IJCRT.ORG ISSN : 2320-2882

INTERNATIONAL JOURNAL OF CREATIVE RESEARCH THOUGHTS (IJCRT)

An International Open Access, Peer-reviewed, Refereed Journal

EXPLORING DIVERSE INSTANCES OF BROOK'S THEOREM AND JOHAN COLOURING IN GRAPHS

S. ANCILA BENCY[1] , Dr. A. AROKIA LANCY[2]

II M.Sc. MATHEMATICS[1], ASSISTANT PROFESSOR [2]

DEPARTMENT OF MATHEMATICS

NIRMALA COLLEGE FOR WOMEN

COIMBATORE- 641018

ABSTRACT:

In this dissertation, we have studied the diverse examples illustrating the application of Brooks' Theorem in graph theory, elucidating its role in determining chromatic numbers for various graph structures and also the Johan colouring determining its rainbow neighbourhood.

KEYWORDS:

Graph, colouring, edge, vertices, chromatic number, Brook's theorem, Johan colouring, Rainbow Neighbourhood.

1.INTRODUCTION:

Brooks' Theorem, established by Leonard Brooks in 1941, is a fundamental result in graph theory. It states that any connected graph with maximum degree Δ that is not a complete graph or an odd cycle has a proper vertex colouring using at most Δ colours. This theorem has implications in network design and optimization, providing insights into the chromatic number of graphs based on their maximum degree. Brooks' Theorem remains a key concept in graph theory and has influenced various areas of computer science and network analysis. Brook's Theorem is one of the most well-known graph colouring theorems. Graph colouring is a subset of graph labelling, in graph theory. It involves the assignment of labels to elements of a graph, commonly referred to as "colours," according to specific constraints. In its most basic form, vertex colouring is a method of colouring the vertices of a graph so that no two neighbouring vertices are the same colour. The Johan colouring was established by J.Kok in the year 2013. It mainly relays on the concept of Rainbow neighbourhood. Now, let's take a look at how Brook's Theorem helps with graph colouring and the condition that satisfies J- colouring .

2.PRELIMINARIES:

Definition 2.1: Vertex and Edge [1]

In a particular function by connecting a set of points is called a Graph. Each node in the graph is called vertices and each line is called an edge.

Definition 2.2: Degree of graph [1]

The number of edges that are incident to the vertex is called the degree of a graph.

Definition 2.3: Colouring [15]

Colouring is the assignment of distinct colours to the vertices of a graph in such a way that no two adjacent vertices have the same colour.

Definition 2.4: Brook's Theorem [2]

Brooks' Theorem states that for any connected graph G, the chromatic number of G is at most $\Delta(G)$ unless G is a complete graph or an odd cycle, where $\Delta(G)$ is the maximum degree of any vertex in G.

Definition 2.5: Chromatic Number [15]

The chromatic number of a graph is the minimum number of colours needed colour its vertices.

Definition 2.6: Vertex Colouring [13]

It is nothing but an assignment of colours to the non-adjacent vertices of the given

Graph. It means that the terminal vertices of the edge should not be assigned by the same

colour.

Definition 2.7: Edge Colouring [12]

It is the way of assigning colours to the non-adjacent edges of the given graph.

Definition 2.8: Rainbow Neighbourhood [5]

A rainbow neighbourhood of a graph G is the closed neighbourhood N[v] of a vertex $v \in V(G)$ which contains at least one coloured vertex of each colour in the chromatic colouring C of G.

Definition 2.9: Johan colouring [7]

A maximal proper colouring of a graph G is a Johan colouring denoted, J-colouring, if and only if every vertex of G belongs to a rainbow neighbourhood of G. The maximum number of colours in a J-colouring is denoted by J(G).

Definition 2.10: Simple andConnected Graph [1]

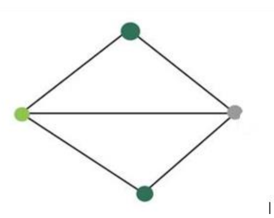
A graph is said to be connected if every pair of vertices in the graph is connected. A graph that is undirected and does not have any loops or multiple edges.

3.APPLICATION OF BROOKS THEOREM

Theorem 3.1:

The DIAMOND graph satisfies Brooks theorem

Proof:



Let us consider the 4 vertices of the diamond graph.

