrak{z}')}{1+\Gamma(\sigma,\mathfrak{z}')}\right)$$

$$+\xi(\sigma)max\{\Gamma(\sigma,\sigma),\Gamma(\sigma,\mathfrak{z}'),\Gamma(\mathfrak{z}',\mathfrak{z}'),\Gamma(\mathfrak{z}',\sigma)\}$$

$\Rightarrow \Gamma(\sigma,\mathfrak{z}') \precsim_{i_3} \alpha(\sigma)\Gamma(\sigma,\mathfrak{z}')+\delta(\sigma)[\Gamma(\mathfrak{z}',\sigma)+\Gamma(\sigma,\mathfrak{z}')]$

$$+\vartheta(\sigma)\left(\frac{\Gamma(\mathfrak{z}',\sigma)\Gamma(\sigma,\mathfrak{z}')}{1+\Gamma(\sigma,\mathfrak{z}')}\right)+\xi(\sigma)\Gamma(\sigma,\mathfrak{z}')$$

$\Rightarrow \|\Gamma(\sigma,\mathfrak{z}')\| \precsim_{i_3} \alpha(\sigma)\|\Gamma(\sigma,\mathfrak{z}')\|+2\delta(\sigma)\Gamma(\sigma,\mathfrak{z}')$

$$+\vartheta(\sigma)\|\sigma(\mathfrak{z}',\sigma)\|\left(\left\|\frac{\Gamma(\sigma,\mathfrak{z}')}{1+\Gamma(\sigma,\mathfrak{z}')}\right\|\right)+\xi(\sigma)\Gamma(\sigma,\mathfrak{z}')$$

$\Rightarrow \|\mathcal{S}(\sigma,\mathfrak{z}')\| \precsim_{i_3} \alpha(\sigma)\|\Gamma(\sigma,\mathfrak{z}')\|+2\delta(\sigma)\|\Gamma(\sigma,\mathfrak{z}')\|+\vartheta(\sigma)\|\Gamma(\sigma,\mathfrak{z}')\|+\xi(\sigma)\Gamma(\sigma,\mathfrak{z}')$

$\Rightarrow \|\Gamma(\sigma,\mathfrak{z}')\| \precsim_{i_3} (\alpha+2\delta+\vartheta+\xi)(\sigma)\|\Gamma(\sigma,\mathfrak{z}')\|$

$\Rightarrow ((1-(\alpha+2\delta+\vartheta+\xi))(\sigma)\|\Gamma(\sigma,\mathfrak{z}')\| \precsim_{i_3} 0$

$\Rightarrow \|\Gamma(\sigma,\mathfrak{z}')\| \precsim_{i_3} 0.$ *Therefore* $\sigma=\mathfrak{z}'.$

Hence σ *is the unique common fixed point of* $F, G, P,$ *and* U . Similarly prove of theorem can follows if $U(\mathfrak{K})$ is a complete subspace of $\mathfrak{K}$.

Example 3.3: Let $\mathfrak{K}=[0,1]$ and $\Gamma:\mathfrak{K}^3 \to \mathbb{C}_3$ be defined by $\Gamma(\tau,\eta)=|\tau-\eta|^2+i_3|\tau-\eta|^2$. then $(\mathfrak{K},\mathcal{S})$ is a *TCVbMS with* $s \geq 1$. Let $F, G, P, U: \mathfrak{K} \to \mathfrak{K}$ be a self maps given by

$F(\tau)=\frac{\tau}{2},\ G(\tau)=\frac{\tau}{8}, P(\tau)=\frac{\tau}{4},\ U(\tau)=\frac{\tau}{5}\ \ for\ all\ \tau\in\mathfrak{K}$, we define the function

$$\alpha,\beta,\gamma,\delta,\Omega,\vartheta,\xi:\mathfrak{K}^3\to\left[0,\frac{1}{s}\right)\ by$$

$$\alpha(\tau)=\frac{\tau}{20},\qquad \beta(\tau)=\frac{\tau}{10},\qquad \gamma(\tau)=\frac{\tau^2}{10},$$

$$\delta(\tau)=\frac{\tau^3}{10},\qquad \Omega(\tau)=\frac{\tau^3}{5},\quad \vartheta(\tau)=\frac{\tau^2}{11},\qquad \xi(\tau)=\frac{\tau}{11}$$

then $\alpha(\tau)+\beta(\tau)+\gamma(\tau)+2\delta(\tau)+\Omega(\tau)+\vartheta(\tau)+\xi(\tau)<1$

$$\leq\frac{\tau}{20}+\frac{\tau}{10}+\frac{\tau^2}{10}+\frac{2\tau^3}{10}+\frac{\tau^3}{5}+\frac{\tau^2}{11}+\frac{\tau}{11}$$

$$\leq\frac{1}{20}+\frac{1}{10}+\frac{1}{10}+\frac{1}{5}+\frac{1}{5}+\frac{1}{11}+\frac{1}{11}$$

$$=.8318<1.$$

Clearly, $\alpha(\tau)+\beta(\tau)+\gamma(\tau)+2s\delta(\tau)+\Omega(\tau)+\vartheta(\tau)+\xi(\tau)<1.$ for all $\tau,\eta\in\mathfrak{K}$.

