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Insertion Operation on Primitive Partial Words

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Abstract. A partial word is a word over a finite alphabet that contain some unknown places known as holes or "do not know" symbols. A partial word w is said to be primitive if there does not exist any word v such that w is contained in v^n with $n \geq 2$. In this paper, we investigate the effect of insertion operation on primitive partial words with a single hole. We characterize a special class of such words and call it as ins-robust primitive partial words. We identify some important properties of such partial words and prove that the language of non-ins-robust primitive partial words with one hole is not context-free.

Keywords: Combinatorics on Partial Words, Primitive Partial Word, Ins-robust Primitive partial Word, Reflective, Context-free Language

1 Introduction

Let Σ be a finite alphabet. A word is a sequence of symbols drawn from the alphabet Σ. Combinatorics on words is an active research area in discrete math- ematics and theoretical computer sciecne. It has been widely studied several research areas including formal language and automata theory [9], coding the- ory [2], graph theory [5], computational biology and DNA computing [14]. Several aspects of words such as, algebraic [17], applied [18] and algorithmic [11], have been extensively explored.

Partial words, a canonical extension of words, are words that may have some unknown symbols known as holes and has been introduced by Berstel and Boas- son [1]. The motivation behind introducing partial words is the comparison of two genes. Alignment of two DNA sequences can be viewed as construction of two partial words that are compatible. A partial word $w = a_1a_2 \ldots a_n$ of length n over the finite alphabet where $a_i \in \Sigma \cup \{\diamond\}$ and $\diamond$ is referred as hole. In the context of combinatorics of words, primitive words are of special interest where a word is said to be primitive if it cannot be represented as an integer power of a smaller word [16]. A partial word is said to be primitive if it is not con- tained in power of a word. The relation between the language of primitive words and conventional formal language classes has been extensively studied [10, 22, 15]. It is still an open problem that whether the language of primitive words is context-free [10, 22].

In [21], G.Păun et al. have studied the robustness of the language of primi- tive words with respect to various point mutation operations such as insertion, deletion or substitution of a symbol and homomorphism. In [7], Dassow et al. consider the word ww' where w is a primitive word and w' is a modification of w and study whether ww' is primitive. Similarly, in [6], Sadri et al. extended the work of Dassow et al. to partial words and studied the operations that preserve primitivity in partial words with one hole. Also the language of primitive partial words have been studied with respect to language classes in Chomsky hierar- chy [19]. In [20], Nayak et al. studied the language of primitive partial words which are robust to deletion operation. In this paper we discuss preservation of primitivity in primitive partial words with one hole with respect to insertion operation. This special class of primitive partial words with one hole is referred to as ins-robust primitive partial words with one hole.

This paper is organized as follows. In Section 2 we review some basic con- cepts and preliminaries which are used in the rest of the paper. In Section 3, we characterize ins-robust primitive partial words with one hole and identify its properties. In Section 4, we prove that the language of ins-robust primitive partial words with one hole is not a regular language and the language of non- ins-robust primitive partial words with one hole is not context-free.

2 Preliminaries

Let Σ be a finite set of symbols known as alphabet. We assume that Σ is a nontrivial alphabet, which means that it has at least two distinct symbols. A total word (referred to as simply a word) $u = a_0a_1a_2 \ldots a_{n-1}$ of length n can be defined by a total function $u : \{0, \ldots, n-1\} \to \Sigma$ where each $a_i \in \Sigma$ [3]. We use string and word interchangeably. The set Σ^* is the free monoid generated by Σ which contains all the strings. The length of a string u is the number of symbols contained in it and is denoted by $|u|$, and $\alpha(u)$ is the set of symbols appearing in u from Σ. The empty word, λ, is a word that does not contain any letter and therefore $|\lambda| = 0$. The notation $|w|_a$ denotes the number of times letter a appears in a word w. The set of all words of length n over Σ is denoted by Σ^n. We define $\Sigma^* = \bigcup_{n\geq 0} \Sigma^n$ where $\Sigma^0 = \{\lambda\}$, and $\Sigma^+ = \Sigma^* \setminus \{\lambda\}$ is the free semigroup generated by Σ. A language L over Σ is a subset of Σ^*.