Let $\chi(G)$ be the maximum degree

Let K be the chromatic number

TO PROVE:

$\chi(G) \geq K$

The maximum degree of the graph is 3

The chromatic number of the graph is 3

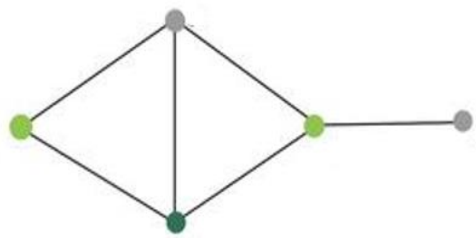
Therefore $\chi(G) \geq K$

Hence the Brooks theorem is true in diamond graph

Theorem 3.2:

The KITE graph satisfies Brooks theorem

Proof:



Let us consider the 5 vertices of the KITE graph.

Let $\chi(G)$ be the maximum degree

Let K be the chromatic number

TO PROVE:

$\chi(G) \geq K$

The maximum degree of the graph is 3

The chromatic number of the graph is 3

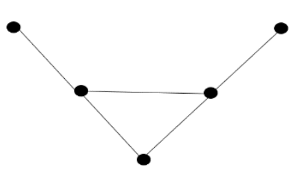
Therefore $\chi(G) \geq K$

Hence the Brooks theorem is true in KITE graph

Theorem 3.3:

The BULL graph satisfies Brooks theorem

Proof:



Let us consider the 5 vertices of the BULL graph.

Let $\chi(G)$ be the maximum degree

Let K be the chromatic number

TO PROVE:

$\chi(G) \geq K$

The maximum degree of the graph is 3

The chromatic number of the graph is 3

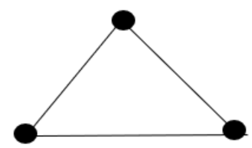
Therefore $\chi(G) \geq K$

Hence the Brooks theorem is true in BULL graph

Theorem 3.4:

The TRIANGULAR graph satisfies Brooks theorem

Proof:



Let us consider the 5 vertices of the TRIANGULAR graph.

Let $\chi(G)$ be the maximum degree

Let K be the chromatic number

TO PROVE:

$\chi(G) \geq K$

The maximum degree of the graph is 3

The chromatic number of the graph is 3

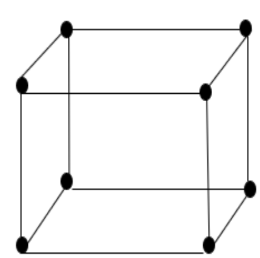
Therefore $\chi(G) \geq K$

Hence the Brooks theorem is true in TRIANGULAR graph

Theorem 3.5:

The CUBE graph satisfies Brooks theorem

Proof:



Let us consider the 8 vertices of the CUBE graph.

Let $\chi(G)$ be the maximum degree

Let K be the chromatic number

TO PROVE:

$\chi(G) \geq K$

The maximum degree of the graph is 3

The chromatic number of the graph is 3

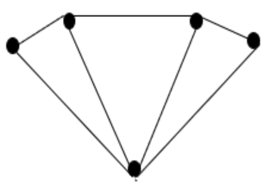
Therefore $\chi(G) \geq K$

Hence the Brooks theorem is true in CUBE graph

Theorem 3.6:

The GEM graph satisfies Brooks theorem

Proof:



Let us consider the 5 vertices of the GEM graph.

Let $\chi(G)$ be the maximum degree

Let K be the chromatic number

TO PROVE:

$\chi(G) \geq K$

The maximum degree of the graph is 4

The chromatic number of the graph is 5

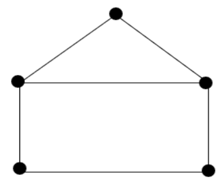
Therefore $\chi(G) \geq K$

Hence the Brooks theorem is true in GEM graph

Theorem 3.8:

The HOUSE graph satisfies Brooks theorem

Proof:



Let us consider the 5 vertices of the HOUSE graph.

Let $\chi(G)$ be the maximum degree

Let K be the chromatic number

TO PROVE:

$\chi(G) \geq K$

The maximum degree of the graph is 3

The chromatic number of the graph is 3

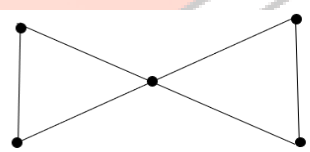
Therefore $\chi(G) \geq K$

Hence the Brooks theorem is true in the HOUSE graph

Theorem 3.7:

The BUTTERFLY graph satisfies Brooks theorem

Proof:



Let us consider the 5 vertices of the BUTTERFLY graph.

