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On Solving Sum of Ratios Fractional Programming Problems by Interval Branch and Bound Method

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Abstract: This article represents an interval branch and bound (*B&B* technique) constrained global optimization method developed by Karmakar&Bhunia(2012) for Sum of Ratios Fractional Programming Problems (FPPs). The constraints of this kind of FPPs involves sum of ratios fractional type constraints. There exist several efficient constraint-handling techniques to find global optimizers for various problems; but, not so effective for general sum of ratios fractional type constraints. A more updated and general interval ranking definitions in terms of decision maker's view point has been used to modify this *B&B* technique. The method is mainly based on the multi-section splitting criterion of the accepted/prescribed search region, calculation of interval inclusion of the fractional objectives and constraints and the selection of subregion depending on the modified interval order relations said earlier. Adequate number of numerical examples selected form the existing literature have been solved and compared in support of this technique.

Keywords: Fractional Programming Problem; Interval Optimization; Interval Arithmetic; Interval Order Relation; Branch and Bound Technique; Decision Theory.

1 Introduction

Fractional programming is a type of mathematical optimization problem where the objective function is a ratio of two functions, and the goal is to minimize or maximize this ratio subject to certain constraints. In several applications of nonlinear programming, a function is to be optimized which is characterized by single or several fractions of given functions.Fractional programming problems (FPPs) can be challenging to solve due to the nonlinearity introduced by the ratio of functions. Various techniques are employed to address these problems, including transforming them into equivalent nonlinear programming problems or using specialized algorithms.

FPPs are initially suggested by Charnas and Cooper (1962). A wide application of FPP is seen in different Economic models. One of the earliest economic models, named, Von Neumann's (1937) model for an expanding economy contains fractional programs which determines the growth rate of an economy as the maximum of the smallest of several output-input ratios. In different economic applications such as maximization of productivity, maximization of return on investment, maximization of return/risk, maximization of input/output, the FPPs are frequently used. Apart from these there are a number of problems involving fractional program in management science and information theory. Among the various types of FPP, the single ratio and the sum of ratios problems are mostly used in aforesaid application areas.

The general mathematical form of sum of ratios FPP is

P1: Minimize $f(x)=\sum_{i=1}^{p} z_i(x)=\sum_{i=1}^{p}\frac{n_i(x)}{d_i(x)}$

subject to $g_k(x) \le (\text{or} =) b_k$, $k = 1, 2, 3, \ldots, m$

$$\Omega=\left\{x \mid 0 \le l_j \le x_j \le u_j,\ j=1,2,\ldots,n\right\}$$

where for each $i = 1, 2, 3, \ldots, p$ and $k = 1, 2, 3, \ldots, m$

$$n_i(x)=\sum_{\tau=1}^{T_i^1}\alpha_{i\tau}^1\prod_{j=1}^{n}x_j^{\gamma_{i\tau j}^1},\ d_i(x)=\sum_{\tau=1}^{T_i^2}\alpha_{i\tau}^2\prod_{j=1}^{n}x_j^{\gamma_{i\tau j}^2}\ \text{ and }\ g_k(x)=\sum_{\tau=1}^{T_k^3}\alpha_{k\tau}^3\prod_{j=1}^{n}x_j^{\gamma_{k\tau j}^3}$$

and p, T_i^1, T_i^2, T_k^3 are all natural numbers, $\alpha_{i\tau}^1, \alpha_{i\tau}^2, \alpha_{k\tau}^3$ are all nonzero real constant coefficients, $\gamma_{i\tau j}^1, \gamma_{i\tau j}^2, \gamma_{k\tau j}^3$ are all real constant exponents, and $n_i(x) \ge 0$, $d_i(x) > 0$.

More specifically, the different types of FPPs are distinguished as follows:

(**P1**) is said to be a single ratio FPP if $p = 1$, otherwise it will be a sum of ratio FPP.
(**P1**) is called a linear fractional program if all functions $n_i(x)$, $d_i(x)$,$g_k(x)$ for all the prescribed values of i and k are affine, *i.e.*, the sum of a linear function and a constant. Otherwise, it will be a nonlinear FPP.
(**P1**) is said to be a quadratic fractional program if $n_i(x)$ and $d_i(x)$ are quadratic and $g_k(x)$ are affine.
(**P1**) is said to be a concave-convex FPP if all $n_i(x)$ are concave and all $d_i(x)$ as well as all $g_k(x)$ are convex.

