IJCRT.ORG ISSN : 2320-2882

INTERNATIONAL JOURNAL OF CREATIVE RESEARCH THOUGHTS (IJCRT)

An International Open Access, Peer-reviewed, Refereed Journal

A STUDY ON APPLICATIONS OF PARTIAL DIFFERENTIAL EQUATIONS USING FOURIER TRANSFORM

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Abstract: The specific objective is to prove the existence and uniqueness of the solution of the proposed PDE. The existence and uniqueness of the solution have been proved. To demonstrate the existence of the solution, the Fourier transformation was used. The variational formulation was used to prove the uniqueness of the solution. The combination of the Fourier transformation and the variational formulation yielded the expected results: the existence and uniqueness of the solution.

Key Words: Partial differential equation, Integral Transform and Fourier transform

I. Introduction:

Joseph Fourier, a French mathematician, had invented a method called Fourier transform in 1801, to explain the flow of heat around an anchor ring. Since then, it has become a powerful tool in diverse fields of science and engineering. It can provide a means of solving unwieldy equations that describe dynamic responses to electricity, heat or light. In some cases, it can also identify the regular contributions to a fluctuating signal, thereby helping to make sense of observations in astronomy, medicine and chemistry. Fourier transform has become indispensable in the numerical calculations needed to design electrical circuits, to analyze mechanical vibrations, and to study wave propagation. Fourier transform techniques have been widely used to solve problems involving semi- infinite or totally infinite range of the variables or unbounded regions. In order to deal with such problems, it is necessary to generalize Fourier series to include infinite intervals and to introduce the concept of Fourier integral.

II. Basic Definitions:

Partial Differential Equation:

An equation which involves several independent variables denoted by $x, y, z, t, \ldots$, a dependentvariable and its partial derivatives with respect to the independent variables such as, $F\left(x, y, z, t, \ldots, u, u_x, u_y, u_z, u_t, \ldots, u_{xx}, u_{yy}, \ldots, u_{xy}, \ldots\right) = 0$ is called *a partial differential equation*.

Integral Transform:

An Integral transform of function, $f(t)$ defined on a finite (or) infinite interval $a < t < b$ are particularly useful in dealing with problems in linear differential equations.

A general linear integral transformation of a function $f(t)$ is represented by the equation $F(s) = T\{f(t)\} = \int_a^b k(s,t).f(t)dt$. i.e., that a given function $f(t)$ is transformed into another function $F(s)$ by means of an integral. The new function $F(s)$ is said to be the transform of $f(t)$, and $k(s,t)$ is called the *kernel of the transformation*. Both $k(s,t)$ and $f(t)$ must satisfy certain conditions to ensure existence of the integral and a unique transform function $F(s)$.

Fourier Transform:

If $u(x,t)$ is a continuous, piecewise smooth, and absolutely integrable function, then the *Fourier transform* of $u(x,t)$ with respect to $x \in \mathbb{R}$ is denoted by $U(k,t)$ and is defined by $\mathcal{F}\{u(x,t)\} = U(k,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ikx}u(x,t)dx$, Where k is called the Fourier transform variable and $\exp(-ikx)$ is called the *kernel of the transform*.

Inverse Fourier Transform:

If $u(x,t)$ is a continuous, piecewise smooth, and absolutely integrable function, then the *Inverse Fourier transform* of $U(k,t)$ with respect to $x \in \mathbb{R}$ is denoted by $u(x,t)$ and is defined by $\mathcal{F}^{-1}\{U(k,t)\} = u(x,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ikx}U(k,t)dk$, Where k is called the Fourier transform variable and $\exp(ikx)$ is called the kernel of the transform.

Fourier Cosine Transformation:

Let $f(x)$ be defined for $0 \le x < \infty$, and extended as an even function in $(-\infty,\infty)$ satisfying the conditions of Fourier Integral formula. Then, the *Fourier Cosine Transform* of $f(x)$ and its inverse transform are defined by,

$$\mathcal{F}_c\{f(x)\} = F_c(k) = \sqrt{\frac{2}{\pi}}\int_0^{\infty} \cos kx\, f(x)dx,\ \mathcal{F}_c^{-1}\{F_c(k)\} = f(x) = \sqrt{\frac{2}{\pi}}\int_0^{\infty} \cos kx\, F_c(k)dk,$$

where $\mathcal{F}_c$ is the *Fourier cosine transformation* and $\mathcal{F}_c^{-1}$ is its inverse transformation respectively.