Let $\chi(G)$ be the maximum degree

Let K be the chromatic number

TO PROVE:

$\chi(G) \geq K$

The maximum degree of the graph is 4

The chromatic number of the graph is 3

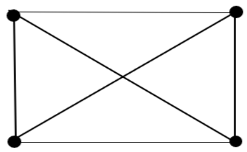
Therefore $\chi(G) \geq K$

Hence the Brooks theorem is true in BUTTERFLY graph

Theorem 3.9:

The SQUARE graph satisfies Brooks theorem

Proof:



Let us consider the 4 vertices of the SQUARE graph.

Let χ(G) be the maximum degree

Let K be the chromatic number

TO PROVE:

$\chi(G) \geq K$

The maximum degree of the graph is 3

The chromatic number of the graph is 4

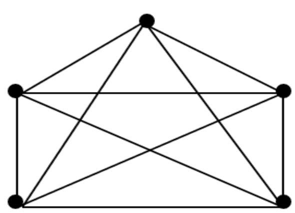
Therefore $\chi(G) \geq K$

Hence the Brooks theorem is true in SQUARE graph

Theorem 3.10:

The STAR graph satisfies Brooks theorem

Proof:



Let us consider the 5 vertices of the STAR graph.

Let χ(G) be the maximum degree

Let K be the chromatic number

TO PROVE:

$\chi(G) \geq K$

The maximum degree of the graph is 3

The chromatic number of the graph is 4

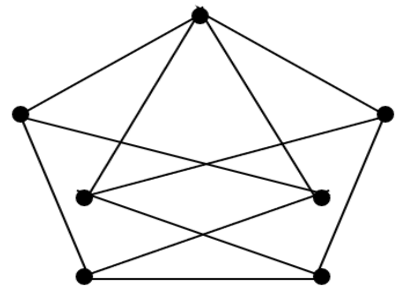
Therefore $\chi(G) \geq K$

Hence the Brooks theorem is true in STAR graph

Theorem 3.11:

The MOSER graph satisfies Brooks theorem

Proof:



Let us consider the 7 vertices of the MOSER graph.

Let χ(G) be the maximum degree

Let K be the chromatic number

TO PROVE:

$\chi(G) \geq K$

The maximum degree of the graph is 4

The chromatic number of the graph is 4

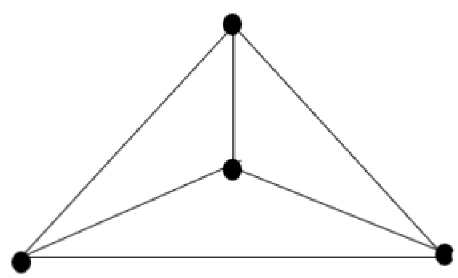
Therefore $\chi(G) \geq K$

Hence the Brooks theorem is true in MOSER graph

Theorem 3.12:

The TRIANGLE graph satisfies Brooks theorem

Proof:



Let us consider the 4 vertices of the TRIANGLE graph.

Let χ(G) be the maximum degree

Let K be the chromatic number

TO PROVE:

$\chi(G) \geq K$ The maximum degree of the graph is 3

The chromatic number of the graph is 4

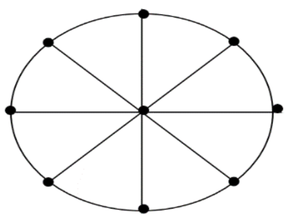
Therefore $\chi(G) \geq K$

Hence the Brooks theorem is true in TRIANGLE graph

Theorem 3.13:

The WHEEL graph satisfies Brooks theorem

Proof:



Proof:

Let us consider the 8 vertices of the WHEEL graph.