A partial word u of length n over alphabet Σ can be defined by a partial function $u : \{0, \ldots, n-1\} \to \Sigma$. The partial word u contains some do not know symbols known as holes along with the usual symbols. For $0 \leq i < n$, if $u(i)$ is defined, then we say $i \in D(u)$ (the domain of u), otherwise $i \in H(u)$ (the set of holes) [3]. A word is a partial word without any hole. If u and v are two partial words of equal length, then u is said to be *contained* in v, if all elements in $D(u)$ are also in the set $D(v)$ and $u(i) = v(i)$ for all $i \in D(u)$ and u is said to be *compatible* to v if there exists a partial word w such that $u \subset w$ and $v \subset w$. The containment and compatibility are denoted as $\subset$ and $\uparrow$ respectively. We denote the extended alphabet as $\Sigma_\diamond = \Sigma \cup \{\diamond\}$.

A word is said to be primitive if it cannot be expressed as a nontrivial power of another word. Formally, a word w is primitive if there does not exist any word v such that $w = v^n$ with $n \geq 2$. Similarly, a partial word u is said to be primitive if there does not exist any word v such that $u \subset v^n$, $n \geq 2$. Note that if u is primitive and $u \subset v$, then v is primitive as well [3]. The language of primitive partial words is denoted as Q_p and the language of primitive partial words with i holes is represented as Q^i_p. We denote the language of non-primitive partial words over an alphabet Σ as Z_p.

Let $w = a_1a_2 \cdots a_n$ be a partial word of length n. A *strong period* of a partial word w is a positive integer p such that $a_i = a_j$ whenever $i, j \in D(w)$ and $i \equiv j \mod p$. A *weak period* or *local period* of a partial word w is a positive integer p such that $a_i = a_{i+p}$ for all $1 \leq i \leq n-p$ whenever $i, i+p \in D(w)$. If a partial word w has a strong period p then we say w is p-periodic, and if w has a local period p then we call w is locally p-periodic. For example, $w = abc\diamond\diamond cacc$ over the alphabet $V = \{a, b, c\}$ is locally 3-periodic but not 3-periodic.

Theorem 1 ([23]). *Every nonempty word w can be expressed uniquely in the form $w = x^n$, where $n \geq 1$ and x is primitive.*

Observe that the above result is not true for partial words, that is, the unique- ness does not hold in case of partial word. For example, $u = a\diamond$, we have $u \subset a^2$ and $u \subset ab$.

Definition 1 (Reflective Language [21]). *A language L is called reflective if $uv \in L$ implies $vu \in L$, for all $u, v \in \Sigma^*$.*

Several facts are known about the language of primitive words Q and the language of primitive partial words Q_p. Let us recall some of them which will be useful later in the paper.

Lemma 1 ([23]). *The languages Q and Z are reflective.*

Theorem 2 ([3]). *Let u and v be partial words. If there exists a primitive word x such that $uv \subset x^n$ for some positive integer n, then there exists a primitive word y such that $vu \subset y^n$. Moreover, if uv is primitive then vu is primitive.*

Corollary 1. *The language Q_p is reflective.*

Let $w = uv$ be a nonempty partial word. Then, the partial words u and v are said to be prefix and suffix of w, respectively. A partial word y is said to be a factor of a word w if w can be written as xyz, where $x, z \in \Sigma_\diamond{}^*$ and $y \in \Sigma_\diamond{}^+$. The partial word y is said to be proper factor if $x \neq \lambda$ or $z \neq \lambda$. A prefix (suffix) of length k of a partial word w is denoted as pref(w, k) (suff(w, k)), respectively, where $k \in \{0, 1, \ldots, |w|\}$ and pref$(w, 0) =$ suff$(w, 0) = \lambda$.