Fourier Sine Transformation:

Let $f(x)$ be defined for $0 \le x < \infty$, and extended as an odd function in $(-\infty,\infty)$ satisfying the conditions of Fourier Integral formula. Then, the *Fourier Sine Transform* of $f(x)$ and its inverse transform are defined by,

$$\mathcal{F}_s\{f(x)\} = F_s(k) = \sqrt{\frac{2}{\pi}}\int_0^{\infty} \sin kx\, f(x)dx,\ \mathcal{F}_s^{-1}\{F_s(k)\} = f(x) = \sqrt{\frac{2}{\pi}}\int_0^{\infty} \sin kx\, F_s(k)dk,$$

where $\mathcal{F}_s$ is the *Fourier cosine transformation* and $\mathcal{F}_s^{-1}$ is its inverse transformation respectively.

Convolution of the two functions:

The function $(f * g)(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x-\xi)g(\xi)d\xi$ is called the *convolution* of the functions f and g over the interval $(-\infty,\infty)$.

Properties of Convolution:

(1) $f * g = g * f$ (Commutative) (2) $f * (g * h) = (f * g) * h$ (Associative).
(3) $f * (ag + bh) = a(f * g) + b(f * h)$, (Distributive), where a and b are constants.

III. Properties of Fourier transform:

Linearity Property: The Fourier transformation $\mathcal{F}$ is linear. [or] If $\mathcal{F}$ be the Fourier transform of $f(x)$ then for any constants a and b it follows that $\mathcal{F}[af(x) + bg(x)] = a\mathcal{F}[f(x)] + b\mathcal{F}[g(x)]$.

Shifting Property: Let $\mathcal{F}[f(x)]$ be a Fourier transform of $f(x)$, then $\mathcal{F}[f(x-c)] = e^{-ikc}\mathcal{F}[f(\xi)]$, where c is a real constant.

Scaling Property: If $\mathcal{F}$ is the Fourier transform of f, then $\mathcal{F}[f(cx)] = \left(\frac{1}{|c|}\right)F(k/c)$, where c is a non-zero constant.

Differentiation Property: Let f be continuous and piecewise smooth in $(=\infty,\infty)$. Let $f(x)$ approach zero as $|x| \to \infty$. If f and f' are absolutely integrable, then$\mathcal{F}[f'(x)] = ik\mathcal{F}[f(x)] = ikF(k)$.

Modulation Property: If $F(k)$ is the Fourier transform of $f(x)$, then the Fourier transform of $f(x)\cos ax$ is $\frac{1}{2}[F(k-a) + F(k+a)]$

Convolution Theorem: If $F(k)$ and $G(k)$ are the Fourier transforms of $f(x)$ and $g(x)$ respectively, then the Fourier transform of the convolution $(f * g)$ is the product $F(k)G(k)$.
i.e., $\mathcal{F}\{f(x) * g(x)\} = F(k)G(k)$ (or) $\mathcal{F}^{-1}\{F(k)G(k)\} = f(x) * g(x)$.

Fourier Cosine Transform: Let $f(x)$ and its first derivative vanish as $x \to \infty$. If $F_c(k)$ is the Fourier Cosine transformation, then $\mathcal{F}_c[f''(x)] = -k^2F_c(k) - \sqrt{\frac{2}{\pi}}f'(0)$.

Fourier Sine Transform: Let $f(x)$ and its first derivative vanish as $x \to \infty$. If $F_s(k)$ is the Fourier sine transform, then $\mathcal{F}_s[f''(x)] = \sqrt{\frac{2}{\pi}}kf(0) - k^2F_s(k)$.