Now also,

$$\alpha(F\tau)\leq\alpha(\tau)=\alpha\left(\frac{\tau}{2}\right)=\frac{\tau}{40}\leq\frac{\tau}{20}=\alpha(\tau);\quad \beta(F\tau)\leq\beta(\tau)=\beta\left(\frac{\tau}{2}\right)=\frac{\tau}{20}\leq\frac{\tau}{10}=\beta(\tau),$$

$\gamma(F\tau)\leq\gamma(\tau)=\gamma\left(\frac{\tau}{2}\right)=\frac{\tau^2}{20}\leq\frac{\tau^2}{10}=\gamma(\tau),\quad \delta(F\tau)\leq\delta(\tau)=\delta\left(\frac{\tau}{2}\right)=\frac{\tau^3}{20}\leq\frac{\tau^3}{10}=\delta(\tau),\quad \Omega(F\tau)\leq \Omega(\tau)=\Omega\left(\frac{\tau}{2}\right)=\frac{\tau^3}{10}\leq\frac{\tau^3}{5}=\Omega(\tau),\quad \vartheta(F\tau)\leq\vartheta(\tau)=\vartheta\left(\frac{\tau}{2}\right)=\frac{\tau^2}{22}\leq\frac{\tau^2}{11}=\vartheta(\tau),\ and\quad \xi(F\tau)=\xi\left(\frac{\tau}{2}\right)=\frac{\tau}{22}\leq\frac{\tau}{11}=\xi(\tau)$

Similarly we can show all the following conditions:

$\alpha(G\tau)\leq\alpha(\tau);\quad \beta(G\tau)\leq\beta(\tau),\quad \gamma(G\tau)\leq\gamma(\tau),\quad \delta(G\tau)\leq\delta(\tau),\Omega(G\tau)\leq\Omega(\tau),\quad \vartheta(G\tau)\leq\vartheta(\tau),\ and\quad \xi(G\tau)\leq\xi(\tau)\quad \alpha(P\tau)\leq\alpha(\tau);\quad \beta(P\tau)\leq\beta(\tau),\quad \gamma(P\tau)\leq\gamma(\tau),\quad \delta(P\tau)\leq\delta(\tau)=,\Omega(P\tau)\leq \Omega(\tau),\quad \vartheta(P\tau)\leq\vartheta(\tau),\ and\quad \xi(P\tau)\leq\xi(\tau)$

$\alpha(U\tau)\leq\alpha(\tau);\quad \beta(U\tau)\leq\beta(\tau),\quad \gamma(U\tau)\leq\gamma(\tau),$
$\delta(U\tau)\leq\delta(\tau),\Omega(U\tau)\leq\Omega(\tau),\quad \vartheta(U\tau)\leq\vartheta(\tau),\ and\quad \xi(U\tau)\leq\xi(\tau)$

Now, for verifying inequality (V), one needs to note that

$$0\precsim_{i_3}\left\{\begin{array}{c}\Gamma(P\tau,U\eta),\Gamma(P\tau,F\tau),\Gamma(U\eta,G\eta),[\Gamma(U\eta,F\tau)+\Gamma(P\tau,G\eta)],\\ \frac{\Gamma(P\tau,F\tau)\Gamma(U\eta,G\eta)}{1+\Gamma(P\tau,U\eta)},\frac{\Gamma(U\eta,F\tau)\Gamma(P\tau,G\eta)}{1+\Gamma(P\tau,U\eta)},max\{\Gamma(P\tau,F\tau),\Gamma(P\tau,U\eta),\Gamma(U\eta,G\eta),\Gamma(U\eta,F\tau)\}\end{array}\right\}$$

Now, it is sufficient to show that $\Gamma(F\tau,G\eta)\precsim_{i_3}\alpha(\tau)\Gamma(P\tau,,U\eta)$

Now consider

$$\Gamma(F\tau,G\eta)=|F\tau-G\eta|^2+i_3|F\tau-G\eta|^2$$

$$=\left|\frac{\tau}{2}-\frac{\eta}{8}\right|^2+i_3\left|\frac{\tau}{2}-\frac{\eta}{8}\right|^2\precsim_{i_3}\frac{\tau}{20}\{|F\tau-G\eta|^2+i_3|F\tau-G\eta|^2\}=\alpha(\tau)\Gamma(P\tau,U\eta)$$

i.e, $\Gamma(F\tau,G\eta)\precsim_{i_3}\alpha(\tau)\Gamma(P\tau,U\eta)$, Thus all the conditions of above theorem are satisfied and $\tau=0$ is the fixed point of $F,G,P,and\ U$.