The robustness of primitive words has been defined in [21]. Given a primitive word, w, the robustness of w is considered with respect to insertion of a symbol from Σ, deletion of a symbol from w, substitution of a symbol in w by another symbol from Σ. We state some of the properties about robustness of primitive words and primitive partial words that will be useful later.

The next result shows the possibility of obtaining primitive partial words by appending a symbol or removing the last symbol in any nonempty partial word. Specifically, if u is nonempty partial word with one hole, then at least one of the u or ua is primitive for $a \in \Sigma$.

Lemma 2 ([3]).

(i) *Let u be a partial word with one hole such that $|\alpha(u)| \geq 2$. If a is any letter then u or ua is primitive.*
(ii) *Let u_1, u_2 be nonempty partial words such that u_1u_2 has one hole such that $|\alpha(u_1u_2)| \geq 2$. Then for any letter a, u_1u_2 or u_1au_2 is primitive.*

The next result is an extension of Lemma 2 in total words.

Lemma 3 ([21]). *For every word $u \in \Sigma^+$ and all symbols $a, b \in \Sigma$, where $a \neq b$, at least one of the words ua, ub is primitive.*

The next proposition holds for partial words with exactly one hole.

Proposition 1 ([3]). *Let u be a partial word with one hole which is not of the form $x\diamond x$ for any word x. If a and b are distinct letters, then ua or ub is primitive.*

The following proposition shows the possibility of obtaining primitive word by deletion of a symbol in a primitive word.

Proposition 2 ([21]). *Every word $w \in Q, |w| \geq 2$, can be written in the form $w = u_1au_2$, for some $u_1, u_2 \in \Sigma^*$, $a \in \Sigma$, such that $u_1u_2 \in Q$.*

3 Insertion Operation on Partial Words with One Hole

In this section, we study a special class of primitive partial words having one hole which remain primitive after insertion of a symbol. We refer to this special class as ins-robust primitive partial words with one hole that are formally defined as follows.

Definition 2 (Ins-Robust Primitive Partial Words). *A primitive partial word w of length n with one hole is said to be ins-robust if and only if the partial word*

$$\mathrm{pref}(w, i)\ .a.\ \mathrm{suff}(w, n - i)$$

is a primitive partial word for all $i \in \{0, 1, \ldots, n - 1\}$.

Note that an ins-robust primitive partial word remains primitive on insertion of a symbol. The number of such words is infinite. For example, $ab\diamond a$ and $a^m\diamond b^n$ for $m, n \geq 2$ are ins-robust primitive partial words with one hole. We denote the set of all ins-robust primitive partial words with one hole by $Q_p^{1\prime}$ over an alphabet $\Sigma_\diamond$. It is obvious that the language of ins-robust primitive partial words with one hole is a subset of Q_p^1 and hence $w \in Q_p^1$ for all $w \in Q_p^{1\prime}$ where Q_p^1 be the set of all primitive partial words with one hole.

Next we give complete structural characterization of those primitive partial words with one hole which are in the set Q_p^1 but not in $Q_p^{1\prime}$, that is, insertion of a symbol or a hole into such words will result in non-primitive partial words.

Theorem 3. *A primitive partial word w with one hole is not ins-robust if and only if w is contained in any word which is of the form $u^{k_1}u_1u_2u^{k_2}$ where $u = u_1au_2$, $a \in \Sigma$, k_1, $k_2 \geq 0$ and $k_1 + k_2 \geq 1$.*

Proof. The necessary and sufficient parts are proved as follows.