IV. Applications of partial differential equation involving Fourier transforms:

Boundary Value Problem: Fourier integrals and Fourier transforms are very useful in solving boundary value problems on infinite domains. Such problems, however, are classified as singular since they contain no infinite boundaries. In this case we normally prescribe boundary conditions of the form, $y(x), y'(x)$finite as $|x| \to \infty,\ (4.1)$. Nonetheless, the use of Fourier transforms often forces us (at least initially) to impose the more stringent requirements. $y(x) \to 0, y'(x) \to 0,$ as $|x| \to \infty,\ (4.2)$. These stronger requirements are necessary to ensure that the Fourier transforms of $y'(x)$ and $y''(x)$ exist. Even so, the formal solutions that we generate by the transform method may not satisfy (4.2). In such cases we normally require $y(x)$ to at least satisfy (4.1).

Heat Conduction is Solids: It is well known that if the temperature u in a solid body is not constant, heat energy flows in the direction of the gradient $-\nabla u$ with magnitude $k|\nabla u|$. The quantity k is called the thermal conductivity of the material and the above principle is called Fourier's law of heat conduction. This law combined with the law of conservation of thermal energy, which states that, "… the rate of heat entering a region plus that which is generated inside the region equals the rate of heat leaving the region plus that which is stored….," leads to the partial differential equation. $\nabla^2 u = a^{-2}u_t - q(x,y,z,t), \to (4.3)$ where a^2 is another physical constant called the diffusivity. Equation (1) is commonly called the het equation (or) diffusion equation. The quantity $\nabla^2 u$ in (4.3) is called the Laplacian and is a measure of the difference between the value of u at a point and the average value of u in a small neighborhood of the point. In rectangular co-ordinates, the Laplacian takes the form $\nabla^2 u = u_{xx} + u_{yy} + u_{zz}, \to (4.4)$.

Heat Equation in an Infinite Line: Let us fist consider the flow of heat in the infinite medium $-\infty < x < \infty$ when the initial temperature distribution $f(x)$ is known and the region is free of any heat sources. Physically, this problem might represent the linear flow of heat in a very long slender rod whose lateral surface is insulated. In such cases, the solution will represent the temperature distribution in the middle portion of the infinitely long rod, prior to the time when such temperatures are greatly influenced by the actual boundary conditions of the rod. The Problem is mathematically characterized by, $u_{xx} = a^{-2}u_t, x < \infty,\ t > 0$ B.C: $u(x,t) \to 0,\ u_x(x,t) \to 0,$ as $|x| \to \infty$ I.C: $u(x,0) = f(x), -\infty < x < \infty$.

Wave Equation: The wave equation $\nabla^2 u = c^{-2}u_{tt} - q(x,y,z,t)$, where c is a constant having the dimension of velocity, describes various wave motions in nature and mechanical systems such as sound waves emanating from a struck bell, surface waves propagingradially outward when a pebble is dropped into a pool, and the deflections of a membrane set in motion. The term q is proportional to an external "force" acting on the system under investigation. A properly-posed problem involving the wave equation consists of two initial conditions and one boundary condition at each boundary point. The one-dimensional wave equation $u_{xx} = c^{-2}u_{tt} - q(x,t)$ is the governing equation for such rudimentary problems as the transverse oscillations of a tightly stretched string or the longitudinal vibrations of a beam.

Potential Theory: Perhaps the single most important Partial Differential Equation in mathematical physics is the equation of Laplace, of potential equation. In two and three dimensions, respectively, we have the rectangular co-ordinate representations, $u_{xx} + u_{yy} = 0$, $u_{xx} + u_{yy} + u_{zz} = 0$ where as in general we write, $\nabla^2 u = 0$ regardless of the co-ordinate system or number of dimensions. A properly-posed problem involving Laplace's equation consists of finding a Harmonic function in a region R subject to a single boundary condition. The most common boundary conditions fall mainly into two categories, giving us two primary types of boundary-value problem. If R denotes a region in the plane and C its boundary curve, then one type of problem is characterized by,$\nabla^2 u = 0$ in R , $u = f$ on C which is called a Dirichlet problem (or) boundary value problem of the first kind. An example of a Dirichlet Problem is to find the steady-state temperature distribution in a region R given that the temperature is known everywhere on the boundary of R. Another problem is characterized by, $\nabla^2 u = 0$ in R. $\frac{\partial u}{\partial n} = f$ on C, which is known as a Neumann Problem (or) boundary value problem of the second kind. The derivative $\frac{\partial u}{\partial n}$ is called the normal derivative of u and is positive in the direction of the outward normal to the boundary curve C. There is also a third boundary value problem, called Robin's problem, in which the boundary conditions is a linear combination of u and its normal derivative.