Example 3.4: Let $\mathfrak{K}=[0,1]$ and $\Gamma:\mathfrak{K}^3\to €_3$ be defined by $\Gamma(\tau,\eta)=|\tau-\eta|^2e^{i_3\frac{\pi}{6}}$. then $(\mathfrak{K},\mathcal{S})$ is a $TCVbMS\ with\ s\geq1$. Let $F,G,P,U:\mathfrak{K}\to\mathfrak{K}$ be a self maps given by

$F(\tau)=\frac{\tau}{7},\ G(\tau)=\frac{\tau}{8}, P(\tau)=\frac{\tau}{6},\ U(\tau)=\frac{\tau}{5}\ \ for\ all\ \tau\in\mathfrak{K}$ By $\alpha,\beta,\gamma,\delta,\Omega,\vartheta,\xi:\mathfrak{K}^3\to\left[0,\frac{1}{s}\right)$

Clearly, $\alpha(\tau)+\beta(\tau)+\gamma(\tau)+2s\delta(\tau)+\Omega(\tau)+\vartheta(\tau)+\xi(\tau)<1.$ for all $\tau,\eta\in\mathfrak{K}$.

Now also,

$$\alpha(F\tau)\leq\alpha(\tau)=\alpha\left(\frac{\tau}{7}\right)=\frac{\tau}{140}\leq\frac{\tau}{20}=\alpha(\tau);\quad \beta(F\tau)\leq\beta(\tau)=\beta\left(\frac{\tau}{7}\right)=\frac{\tau}{70}\leq\frac{\tau}{10}=\beta(\tau),$$

$\gamma(F\tau)\leq\gamma(\tau)=\gamma\left(\frac{\tau}{7}\right)=\frac{\tau^2}{70}\leq\frac{\tau^2}{10}=\gamma(\tau),\quad \delta(F\tau)\leq\delta(\tau)=\delta\left(\frac{\tau}{7}\right)=\frac{\tau^3}{70}\leq\frac{\tau^3}{10}=\delta(\tau),\quad \Omega(F\tau)\leq \Omega(\tau)=\Omega\left(\frac{\tau}{7}\right)=\frac{\tau^3}{70}\leq\frac{\tau^3}{5}=\Omega(\tau),\quad \vartheta(F\tau)\leq\vartheta(\tau)=\vartheta\left(\frac{\tau}{7}\right)=\frac{\tau^2}{77}\leq\frac{\tau^2}{11}=\vartheta(\tau),\ and\quad \xi(F\tau)=\xi\left(\frac{\tau}{7}\right)=\frac{\tau}{77}\leq\frac{\tau}{11}=\xi(\tau)$

Similarly we can show all the following conditions:

$\alpha(G\tau)\leq\alpha(\tau);\quad \beta(G\tau)\leq\beta(\tau),\quad \gamma(G\tau)\leq\gamma(\tau),\quad \delta(G\tau)\leq\delta(\tau),\Omega(G\tau)\leq\Omega(\tau),\quad \vartheta(G\tau)\leq\vartheta(\tau),\ and\quad \xi(G\tau)\leq\xi(\tau)\quad \alpha(P\tau)\leq\alpha(\tau);\quad \beta(P\tau)\leq\beta(\tau),\quad \gamma(P\tau)\leq\gamma(\tau),\quad \delta(P\tau)\leq\delta(\tau)=,\Omega(P\tau)\leq \Omega(\tau),\quad \vartheta(P\tau)\leq\vartheta(\tau),\ and\quad \xi(P\tau)\leq\xi(\tau)$

$\alpha(U\tau)\leq\alpha(\tau);\quad \beta(U\tau)\leq\beta(\tau),\quad \gamma(U\tau)\leq\gamma(\tau),$
$\delta(U\tau)\leq\delta(\tau),\Omega(U\tau)\leq\Omega(\tau),\quad \vartheta(U\tau)\leq\vartheta(\tau),\ and\quad \xi(U\tau)\leq\xi(\tau)$