($\Leftarrow$) Let us consider a primitive partial word with one hole $w \subset u^{k_1}u_1u_2u^{k_2}$ where $u_1au_2 = u$ and $a \in \Sigma$. If the symbol a is inserted in the word w which is contained in the word $u^{k_1}u_1au_2u^{k_2}$ makes the partial word non-primitive. Hence, w is not an ins-robust primitive partial word.
($\Rightarrow$) Let w be a primitive partial word with one hole which is not ins-robust. Therefore w can be written as $w = w_1w_2$ such that w_1cw_2 for some $c \in \Sigma_\diamond$ is not a primitive partial word. That is, $w_1cw_2 \subset v^n$ for $v \in Q$ and $n \geq 2$. Hence, $w_1 \subset v^rv_1$ and $w_2 \subset v_2v^s$ for r, $s \geq 0$ and $r + s \geq 1$ such that $v_1cv_2 = v$. Therefore, $w \subset u^{k_1}u_1u_2u^{k_2}$. ◻

Next we prove that the language of ins-robust primitive partial words with one hole $Q_p^{1\prime}$ is reflective.

Lemma 4. *$Q_p^{1\prime}$ is reflective.*

Proof. We prove it by contradiction. Let a partial word $w = xy \in Q_p^{1\prime}$ such that $yx \notin Q_p^{1\prime}$. Since $w \in Q_p^{1\prime}$, hence $w \in Q_p^1$. So $yx \subset u^{k_1}u_1u_2u^{k_2}$ where $u = u_1au_2$,

$a \in \Sigma$ and $k_1 + k_2 \geq 1$. By Theorem 3, $yx \notin Q_p^{1\prime}$. We consider the following cases.

Case 1 If $y \subset u^{k_1}u_1u_2u^r u'_2$ and $x \subset u'_1 u^s$ where $u'_1 a u'_2 = u$, $r + s + 1 = k_2$ then $xy \subset u'_1 u^s u^{k_1} u_1 u_2 u^r u'_2$ which will not be ins-robust after insertion of the symbol $a \in \Sigma$ and the obtained partial word will be non-primitive. Hence, this is a contradiction to the assumption that $xy \in Q_p^{1\prime}$.

Case 2 If $y \subset u^{k_1}u_1u'_2$ and $x \subset u''_2 u^{k_2}$ where $u = u_1 a u_2$ for $a \in \Sigma$ and $u_2 = u'_2 u''_2$. Now $xy \subset u''_2 u^{k_2} u^{k_1} u_1 u'_2$ which will not result in an ins-robust primitive partial word after insertion of a letter a. Moreover, the partial word will be contained in $(u''_2 u_1 u'_2)^{k_1+k_2+1}$ and xy is a non-primitive partial word. Hence it is a contradiction.

Hence the language of ins-robust primitive partial words with one hole $Q_p^{1\prime}$ is reflective. □

Next we study the subset of primitive partial words with one hole into which insertion of a symbol results in a non-primitive partial word. We call such partial words as non-ins-robust primitive partial words with one hole and the set is denoted by $\overline{Q}_p^{1\prime}$.

Definition 3 (Non-Ins-Robust Primitive Partial Words). *A primitive partial word w with one hole is said to be non-ins-robust if and only if insertion of a symbol $a \in \Sigma \cup \{\diamond\}$ into w makes it non-primitive.*

Thus $\overline{Q}_p^{1\prime} = Q_p^{1} \setminus Q_p^{1\prime}$ where '\' is the set difference operator. The number of such non-ins-robust partial words are infinite. For example, $a\diamond b$, $a^m b a^m \diamond a^m$ for $m \geq 1$ are non-ins-robust words.

Corollary 2. *$\overline{Q}_p^{1\prime}$ is reflective.*

Theorem 4. *A primitive partial word w with one hole is non-ins-robust if and only if w is contained in $u^n u'$ or is contained in its cyclic permutation for some for some $u \in Q_p$ where $u = u'a$ for some $a \in \Sigma$, $n \geq 2$, and $|\alpha(u)| \geq 2$.*

Proof. We prove the necessary and sufficient conditions as follows:

($\Rightarrow$) Let $w \in Q_p^1$ be non-ins-robust, that is, $w \in \overline{Q}_p^{1\prime}$. So w is contained in the word which of the form $u^p u_1 u_2 u^q$ for some $u = u_1 a u_2 \in Q$ and $a \in \Sigma$. Since $\overline{Q}_p^{1\prime}$ is reflective, so a permutation of w which is contained in $u_2 u^q u^p u_1 = (u_2 a u_1)^{p+q} u_2 u_1$ is also in $\overline{Q}_p^{1\prime}$. Hence $w \subset (u_2 a u_1)^{p+q} u_2 u_1 = u^{p+q} u'$.