Potential Problems in the Infinite Strip: Suppose we now consider the Dirichlet problem $u_{xx}+u_{yy}=0,\ -\infty<x<\infty,\ 0<y<a,\ \to$ (4.5)B.C: $u(x,0)=f(x), u(x,a)=g(x),\ -\infty<x<\infty.$
Physically, this problem might correspond to finding the steady-state temperature in an infinite slab whose faces are mentioned at prescribed temperature[See Fig.1]. Again, we use the Fourier transform, $\mathcal{F}\{u(x,y);x\to s\}=U(s,y)$ which reduces to (4.5) to $u_{yy}-s^2U=0,\ 0<y<a$
B.C; $U(s,0)=F(s),\ U(s,a)=G(s)$
where $F(s)$ and $G(s)$ are Fourier transforms, respectively, of $f(x)$ and $g(x)$. The solution of this boundary value problem is easily shown to be $U(s,y)=F(s)\frac{\sinh s(a-y)}{\sinh sa}+G(s)\frac{\sinh sy}{\sinh sa},\ \to$ (4.6)
Recalling the Fourier transforms, inverse transforms and cosine transforms are all the same for even functions,

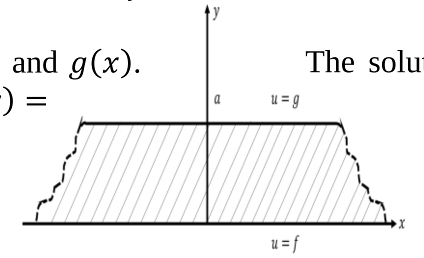

Fig.1: Infinite Slab

we deduce that $k(x,y)=\mathcal{F}^{-1}\left\{\frac{\sinh sy}{\sinh sa},\ s\to x\right\}=\frac{1}{a}\sqrt{\frac{\pi}{2}}\frac{\sin(\pi y/a)}{\cos(\pi x/a)+\cos(\pi y/a)}$.
Hence, using the convolution theorem we can express the inverse transform of (4.6) in the form, $u(x,y)=\frac{1}{\sqrt{2\pi}}\left[\int_{-\infty}^{\infty}f(\xi)k(x-\xi,a-y)d\xi+\int_{-\infty}^{\infty}g(\xi)k(x-\xi,y)d\xi\right]$

Hydrodynamics: A fluid flow in three-dimensional space is called two dimensional if the velocity vector v is always parallel to a fixed plane (xy plane), and if the velocity components parallel to this plane along with the pressure p and fluid density ρ are all constant along any normal to the plane. This permits us to confine our attention to just a single plane which we interpret as across section of the three-dimensional region under consideration. Our discussion here will be limited to two-dimensional flow problems. An ideal fluid is one in which the stress on an element of area is wholly normal and independent of the orientation of the area. In contrast, the stress on a small area is no longer normal to that area for a viscous fluid in motion. If the density ρ is constant, we say the flow is incompressible. Of course, the notions of an ideal fluid or incompressible fluid are only idealizations that are valid when certain effects can be safely neglected in the analysis of a real fluid. The velocity, pressure, and fluid density are all interrelated through a set of differential equations consisting of a continuity equation, equation of motion, and an equation of state (such as the density equal to a constant, etc.,).

The continuity equation is an expression of the conservation of mass out of it, and assumes the form, $\frac{\partial\rho}{\partial t}+\nabla(\rho v)=0$ For an incompressible fluid, this reduces to $\nabla.v=0$ which implies there are no sources nor sinks with in the region of interest (i.e., points at which fluid appears (or) disappears).