Now, for verifying inequality (V), one needs to note that

$$0 \precsim_{i_3} \left\{ \begin{array}{c} \Gamma(P\tau, U\eta), \Gamma(P\tau, F\tau), \Gamma(U\eta, G\eta), [\Gamma(U\eta, F\tau) + \Gamma(P\tau, G\eta)], \\ \frac{\Gamma(P\tau, F\tau)\Gamma(U\eta, G\eta)}{1+\Gamma(P\tau, U\eta)}, \frac{\Gamma(U\eta, F\tau)\Gamma(P\tau, G\eta)}{1+\Gamma(P\tau, U\eta)}, max\{\Gamma(P\tau, F\tau), \Gamma(P\tau, U\eta), \Gamma(U\eta, G\eta), \Gamma(U\eta, F\tau)\} \end{array} \right\}$$

Now, it is sufficient to show that $\Gamma(F\tau, G\eta) \precsim_{i_3} \alpha(\tau)\Gamma(P\tau,, U\eta)$

Now consider

$$\Gamma(F\tau, G\eta) = \Gamma\left(\frac{\tau}{7}, \frac{\eta}{8}\right)$$

$$= \left|\frac{\tau}{7} - \frac{\eta}{8}\right|^2 e^{i_3\frac{\pi}{6}} \precsim_{i_3} \frac{\tau}{7}\left\{\left|\frac{\tau}{7} - \frac{\eta}{8}\right|^2 e^{i_3\frac{\pi}{6}}\right\} = \alpha(\tau)\Gamma(P\tau, U\eta)$$

i.e, $\Gamma(F\tau, G\eta) \precsim_{i_3} \alpha(\tau)\Gamma(P\tau, U\eta)$, Thus all the conditions of above theorem are satisfied and $\tau = 0$ is the fixed point of $F, G, P, and\ U$.

By setting $\xi = 0$ in theorem (3.3) then we have obtain following:

Corollary 3.5: Let $(\mathfrak{K}, \mathcal{S})$ be a $TCVbMS$ and $F, G, P, U: \mathfrak{K} \to \mathfrak{K}$ satisfying the conditions:

(i) $G(\mathfrak{K}) \subseteq P(\mathfrak{K})\ and\ F(\mathfrak{K}) \subseteq U(\mathfrak{K})$,

(ii) The pair $(F, P)\ and\ (G, U)$ are weakly compatible,

(iii) $P(\mathfrak{K})\ or\ U(\mathfrak{K})$ is a complete subspace of $\mathfrak{K}$,

(iv) If there exist mapping $\alpha, \beta, \gamma, \delta, \Omega, \vartheta, \xi: \mathfrak{K}^3 \to [0,1)\ such\ that$

$$\alpha(F\tau) \le \alpha(\tau); \quad \beta(F\tau) \le \beta(\tau), \quad \gamma(F\tau) \le \gamma(\tau), \quad \delta(F\tau) \le \delta(\tau),\ \Omega(F\tau) \le \Omega(\tau)$$

$$\alpha(G\tau) \le \alpha(\tau); \quad \beta(G\tau) \le \beta(\tau), \quad \gamma(G\tau) \le \gamma(\tau), \quad \delta(G\tau) \le \delta(\tau),\ \Omega(G\tau) \le \Omega(\tau)$$

$$\alpha(P\tau) \le \alpha(\tau); \quad \beta(P\tau) \le \beta(\tau), \quad \gamma(P\tau) \le \gamma(\tau), \quad \delta(P\tau) \le \delta(\tau), \Omega(P\tau) \le \Omega(\tau), \quad \vartheta(P\tau) \le \vartheta(\tau),$$

$$\alpha(U\tau) \le \alpha(\tau); \quad \beta(U\tau) \le \beta(\tau), \quad \gamma(U\tau) \le \gamma(\tau), \quad \delta(U\tau) \le \delta(\tau), \Omega(U\tau) \le \Omega(\tau), \quad \vartheta(U\tau) \le \vartheta(\tau),$$

$$for\ all\ \tau, \eta\ \in \mathfrak{K}\ and\ for\ a\ fixed\ point\ element\ a \in \mathfrak{K},$$