($\Leftarrow$) Let w be either contained in the word $u^n u'$ or its cyclic permutation. Inser- tion of a symbol a into w which generates a partial word that is contained in $u^n u' a$ gives a non-primitive partial word. That is, $w' \subset u^n$ which is non-primitive (Z_p is reflective). Hence, $w \in \overline{Q}_p^{1\prime}$. □

Let us prove the following observation which shows that if a primitive partial word w with one hole is ins-robust then the reverse of w denoted by $rev(w)$ is also ins-robust.

Lemma 5. *If $w \in Q_p^{1\prime}$ then $rev(w) \in Q_p^{1\prime}$.*

Proof. We prove this by contradiction. Let w be a primitive partial word with one hole which is ins-robust, that is, $w \in Q_p^{1\prime}$ but $rev(w) \notin Q_p^{1\prime}$. Using the structural characterization of non-ins-robust words, we have $rev(w) \subset u^m u_1 u_2 u^n$ where $u = u_1 a u_2$, $m + n \geq 1$. Now,

$$rev(rev(w)) = w \subset rev(u^m u_1 u_2 u^n)\ rev(u^m u_1 u_2 u^n) = (rev(u))^n rev(u_2) rev(u_1) (rev(u))^m$$

Since $u = u_1 a u_2$, we have $rev(u) = rev(u_2)\ a\ rev(u_1)$. Thus,

$$w \subset (rev(u_2)\ a\ rev(u_1))^n rev(u_2) rev(u_1) (rev(u_2)\ a\ rev(u_1))^m.$$

It is clear that w is non-ins-robust which is a contradiction to the assumption. □

There is an algorithm in [4] that recognizes whether a given partial word, w, with at most one hole is primitive or not. It uses the fact that if w is primitive and $ww \uparrow xwy$ then it implies that either $x = \lambda$ or $y = \lambda$.

We know that a primitive partial word with one hole is robust to insertion operation if it is not contained in a word of the form $u^p u_1 u_2 u^s$ for $u = u_1 a u_2$, $u \in Q$ for some $a \in \Sigma$, $p + s \geq 1$.

Theorem 5. *Let w be a primitive partial word with one hole. Then w is non- ins-robust if and only if ww contains at least one strongly p-periodic partial word of length $|w|$ such that p divides $|w| + 1$ and $p \leq |w|$.*

Proof. The necessary and sufficient conditions are proved below.

(⇒) Let w be a non-ins-robust primitive partial word with one hole. Then $w \subset u^r u_1 u_2 u^s$ for some primitive word $u \in Q$ with $r + s \geq 1$ and $u = u_1 a u_2$ for some $a \in \Sigma$. Now $ww \subset u^r u_1 u_2 u^s u^r u_1 u_2 u^s$. Observe that there is a substring of ww which is contained in $u_2 u^s u^r u_1$ of length $|w|$, that is, x is a substring of ww and $x \subset (u_2 u_1 a)^{r+s} u_2 u_1$ which is a $|u|$-strongly periodic and divides $|w| + 1$.

(⇐) Let us assume that ww has a substring of length $|w|$ with strong period of length p which divides $|w| + 1$ and w is a primitive partial word. Now $ww \subset u_1 u^r u' u_2$ where $u_1, u_2 \in \Sigma^*$, $|u^r u'| = |w|$, $u \in Q$ and $u = u' a$ for some $a \in \Sigma$. We Consider the following cases depending upon whether w is contained in $u^r u'$ or in some portion of $u^r u'$.

Case 1 Let w is entirely contained in $u^r u'$. Then w is not ins-robust as inserting a symbol in w will make it a nonprimitive partial word.