Irrotational Flow of an Ideal Fluid: If the vorticity Ω is zero at every point with in the region of interest, we say the flow is irrotational. This means that $\nabla\times v=0$, which in two dimensions is described by $\omega=v_x-u_y=0$. This relation combined with $u=-\psi_y$ and $v=\psi_x$ leads to Laplace's equation $\psi_{xx}+\psi_{yy}=0,\ \to$ (4.7) Clearly, solutions of Laplace's equation $\nabla^2\psi=0$ are solutions of the equations of motion given by, $\frac{\partial}{\partial t}\nabla^2\psi+\frac{\partial\psi}{\partial y}\left(\frac{\partial}{\partial x}-\frac{\partial\psi}{\partial x}\frac{\partial}{\partial y}\right)\nabla^2\psi=v\nabla^4\psi.$

The irrotational flow of an fluid can also be described in terms of a velocity potential functions ϕ. That is, the condition $\nabla\times v=0$ implies the existence of a potential function ϕ such that $v=-\nabla\phi$, (or) $u=-\phi_x,\ v=-\phi_y,\ \to$ (4.8) . By combining (4.8) and $(u_x+u_y=0)$, we find that the velocity potential ϕ is likewise a solution of Laplace's equation $\phi_{xx}+\phi_{yy}=0,\ \to$ (4.9) Thus, for irrotational flows we have the choice of solving (4.7) for the stream function ψ (or) solving (4.9) for the potential function ϕ.

Elasticity in Two-Dimensions: The effect of body or surface forces on a two-dimensional body will be to produce internal forces between various parts of the body. The magnitude of these internal forces are defined by the ratio of the force to the area over which it acts, called the average stress. In the limit as the area shrinks to zero, we obtain the components of stress at a point in the elastic medium. This stress is composed of two normal components σ_{xx} and σ_{yy}, and two shearing components, σ_{xy} and σ_{yx}, for which $\sigma_{xy}=\sigma_{yx}$. We adopt the convention that stresses are positive when a tension is produced and negative when a compression occurs. The Differential Equations satisfied by the components of stress in an elastic medium under the action of a force per unit mass having components (F_x,F_y) may be obtained by applying Newton's 2nd law of motion to a small rectangular of the medium. Writing the displacement

vector as, $u = (u, v)$, these equations of motion are, $\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \rho F_x = \rho \frac{\partial^2 u}{\partial t^2}$; $\frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \rho F_y = \rho \frac{\partial^2 v}{\partial t^2}$, where ρ is the mass density of the elastic body, for equilibrium problems the time derivatives on the right-hand sides can be set to zero. Also, in the absence of body forces we have$F_x = F_y = 0$, and in these cases the equation of equilibrium take the simpler form, $\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} = 0$, → (4.10) $\frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} = 0$, → (4.11). The above equations of equilibrium, together with appropriate boundary conditions are still not sufficient for the determination of the stresses. That is, the complete solution requires us to take into account a compatibility condition for the distribution of stress that arises from the stresses, is given by the relation, $\frac{\partial^2}{\partial y^2}\left[\sigma_{xx} - v\left(\sigma_{xx} + \sigma_{yy}\right)\right] + \frac{\partial^2}{\partial x^2}\left[\sigma_{yy} - v\left(\sigma_{xx} + \sigma_{yy}\right)\right] = 2\frac{\partial \sigma_{xy}}{\partial x \partial y}$, → (4.12) where v is the Poisson ratio of the material. To solve this system of equations is it convenient to introduce a scalar function χ, called the Airy stress function, by setting, $\sigma_{xy} = -\frac{\partial^2 \chi}{\partial x \partial y}$. Using this relation, we find the equations of equilibrium (4.10), (4.11) reduce to $\frac{\partial}{\partial x}\left(\sigma_{xx} - \frac{\partial^2 \chi}{\partial y^2}\right) = 0$: $\frac{\partial}{\partial y}\left(\sigma_{yy} - \frac{\partial^2 \chi}{\partial x^2}\right) = 0$. Hence it follows immediately that the equations of equations of equilibrium (4.10), (4.11) are satisfied by
$\sigma_{xx} = \frac{\partial^2 \chi}{\partial y^2}, \sigma_{yy} = \frac{\partial^2 \chi}{\partial x^2}, \sigma_{xy} = -\frac{\partial^2 \chi}{\partial x \partial y}$, → (4.13) Lastly, the substitution of these expressions into the compatibility condition (3) yields the biharmonic equation, $\frac{\partial^4 \chi}{\partial x^4} + 2\frac{\partial^4 \chi}{\partial x^2 \partial y^2} + \frac{\partial^4 \chi}{\partial y^4} = 0$, → (4.14). Solving (4.14) for the Airy stress function, subject to appropriately prescribed boundary conditions, leads to the stress components through use of (4.13).