Let $\chi(G)$ be the maximum degree

Let K be the chromatic number

TO PROVE:

$\chi(G) \geq K$

The maximum degree of the graph is 8

The chromatic number of the graph is 4

Therefore $\chi(G) \geq K$

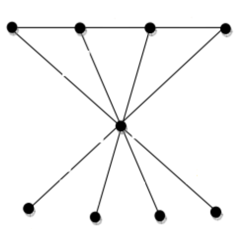
Hence the Brooks theorem is true in WHEEL graph

4. verification of Johan Colouring in various graphs:

Theorem 4.1:

The OCTOPUS graph is Johan colourable:

Proof:



Let us consider an octopus graph

Condition for rainbow neighbourhood:

$\chi(G) \geq r_x(G)$

The chromatic number $\chi(G)$ is 5

$r_x(G)$ = min {Number of neighbourhood of every vertex}

$r_x(G) = \min\{8,3,3,3,3,3,3,3\}$

Therefore $r_x(G)=3$

$5 > 3$

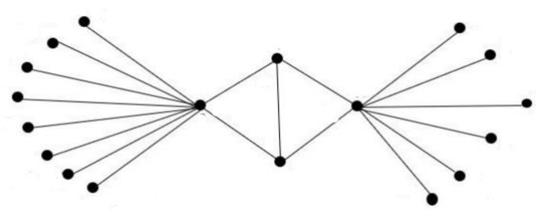
The condition is satisfied.

Therefore the Octopus graph is Johan colourable.

Theorem 4.2:

The JELLY FISH graph is Johan colourable:

Proof:



Let us consider a Jelly Fish graph

Condition for rainbow neighbourhood:

$\chi(G) \geq r_x(G)$

The chromatic number $\chi(G)$ is 2

$r_x(G)$ = min {Number of neighbourhood of every vertex}

$r_x(G) = \min\{3,3,3,3,3,3,3,3,3,3,3,3,3,2,2,10,10\}$

Therefore $r_x(G)=2$

$2=2$

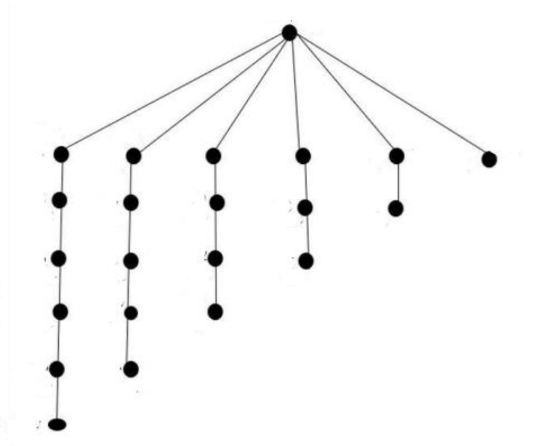
The condition is satisfied.

Therefore the Jelly Fish graph is Johan colourable.

Theorem 4.3:

The OLIVE TREE graph is Johan colourable:

Proof:



Let us consider an olive tree graph

Condition for rainbow neighbourhood:

$\chi(G) \geq r_x(G)$

The chromatic number $\chi(G)$ is 2

$r_x(G)$ = min {Number of neighbourhood of every vertex}

$r_x(G)$ = min{6,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,1}

Therefore $r_x(G)$= 1

2 > 1

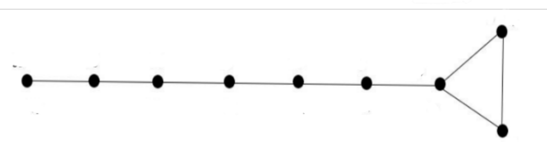
The condition is satisfied.

Therefore the Olive tree graph is Johan colourable.

Theorem 4.4:

The TADPOLE graph is Johan colourable:

Proof:



Let us consider a Tadpole graph

Condition for rainbow neighbourhood:

$\chi(G) \geq r_x(G)$

The chromatic number $\chi(G)$ is 3

$r_x(G)$ = min {Number of neighbourhoods of every vertex}

$r_x(G)$ = min {2,2,2,2,2,2,2,2,2,2}

Therefore $r_x(G)$=2

3 > 2

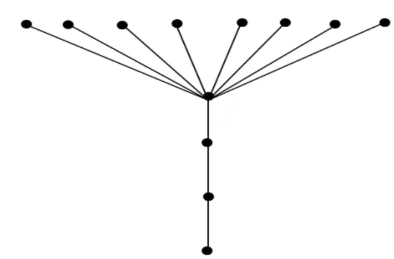
The condition is satisfied.

Therefore, the Tadpole graph is Johan colourable.