(v) $\Gamma(F\tau, G\eta) \precsim_{i_3} \alpha(P\tau)\Gamma(P\tau, U\eta) + \beta(P\tau)\Gamma(P\tau, F\tau)$

$$+\gamma(P\tau)\Gamma(U\eta, G\eta) + \delta(P\tau)[\Gamma(U\eta, F\tau) + \Gamma(P\tau, G\eta)] + \Omega(P\tau)\left(\frac{\Gamma(P\tau, F\tau)\Gamma(U\eta, G\eta)}{1+\Gamma(P\tau, U\eta)}\right)$$

$$+\vartheta(P\tau)\left(\frac{\Gamma(U\eta, F\tau)\Gamma(P\tau, G\eta)}{1+\Gamma(P\tau, U\eta)}\right)$$

$$for\ all\ \tau, \eta \in \mathfrak{K}\ ,$$

And $\alpha(\tau) + \beta(\tau) + \gamma(\tau) + 3s\delta(\tau) + \Omega(\tau) + \vartheta(\tau) < 1$

Then $F, G, P, and\ U$ have a unique common fixed point.

By setting $\vartheta, \xi = 0$ in theorem (3.3) then we have obtain following:

Corollary 3.6: Let $(\mathfrak{K}, \mathcal{S})$ be a $TCVbMS$ and $F, G, P, U: \mathfrak{K} \to \mathfrak{K}$ satisfying the conditions:

(i) $G(\mathfrak{K}) \subseteq P(\mathfrak{K})\ and\ F(\mathfrak{K}) \subseteq U(\mathfrak{K})$,

(ii) The pair $(F, P)\ and\ (G, U)$ are weakly compatible,

(iii) $P(\mathfrak{K})\ or\ U(\mathfrak{K})$ is a complete subspace of $\mathfrak{K}$,

(iv) If there exist mapping $\alpha, \beta, \gamma, \delta, \Omega, \vartheta, \xi: \mathfrak{K}^3 \to [0,1)\ such\ that$

$$\alpha(F\tau) \le \alpha(\tau); \quad \beta(F\tau) \le \beta(\tau), \quad \gamma(F\tau) \le \gamma(\tau),\ \delta(F\tau) \le \delta(\tau),\ \Omega(G\tau) \le \Omega(\tau)$$

$$\alpha(G\tau) \le \alpha(\tau); \quad \beta(G\tau) \le \beta(\tau), \quad \gamma(G\tau) \le \gamma(\tau), \delta(G\tau) \le \delta(\tau),\ \Omega(G\tau) \le \Omega(\tau)$$

$$\alpha(P\tau) \le \alpha(\tau); \quad \beta(P\tau) \le \beta(\tau), \quad \gamma(P\tau) \le \gamma(\tau), \delta(P\tau) \le \delta(\tau),\ \Omega(P\tau) \le \Omega(\tau)$$

$$\alpha(U\tau) \le \alpha(\tau); \quad \beta(U\tau) \le \beta(\tau), \quad \gamma(U\tau) \le \gamma(\tau), \delta(U\tau) \le \delta(\tau),\ \Omega(U\tau) \le \Omega(\tau)$$

$$for\ all\ \tau, \eta\ \in \mathfrak{K}\ and\ for\ a\ fixed\ point\ element\ a \in \mathfrak{K},$$

(v) $(\Gamma(F\tau, G\eta) \precsim_{i_3} \alpha(P\tau)\Gamma(P\tau, U\eta) + \beta(P\tau)\Gamma(P\tau, F\tau)$

$$+\gamma(P\tau)\Gamma(U\eta, G\eta) + \delta(P\tau)[\Gamma(U\eta, F\tau) + \Gamma(P\tau, G\eta)] + \Omega(P\tau)\left(\frac{\Gamma(P\tau, F\tau)\Gamma(U\eta, G\eta)}{1+\Gamma(P\tau, U\eta)}\right)$$

$$+\vartheta(P\tau)\left(\frac{\Gamma(U\eta, F\tau)\Gamma(P\tau, G\eta)}{1+\Gamma(P\tau, U\eta)}\right)$$

$$for\ all\ \tau, \eta \in \mathfrak{K}\ ,$$

And $\alpha(\tau) + \beta(\tau) + \gamma(\tau) + 3s\delta(\tau) + \Omega(\tau) + \vartheta(\tau) < 1$

Then $F, G, P, and\ U$ have a unique common fixed point.