Probability and Statistics: Suppose that X is a random variable. The function $P(x)$, called the distribution function, represents the probability that $X < x$, where x is a real number. The distribution function has the following properties: (1) $\lim_{x \to -\infty} P(x) = 0$ (2) $\lim_{x \to \infty} P(x) = 1$
(3) $P(x_1) \leq P(x_2)$ when $x_1 \leq x_2$. If we think of X as a continuous variable, then there usually exists a related function $p(x)$ such that $P(x) = \int_{-\infty}^{x} p(u)du$. The function $p(x)$ is called the probability density function (PDF) of the random variable X. Once $p(x)$ has been determined, various properties of the random variable X can be calculated, such as the statistical moments of X. In statistics, the moments $m_1, m_2, \ldots$, of the random variable X are defined in terms of the expectation operator E. For Example,
$m_1 = E[x] = \int_{-\infty}^{\infty} xp(x)dx$; $m_2 = E[x^2] = \int_{-\infty}^{\infty} x^2 p(x)dx$ where as in general,

$$m_k = E[x^k] = \int_{-\infty}^{\infty} x^k p(x)dx, \qquad k = 1,2,3, \ldots, \qquad \to (4.15)$$

The first moment gives the average value of a random variable, and the higher order moments give additional information about the spread of the distribution defining the random variable. The variance of the distribution is defined by $\sigma^2 = \int_{-\infty}^{\infty} (x - m_1)^2 p(x)dx = m_2 - m_1^2$ which follows by expanding the square and using relations (4.15).

Characteristic Functions: In many situations it is convenient to introduce the notion of a characteristic function $C(t)$ from which the statistical moments of X also can be found in certain applications the characteristic function of a random variables is easier to calculate than its PDF, and thus this function can be useful in such cases, we define this new function by the expectation $C(t) = E\left[e^{itx}\right] = \int_{-\infty}^{\infty} e^{itx} p(x)dx$ which we recognize as a Fourier transform relation given by, $C(t) = \sqrt{2\pi}\mathcal{F}\{p(x); t\}$
Hence, it follows that,
$C(0) = \int_{-\infty}^{\infty} p(x)dx = 1; C'(0) = i\int_{-\infty}^{\infty} xp(x)dx = im_1; C''(0) = -\int_{-\infty}^{\infty} x^2 p(x)dx = -m_2$
while in general, we deduce, $m_k = (-i)^k c^k(0)$, $k = 1,2,3, \ldots$ This says that m_k is the co-efficient of $(it)^k/k!$ In the Maclaurin series expansion of the characteristic function. Finally, it also follows from properties of inverse Fourier transforms, that if the characteristic function of a random variable is known, its probability density function is reduced by

$$p(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-itx}C(t)dt, \qquad \to (4.16)$$

V. Problems on Partial differential equations using Fourier transform:

Problem: 5.1

Given that X and Y are independent normal random variables with means zero and unit variances, find the probability density function (PDF) of the random variable Z= XY.