Case 2 Let w is contained in some portion of $u^r u'$. Since $ww \subset u_1 u^r u' u_2$ and the language of nonprimitive partial words Z_p is reflective then the cyclic permutation of ww will be contianed in $u_2 u_1 u^r u' = u'' u''$ where the cyclic permutation of w is contained in u''. The cyclic permutation of w which is contained in $u^r u'$ is non-ins-robust. As the language of non-ins-robust primitive partial word Q_p^{1I} is reflective then w is also a non-ins-robust primitive partial word.

□

4 Relation between Chomsky Hierarchy and $\overline{Q_p^{1I}}$

In this section we investigate the relation between the language of non-ins-robust primitive partial words with one hole Q_p^{1I} and the conventional language classes in Chomsky hierarchy. In particular, we show that $\overline{Q_p^{1I}}$ is not a regular language as well as not a Context-Free Language (CFL) over a nontrivial alphabet $\Sigma \cup \{\diamond\}$. Let us recall the pumping lemma for context-free language which is required to prove this result.

Lemma 6 (Pumping Lemma for Regular Languages [12]). *For a regular language L, there exists an integer $n > 0$ such that for every word $w \in L$ with $|w| \geq n$, there exist a decomposition of w as $w = xyz$ such that the following conditions holds.*

(i) $|y| > 0$,
(ii) $|xy| \leq n$, and
(iii) $xy^iz \in L$ *for all* $i \geq 0$.

Let us recall a result which will be used in proving that the language of ins-robust primitive words is not regular.

Lemma 7 ([8]). *For any fixed integer k, there exist a positive integer m such that the equation system $(k-j)x_j + j = m, j = 0, 1, 2, \ldots, k-1$ has a nontrivial solution with appropriate positive integers $x_1, x_2, \ldots, x_j > 1$.*

Theorem 6. *The language of ins-robust primitive partial words with one hole $Q^{1\prime}_p$ is not regular.*

Proof. We prove this result by method of contradiction. Assume that the lan- guage of ins-robust primitive partial words with one hole $Q^{1\prime}_p$ is regular. Then there exists $m > 0$ by pumping lemma. Now consider a primitive partial word with one hole $w = a^m ba^n\diamondsuit$, $n \geq m+1$ and $m \neq 2n$. Observe that w is an ins-robust primitive partial word with one hole. By pumping lemma for regular languages, there exists a decomposition of $w = xyz$ such that $|xy| \leq m$, $|y| \geq 0$ and $xy^i z \in Q^{1\prime}_p$ for all $i \geq 0$.

Let $x = a^p$, $y = a^{(m-j)}$ and $z = a^{j-p}ba^n\diamondsuit$ where $q \geq 0$ and $p+q+r = m$. Now choose $i = x_j$ and by Lemma 7 that for every $j \in \{0, 1, \ldots, m-1\}$, there exists a positive integer $x_j > 1$ such that $xy^{x_j}z = a^p a^{(m-j)x_j} a^{j-p} ba^n\diamondsuit = a^{(m-j)x_j+j}ba^n\diamondsuit = a^n ba^n\diamondsuit \subset (a^n b)^2 \not\subseteq Q^{1\prime}_p$ which is a contradiction.
Hence the language of ins-robust primitive partial words $Q^{1\prime}_p$ is not a regular language. □

Lemma 8 (Pumping Lemma for Context-Free Languages [13]). *Let L be a CFL. Then there exists an integer $n > 0$ such that for every $u \in L$ with $|u| \geq n$, u can be decomposed into $vwxyz$ such that the following conditions hold:*

(a) $|wxy| \leq n$.
(b) $|wy| > 0$.
(c) $vw^i xy^i z \in L$ *for all* $i \geq 0$.