By setting $\Omega, \vartheta, \xi = 0$ in theorem (3.3) then we have obtain following:

Corollary 3.7: Let $(\mathfrak{K}, \mathcal{S})$ be a $TCVbMS$ and $F, G, P, U: \mathfrak{K} \to \mathfrak{K}$ satisfying the conditions:

(i) $G(\mathfrak{K}) \subseteq P(\mathfrak{K})$ and $F(\mathfrak{K}) \subseteq U(\mathfrak{K})$,

(ii) The pair (F, P) and (G, U) are weakly compatible,

(iii) $P(\mathfrak{K})$ or $U(\mathfrak{K})$ is a complete subspace of $\mathfrak{K}$,

(iv) If there exist mapping $\alpha, \beta, \gamma, \delta, \Omega, \vartheta, \xi: \mathfrak{K}^3 \to [0,1)$ such that

$$\alpha(F\tau) \le \alpha(\tau); \quad \beta(F\tau) \le \beta(\tau), \quad \gamma(F\tau) \le \gamma(\tau), \delta(F\tau) \le \delta(\tau)$$
$$\alpha(G\tau) \le \alpha(\tau); \quad \beta(G\tau) \le \beta(\tau), \quad \gamma(G\tau) \le \gamma(\tau), \delta(G\tau) \le \delta(\tau)$$
$$\alpha(P\tau) \le \alpha(\tau); \quad \beta(P\tau) \le \beta(\tau), \quad \gamma(P\tau) \le \gamma(\tau), \delta(P\tau) \le \delta(\tau)$$
$$\alpha(U\tau) \le \alpha(\tau); \quad \beta(U\tau) \le \beta(\tau), \quad \gamma(U\tau) \le \gamma(\tau), \delta(U\tau) \le \delta(\tau)$$

(v) for all $\tau, \eta \in \mathfrak{K}$ and for a fixed point element $a \in \mathfrak{K}$,
$$\Gamma(F\tau, G\eta) \precsim_{i_3} \alpha(P\tau)\Gamma(P\tau, U\eta) + \beta(P\tau)\Gamma(P\tau, F\tau) + \gamma(P\tau)\Gamma(U\eta, G\eta) + \delta(P\tau)[\Gamma(U\eta, F\tau) + \Gamma(P\tau, G\eta)]$$
for all $\tau, \eta \in \mathfrak{K}$,

And $\alpha(\tau) + \beta(\tau) + \gamma(\tau) + 3s\delta(\tau) < 1$

Then $F, G, P,$ and U have a unique common fixed point.

By setting $\delta, \Omega, \vartheta, \xi = 0$ in theorem (3.3) then we have obtain following:

Corollary 3.8: Let $(\mathfrak{K}, \mathcal{S})$ be a $TCVbMS$ and $F, G, P, U: \mathfrak{K} \to \mathfrak{K}$ satisfying the conditions:

(i) $G(\mathfrak{K}) \subseteq P(\mathfrak{K})$ and $F(\mathfrak{K}) \subseteq U(\mathfrak{K})$,

(ii) The pair (F, P) and (G, U) are weakly compatible,

(iii) $P(\mathfrak{K})$ or $U(\mathfrak{K})$ is a complete subspace of $\mathfrak{K}$,

(iv) If there exist mapping $\alpha, \beta, \gamma, \delta, \Omega, \vartheta, \xi: \mathfrak{K}^3 \to [0,1)$ such that

$$\alpha(F\tau) \le \alpha(\tau); \quad \beta(F\tau) \le \beta(\tau), \quad \gamma(F\tau) \le \gamma(\tau)$$
$$\alpha(G\tau) \le \alpha(\tau); \quad \beta(G\tau) \le \beta(\tau), \quad \gamma(G\tau) \le \gamma(\tau)$$
$$\alpha(P\tau) \le \alpha(\tau); \quad \beta(P\tau) \le \beta(\tau), \quad \gamma(P\tau) \le \gamma(\tau)$$
$$\alpha(U\tau) \le \alpha(\tau); \quad \beta(U\tau) \le \beta(\tau), \quad \gamma(U\tau) \le \gamma(\tau)$$

for all $\tau, \eta \in \mathfrak{K}$ and for a fixed point element $a \in \mathfrak{K}$,

(vi) $$\Gamma(F\tau, G\eta) \precsim_{i_3} \alpha(P\tau)\Gamma(P\tau, U\eta) + \beta(P\tau)\Gamma(P\tau, F\tau) + \gamma(P\tau)\Gamma(U\eta, G\eta)$$

for all $\tau, \eta \in \mathfrak{K}$,

And $\alpha(\tau) + \beta(\tau) + \gamma(\tau) < 1$

Then $F, G, P,$ and U have a unique common fixed point.