Solution:

To find $p_z(z)$, we will first determine the characteristic function for z and then invert it according to (5.1). Since $Z = XY$, the characteristic function of Z can be determined by calculating $C_z(t) = E\left[e^{itz}\right] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{itxy}p_{X,Y}(x,y)dx\,dy$ where $p_{X,Y}(x,y)$ is the joint PDF of X and Y. Because X and Y are assumed to be independent, it follows that the joint PDF is simply the product of their individual PDFs. Hence we have,

$P_{X,Y}(x,y) = p_X(x)p_Y(y) = \frac{1}{\sqrt{2\pi}}e^{-x^2/2}.\frac{1}{\sqrt{2\pi}}e^{-y^2/2} = \frac{1}{\sqrt{2\pi}}e^{-(x^2+y^2)/2}$ which leads to the double integral $C_z(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-y^2/2}\int_{-\infty}^{\infty} e^{itxy}e^{-x^2/2}dx\,dy$ Working with the inner most integral, we find $\int_{-\infty}^{\infty} e^{itxy}e^{-x^2/2}dx = \sqrt{2\pi}\mathcal{F}\{e^{-x^2/2}; ty\} = \sqrt{2\pi}e^{-t^2y^2/2}$. Using this result, the remaining integral yields, $C_z(t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-(1+t^2)y^2/2}dy = \frac{1}{\sqrt{1+t^2}}$. Substituting this expression for $C_z(t)$ into (5.1), we obtain, $p_z(z) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{itz}C_z(t)dt = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{e^{-itz}}{\sqrt{1+t^2}}dt = \frac{1}{\pi}\int_0^{\infty}\frac{\cos zt}{\sqrt{1+t^2}}dt$ where we have used the fact that $C_z(t)$ is an even function. This last integral is not an elementary integral nor does it lend itself to evaluation by conventional means using basic complex variable theory. Nonetheless, it is a well-known integral which leads to the final result, $p_z(z) = \frac{1}{\pi}k_0(|z|)$, where $k_0(x)$ is a modified Bessel function of the second kind and order zero.

Problem: 5.2

Solve the heat conduction equation given by **PDE**: $k\frac{\partial^2 f}{\partial x^2} = \frac{\partial u}{\partial t}$, $-\infty < x < \infty, t > 0$

Subject to **BCs**: $u(x,t)$& $u_x(x,t)$both $\to 0$ as $|x| \to \infty$ **IC**: $u(x,0) = f(x)$, $-\infty < x < \infty$,

Solution:

Taking the Fourier transform $*$ of PDE, we get $-k\alpha^2\mathcal{F}[u(x,t); x \to \alpha] = \mathcal{F}[u_t(x,t); x \to \alpha]$ or $U_t(\alpha,t) + k\alpha^2U(\alpha,t) = 0$ (5.2) . In deriving this, the BCs are already utilized (as can be seen from equations). The Fourier transform of the IC gives $U(\alpha,0) = F(\alpha)$, $-\infty < \alpha < \infty$ (5.3). The solution of equation (5.2) can be readily seen to be $U = \mathcal{A}e^{-k\alpha^2 t}$. When $t = 0$, we have equation (2) the relation $U = F(\alpha)$, implying $\mathcal{A} = F(\alpha)$. Therefore, $U(\alpha,t) = F(\alpha)e^{-k\alpha^2 t}$(5.4) Inverting this relation, we obtain

$u(x,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(\alpha)e^{-k\alpha^2 t}e^{-i\alpha x}\,d\alpha = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(\alpha)\,exp(-k\alpha^2 t - i\alpha x)\,d\alpha$ (5.5)

The product form of the integrand in equation (5.4) suggests the use of convolution. If the Fourier transform of $g(x)$is $e^{-k\alpha^2 t}$, then $g(x)$ will be given by $g(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-k\alpha^2 t}e^{-i\alpha x}\,d\alpha$. But, if $a > 0, b$is real or complex, and we know that

$\int_{-\infty}^{\infty} exp(-ax^2 - i\alpha x)\,dx = \frac{\sqrt{\pi}}{\sqrt{a}}exp(b^2/a)$ (5.6) . Here, $a = kt, 2b = ix$. Therefore,