Theorem 4.5:

The COCONUT TREE graph is Johan colourable:

Proof:



Let us consider a Coconut tree graph

Condition for rainbow neighbourhood:

$\chi(G) \geq r_x(G)$

The chromatic number $\chi(G)$ is 2

$r_x(G)$ = min {Number of neighbourhoods of every vertex}

$r_x(G)$ = min {8,2,2,2,2,2,2,2,2,2,2,2}

Therefore $r_x(G)$=2

2=2

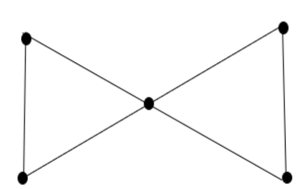
The condition is satisfied.

Therefore, the Coconut graph is Johan colourable.

Theorem 4.6:

The BUTTERFLY graph is Johan colourable

Proof:



Let us consider a butterfly graph

Condition for rainbow neighbourhood:

$\chi(G) \geq r_x(G)$

The chromatic number $\chi(G)$ is 3

$r_x(G)$ = min {Number of neighbourhoods of every vertex}

$r_x(G)$ = min {2,2,2,2,2}

Therefore $r_x(G)=2$

$3 > 2$

The condition is satisfied.

Therefore, the Butterfly graph is Johan colourable.

5. CONCLUSION:

The Brook's theorem, highlights the exceptional cases and offers a nuanced perspective on the complexity of graph colouring problems. Brooks' Theorem remains a cornerstone in graph theory, shaping the way researchers and practitioner's approach and understand the intricacies of colouring graphs in diverse application. The purpose of the graph is to show numerical facts in visual form so that they can be understood quickly, easily and clearly. Thus, graphs are visual representations of data collected. Data can also be presented in the form of a table. However, a graphical presentation is easier to understand. Charts and graphs are visual representations of data. They are important and useful because they are powerful tools that can be used for things like analysing data, emphasizing a point, or comparing multiple sets of data in a way that is easy to understand and remember.

Graph colouring is one of the best known, popular and extensively researched subject in the field of graph theory, having many applications and conjectures, which are still open and studied by various mathematicians and computer scientists along the world. These algorithms not only contribute to solving combinatorial optimization problems but also play a crucial role in improving the efficiency and performance of numerous systems. As technology continues to advance, the significance of graph colouring is likely to grow, providing valuable solutions to complex problems in diverse fields.

6. REFERENCE:

[1] J. A. Bondy and U. S. R. Murty, Graph Theory with Applications, Macmillan Press, London, 1976.

[2] M. Borowiecki, On chromatic number of products of two graphs, Colloquium Mathematicum 25 (1972), 49-52.

[3] G. Chartrand and P. Zhang, Chromatic graph theory, CRC Press, (2009).

[4] F. Harary, Graph Theory, Addison-Wesley, Reading MA, (1969).

[5] J. Kok, S. Naduvath and M. K. Jamil, Rainbow Neighbourhoods of Graphs, arXiv:1703.01089v1 [math.GM], 3 Mar 2017.

[7] J. Kok and S. Naduvath, Johan colouring of graph operations, Preprint, arXiv:1704.02869, 2017.

[8] U. Mary, M. Jerlin Seles, J. Kok and N. K. Sudev, On Chromatic core Subgraph of simple graphs, Contemporary Studies in Discrete Mathematics 2(1), (2018) 1-8.

[9] D. Muthuramakrishnan and G. Jayaraman, Total Coloring of Splitting Graph of Path, Cycle and Star Graphs International Journal of Mathematics And its Applications 6(1D) (2018), 659-664.

[10] D. B. West, Introduction to Graph Theory, Prentice-Hall, Upper Saddle River, 1996.

[11] Kok J, Sudev NK, Jamil MK (2017) Rainbow neighbourhood number of graphs. Proy J Math.

[12] Shirinivas S G, S.Vetrivel and N.M.Elango, Application of Graph theory in Computer

Science an overview, International Journal of Engineering Science and Technology,

(2010)

[13] Bandt C,Graf S and Zähle M. Fractal geometry and stochastics, 1, Birkhäuser, (1995)

[14] Bruhn H, Georgakopoulos A, Sprüssel P, Every rayless graph has an unfriendly

partition , Combinatorica, 521-532, (2010).

[15] Guichard D.R, No-hole k tuple, (r+1) distant colorings, Discrete Appied Mathematics,

64(1), 87-92, (1996).

[16] Gyarfas A, Lehel J, Online and first fit coloring of graphs, Graph theory, 12(2), 217-227, (1988).