By setting $\gamma, \delta, \Omega, \vartheta, \xi = 0$ in theorem (3.3) then we have obtain following:

Corollary 3.9: Let $(\mathfrak{K}, \mathcal{S})$ be a $TCVbMS$ and $F, G, P, U: \mathfrak{K} \to \mathfrak{K}$ satisfying the conditions:

(i) $G(\mathfrak{K}) \subseteq P(\mathfrak{K})$ and $F(\mathfrak{K}) \subseteq U(\mathfrak{K})$,

(ii) The pair (F, P) and (G, U) are weakly compatible,

(iii) $P(\mathfrak{K})$ or $U(\mathfrak{K})$ is a complete subspace of $\mathfrak{K}$,

(iv) If there exist mapping $\alpha, \beta, \gamma, \delta, \Omega, \vartheta, \xi: \mathfrak{K}^3 \to [0,1)$ such that

$$\alpha(F\tau) \le \alpha(\tau); \quad \beta(F\tau) \le \beta(\tau)$$
$$\alpha(G\tau) \le \alpha(\tau); \quad \beta(G\tau) \le \beta(\tau)$$
$$\alpha(P\tau) \le \alpha(\tau); \quad \beta(P\tau) \le \beta(\tau)$$
$$\alpha(U\tau) \le \alpha(\tau); \quad \beta(U\tau) \le \beta(\tau)$$ or all $\tau, \eta \in \mathfrak{K}$ and for a fixed point element $a \in \mathfrak{K}$

(v) $$\Gamma(F\tau, G\eta) \precsim_{i_3} \alpha(P\tau)\Gamma(P\tau, U\eta) + \beta(P\tau)\Gamma(P\tau, F\tau)$$
for all $\tau, \eta \in \mathfrak{K}$,

And $\alpha(\tau) + \beta(\tau) < 1$

By setting $\beta, \gamma, \delta, \Omega, \vartheta, \xi = 0$ in theorem (3.3) then we have obtain following:

Corollary 3.10: Let $(\mathfrak{K}, \mathcal{S})$ be a $TCVbMS$ and $F, G, P, U: \mathfrak{K} \to \mathfrak{K}$ satisfying the conditions:

(i) $G(\mathfrak{K}) \subseteq P(\mathfrak{K})$ and $F(\mathfrak{K}) \subseteq U(\mathfrak{K})$,

(ii) The pair (F, P) and (G, U) are weakly compatible,

(iii) $P(\mathfrak{K})$ or $U(\mathfrak{K})$ is a complete subspace of $\mathfrak{K}$,

(iv) If there exist mapping $\alpha, \beta, \gamma, \delta, \Omega, \vartheta, \xi: \mathfrak{K}^3 \to [0,1)$ such that

$$\alpha(F\tau) \le \alpha(\tau); \quad \beta(F\tau) \le \beta(\tau)$$
$$\alpha(G\tau) \le \alpha(\tau); \quad \beta(G\tau) \le \beta(\tau)$$
$$\alpha(P\tau) \le \alpha(\tau); \quad \beta(P\tau) \le \beta(\tau)$$
$$\alpha(U\tau) \le \alpha(\tau); \quad \beta(U\tau) \le \beta(\tau)$$ or all $\tau, \eta \in \mathfrak{K}$ and for a fixed point element $a \in \mathfrak{K}$

(v) $$\Gamma(F\tau, G\eta) \precsim_{i_3} \alpha(P\tau)\Gamma(P\tau, U\eta) + \beta(P\tau)\Gamma(P\tau, F\tau)$$

for all $\tau, \eta \in \mathfrak{K}$,

And $\alpha(\tau) < 1$

Then $F, G, P, and\ U$ have a unique common fixed point.

4. Utilization: Let $\mathfrak{K} = \mathcal{C}[\Omega_1, \Omega_2]$ be the set of all continues factions on $[\Omega_1, \Omega_2]$ armed with metric $\Gamma(\tau,\eta) = (1+i_3)[|\tau(\mathfrak{z}) - \eta(\mathfrak{z})|^2]$ for all $\tau, \eta \in \mathcal{C}[\Omega_1, \Omega_2]\ and\ \mathfrak{z} \in [\Omega_1, \Omega_2]$, where $|.|$ is the real modulus and $\boldsymbol{\alpha}: \mathfrak{K}^3 \to \left[0, \frac{1}{s}\right)$ be defined by $\alpha(\tau) = \frac{\tau}{20}, \beta(\tau) = \frac{\tau}{10}, \gamma(\tau) = \frac{\tau^2}{10}, \delta(\tau) = \frac{\tau^3}{10},$

$$\Omega(\tau) = \frac{\tau^3}{5},\ \ \vartheta(\tau) = \frac{\tau^2}{11},\ \xi(\tau) = \frac{\tau}{11},$$