$g(x) = \frac{1}{\sqrt{2\pi}}\frac{\sqrt{\pi}}{\sqrt{kt}}exp\left(-\frac{x^2}{4kt}\right) = \frac{1}{\sqrt{2kt}}exp\left(-\frac{x^2}{4kt}\right)$ Using the convolution theorem, we have $u(x,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(\alpha)g(x-\alpha)\,d\alpha$ (5.7). Hence, $u(x,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(\alpha)\frac{1}{\sqrt{2kt}}exp\left[\frac{-(x-\alpha)^2}{4kt}\right]d\alpha = \frac{1}{\sqrt{4\pi kt}}\int_{-\infty}^{\infty} f(\alpha)\,exp\left[-\frac{(x-\alpha)^2}{4kt}\right]d\alpha$ (5.8)

Introducing the change of variable $z = \frac{\alpha - x}{\sqrt{4kt}}$, we can rewrite solution (5.8) in the form $u(x,t) = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} f\left(x + \sqrt{4kt}\,z\right)e^{-z^2}\,dz$ (5.9)

Problem: 5.3

(Flow of heat in a semi-infinite medium). Solve the heat conduction problem described by

PDE: $k\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}$, $0 < x < \infty, t > 0$ **BCs**: $u(0,t) = u_0, t \geq 0$ **IC**: $u(x,0) = 0, 0 < x < \infty$,

u and $\partial u/\partial x$ both tend to zero as $x \to \infty$.

Solution:

Since u is specified at $x = 0$, the Fourier sine transform is applicable to this problem. Taking the Fourier sine transform of the given PDE and using notation. $U_s(\alpha,t) = \sqrt{\frac{2}{\pi}}\int_0^\infty u(x,t)\, Sin\, \alpha x\, dx$, We obtain from known the relation

$$k\left[\sqrt{\frac{2}{\pi}}\alpha u(x,t)\,|_{x=0} - \alpha^2\mathcal{F}_s[u(x,t); x\to\alpha]\right] = \frac{\partial U_s}{\partial t}(\alpha,t) \text{ or } \frac{dU_s}{dt} + k\alpha^2 U_s = \sqrt{\frac{2}{\pi}}k\alpha u_0 (5.10)$$

Its general solution is found to be $U_s(\alpha,t) = \sqrt{\frac{2}{\pi}}\frac{u_0}{\alpha}\left(1 - e^{-k\alpha^2 t}\right)$(5.11). Inverting by Fourier inverse sine transform, we obtain $u(\alpha,t) = \sqrt{\frac{2}{\pi}}\int_0^\infty U_s(\alpha,t)\, Sin\, \alpha x\, d\alpha$. Therefore,

$u(x,t) = \frac{2}{\pi}u_0\int_0^\infty \frac{Sin\,\alpha x}{\alpha}\left(1 - e^{-k\alpha^2 t}\right)d\alpha (5.12)$. Nothing that $\text{erf}(y) = \frac{2}{\sqrt{\pi}}\int_0^y e^{-u^2}\,du$

And using the standard integral $\int_0^\infty e^{-\alpha^2}\frac{Sin(2\alpha y)}{\alpha}d\alpha = \frac{\pi}{2}\text{erf}(y)$

We have solution (5.11) in the form $u(x,t) = \frac{2u_0}{\pi}\left[\frac{\pi}{2} - \frac{\pi}{2}\text{erf}\left(\frac{x}{\sqrt{2kt}}\right)\right]$ (5.13). Finally, the solution of the heat conduction problem is

$$u(x,t) = u_0\left(1 - \frac{2}{\sqrt{\pi}}\int_0^{x/\sqrt{2kt}} e^{-u^2}\,du\right) = u_0\,\text{erfc}\left(\frac{x}{\sqrt{2kt}}\right) (5.14)$$

VI. Conclusion:

In this paper, we have discussed some solution of Heat equations using Fourier transform. Fourier transforms played major roles in Space science technology and computer science. So, I have chosen heat equation solutions with the help of Fourier transforms, It is very helpful for my future research studies. I have extended this Fourier transforms idea to Space science technology.

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