Theorem 7. *$\overline{Q^{1\prime}_p}$ is not a context-free language.*

Proof. We prove it by contradiction. Let us assume that $\overline{Q^{1\prime}_p}$ be a Context-free language. Let $n > 0$ be an integer which is the pumping length that exist by pumping lemma. Since $Q^{1\prime}_p$ is context-free, then it satisfies all the conditions of Lemma 8. Consider a string $w = a^n b^n a^n b^n a^n \diamondsuit b^{n-2}$ where $a, b \in \Sigma$ and are distinct. It is clear that $w \in Q^{1\prime}_p$ and of length at least n.
Hence, by the pumping lemma for CFL, w can be factorized into $uvxyz$ such that $|vy| \geq 1$, $|vxy| \leq n$ and for all $i \geq 0$, $uv^i xy^i z \in Q^{1\prime}_p$. There are several possibilities, that we consider below, depending on whether the substrings v and y contain more than one alphabet symbol or hole.

Case 1 When both v and y contain one type of symbol, that is v does not contain both a's and b's, and same holds for y. Consider one such case. Let v and y contain only a's from the first set of a's. Let $vy = a^k$ for some

$k > 0$. Let $u = a^j$, $vxy = a^p$ and $z = a^q b^n a^n b^n a^n \diamond b^{n-2}$ such that $j \geq 0$ and $j + p + q = n$. Now for $i = 0$, we have $uv^ixy^iz = a^j a^{p-k} a^q b^n a^n b^n a^n \diamond b^{n-2} = a^{j+p+q-k} b^n a^n b^n a^n \diamond b^{n-2} = a^{n-k} b^n a^n b^n a^n \diamond b^{n-2} \notin Q_p^{1\prime}$.

Similar cases can be handled if both v and y contain only symbol b.

Case 2 If v and y contain more than one type of symbol. There will be several cases depending upon whether v contains combinations of a's and b's and y contains only one type of symbol or v contains one type of symbol and y contains the combination of symbols or x contains combinations of a's and b's. Let us consider one such case.

Let $vxy = a^j b^k$ for some j and k such that $0 < j + k \leq n$. Observe that $j,\ k > 0$ otherwise it will fall into Case 1. Suppose $u = a^l$, $v = a^{j_1}$, $x = a^{j_2}$, $y = a^{j_3} b^k$ and $z = b^p a^n b^n a^n \diamond b^{n-2}$ such that $j_1 + j_2 + j_3 = j$, $l + j = n$, and $k + p = n$. For $i = 0$, the string $uv^ixy^iz = a^{l+j_2} b^p a^n b^n a^n \diamond b^{n-2} \notin Q_p^{1\prime}$ as $l + j_2 < n$ and $p < n$.

Similarly other cases in which v and y contain more than one symbol can be handled.

Case 3 Let us consider the last case. If $vxy = \diamond b^p$ then there are following possibilities:

(a) If the symbol $\diamond$ is in vy then $vy = \diamond b^l$ and $x = b^{p-l}$. For $i = 0$, $uv^ixy^iz = a^n b^n a^n b^n a^n b^{n-2-p} \notin Q_p^{1\prime}$.

(b) If the symbol $\diamond$ is in x then $v = \lambda$, $y = b^l$ and $x = \diamond b^{p-l}$ and $l \geq 1$. Now, $uv^ixy^iz = a^n b^n a^n b^n a^n b^{n-2-l} \diamond \notin Q_p^{1\prime}$ for $i = 0$.

Observe that one of the above cases will occur. Since all the above cases result in a contradiction, the assumption that the language of non-ins-robust primitive partial words with one hole $Q_p^{1\prime}$ is not context-free. $\square$

5 Conclusions

In this paper, we have discussed a special subclass of primitive partial words with one hole referred to as ins-robust primitive partial words. We have characterized such partial words and identified several properties. We have also proved that the language of non-ins-robust primitive partial words with one hole $Q_p^{1\prime}$ over a nontrivial alphabet is not context-free.

Several interesting questions for the language of ins-robust primitive partial words needs further exploration. We mention a few of them: (1) Generalizing the insertion operation on primitive partial words with two or more holes. (2) Is the language of ins-robust primitive partial words with one hole $Q_p^{1\prime}$ context-free?

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