Clearly verified all conditions i.e $\alpha(\tau) < 1\ for\ all\ \tau, \eta \in \mathfrak{K}$. then $(\mathfrak{K}, \mathcal{S})$ is a *CTVbMS*. Consider the integral equations of the non-linear Fredholm type integral equation

$$\tau(\mathfrak{z}) = t(\mathfrak{z}) + \frac{1}{\Omega_2 - \Omega_1}\int_{\Omega_1}^{\Omega_2} \Theta_1(\mathfrak{z}, s, \tau(s))ds \ldots\ldots\ldots\ldots\ldots(a)$$

and

$$\tau(\mathfrak{z}) = t(\mathfrak{z}) + \frac{1}{\Omega_2 - \Omega_1}\int_{\Omega_1}^{\Omega_2} \Theta_2(\mathfrak{z}, s, \tau(s))ds \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(b)$$

Where $t: [\Omega_1, \Omega_2] \to \mathbb{R}$ and $\Theta_1, \Theta_2: [\Omega_1, \Omega_2] \times [\Omega_1, \Omega_2] \times \mathbb{R} \to \mathbb{R}$ are continuous for $\mathfrak{z} \in [\Omega_1, \Omega_2] \to \mathbb{R}$ are continuous, and $t(\mathfrak{z})$ is function of $\mathfrak{K}$. Now we define partial order $\precsim_{i_3}\ in\ \epsilon_3\ as\ follows\ \tau(\mathfrak{z}) \precsim_{i_3} \eta(\mathfrak{z})\ iff\ \tau \le \eta.$

Theorem 3.11: Consider the following conditions:

$\left|\Theta_1(\mathfrak{z}, s, \tau(s)) = \Theta_1((\mathfrak{z}, s, \tau(s))\right| \le \alpha(\boldsymbol{\tau})|\tau(s) - \eta(s)|$ Holds, for all $\tau, \eta \in \mathfrak{K}\ with\ \tau \ne \eta$ and for control fuction $\alpha: \mathfrak{K} \times \mathfrak{K} \to [0, \frac{1}{s})$ then the interval operators defended by (a) and (b) have a unique common solution.

Proof: Define a continuous mapping $F, G: \mathfrak{K} \to \mathfrak{K}$ by

$$F\tau(\mathfrak{z}) = t(\mathfrak{z}) + \frac{1}{\Omega_2 - \Omega_1}\int_{\Omega_1}^{\Omega_2} \Theta_1(\mathfrak{z}, s, \tau(s))ds \ldots\ldots\ldots\ldots\ldots (a)$$

and

$$G\,\tau(\mathfrak{z}) = t(\mathfrak{z}) + \frac{1}{\Omega_2 - \Omega_1}\int_{\Omega_1}^{\Omega_2} \Theta_2(\mathfrak{z}, s, \tau(s))ds \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ \ (b)$$

For all $t \in [\Omega_1, \Omega_2]$. Consider $\Gamma(\tau,\eta) = (1+i_3)[|\tau(\mathfrak{z}) - \eta(\mathfrak{z})|^2]$

$$\Gamma(F\tau, G\eta) = (1+i_3)\left\{\frac{1}{\Omega_2 - \Omega_1}\left|\int_{\Omega_1}^{\Omega_2} \Theta_1(\mathfrak{z}, s, \tau(s))ds - \int_{\Omega_1}^{\Omega_2} \Theta_2(\mathfrak{z}, s, \tau(s))ds\right|^2\right\}$$

$$\le \frac{(1+i_3)}{\Omega_2 - \Omega_1}\left\{\left|\int_{\Omega_1}^{\Omega_2} \Theta_1(\mathfrak{z}, s, \tau(s))ds - \int_{\Omega_1}^{\Omega_2} \Theta_2(\mathfrak{z}, s, \tau(s))ds\right|^2\right\}$$

$$\le \frac{(1+i_3)\alpha(\boldsymbol{\tau})}{\Omega_2 - \Omega_1}\left\{\int_{\Omega_1}^{\Omega_2}|\tau(s) - \eta(s)|^2\ ds\right\}$$

$$\Gamma(F\tau, G\eta) \le \alpha(\boldsymbol{\tau})\Gamma(\tau,\eta))$$

Hence all the hypothesis is of theorem 3.3 are fulfilled with $\alpha(\tau, \tau, \eta) < 1$ so that integral equation (a) and (b) have the unique common solution.

IV. **Conclusions:** In this article, we have utilized the notion of tricomplex valued b-metric space *(TCVbMS)* and established common fixed point results for mixed type rational contractions on tricomplex valued b-metric space involving control function. We predict that the proved results in this paper will demonstrate some new relationship. Still there are some open problems that can be transmitted in future new work.

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