The main goal of this article is to apply the earlier developed interval branch and bound constrained handling technique for generalsum of ratios FPPs. The mentioned Interval oriented constrained global optimization (ICCGO) technique has been developed by Karmakar and Bhunia(2012). It has already been stated that there are so many different global optimization techniques which are very efficient in finding global optimizer points for constraint handling problems but cannot find the same for general FPPs. From the literature review (**ref.** Section **3**), it is also clear that most of the optimization techniques used for FPP have been developed based on the *B & B* algorithm. Neither any type of interval-oriented algorithm nor any direct application of interval numbers and interval analysis have been developed till now for non-interval valued FPPs. An updated interval-oriented constraint satisfaction rulesestablished byKarmakar and Bhunia(2012, 2013) are used for equality and inequality constraints. The modified multi-section division criterion with some new interval-oriented constraint satisfaction rules for equality and inequality constraints and some novel interval order relations in the context of the decision makers' point of view developed by Sahoo *et. al*. (2012) have been applied to increase the efficiency of the proposed algorithm. Finally, to show the effectiveness of the technique, it is applied to some test problems taken from the existing literature and the results are compared with the same obtained from the existing methods.

The organization of the rest of the article as follows: In the next section, a brief survey of different types of FPPs and their solution techniques have been given. We shall discuss the important parts of interval mathematics including interval inclusion functions and fundamental theorem of interval analysis in **Section 3**. In **Section 4**, a brief comparative discussion of some interval ordering definitions has been given. **Section 5** provides the statement of the problem and the details of the proposed solution technique and **Section 6** includes the details of numerical experiments and comparative discussions.

2. Literature Survey

Fractional programming provides a flexible framework for handling optimization problems where the objective involves a trade-off between conflicting criteria. Solving fractional programming problems often requires specialized algorithms and techniques, and various methods such as Dinkelbach's algorithm and outer approximation methods have been developed for this purpose. The pioneering work on this topic has been delivered by Charnas and Cooper (1962). After that development of linear FPP, a considerable number of articles have been published by several researchers. In an extensive treatment of the subject, Schaible(1978) tried to relate those works to each other and discussed about their applications in contemporary scientific and industrial fields. Moreover, the theoretical and algorithmic developments of FPP are discussed in details. However, the study of only single ratio FPP has largely dominated the literature in this field near about 1980. The two monographs of one has been given by Craven (1988) and the other by Stancu – Minasian(1997) are completely devoted to general FPP. FPPs with single ratio or sum of ratios objective function have often been studied in the broader context of generalized convex programming (**ref.**Avriel et. al. (1988), Frenk and Schaible(2004)). FPPs are always a part of general global optimization problems. However, there are a large number of global optimization techniques which fail to search better optimizer points compared to general FPPs. As a result, a number of distinguished optimization techniques have been developed theoretically and also algorithmically for FPPs.

During the previous years, various algorithms have been proposed depending on branch-and-bound (*B&B*) technique. Konno *et al.*(1991) proposed an algorithm for sum of ratios linear FPPs by using the parametric simplex method. Another two algorithms for linear FPP which search iteratively the non-convex outcome space until a global optimal solution is found have been proposed by Konno and Yamashita (1999) and Falk and Palocsay (1994). A branch and bound algorithm have been proposed by Shen and Wang (2006) for the sum of linear ratios problem with coefficients. They reduced the initial non-convex programming problem to a series of LPPs by utilizing the equivalent transformation and linearization process. Another *B&B* algorithm developed by Jiao *et al.*(2006) construct the linear lower bounding functions of the objective function and constrained function of the generalized FPP over the feasible region. Qu *et al.*(2007) used Lagrangian relaxation to develop another algorithm for the sum of quadratic ratios problems with non-convex quadratic constraints. A new type of algorithm called Simplicial branch and bound duality-bounds algorithm has been presented by Benson (2007) for the linear-sum-of-ratios problem. Shen and Wang (2008) have generalized the *B & B* technique developed by Shen and Wang (2007) to search the global optimizer of sum of fractional functions containing a sum or product of a finite number of ratios of linear functions, polynomial fractional programming, generalized geometric programming, etc. over a polytope. By using linearization method and a new pruning technique, Jiao *et al.*(2008) proposed a different deterministic global optimization algorithm for the sum of general ratios problem. For quadratic ratios fractional programming with non-convex quadratic constraints, a branch-reduce-bound algorithm has been proposed by Shen *et al.*(2009). Also,Shen *et al.*(2009) developed the simplicial branch and duality-bound algorithm for the sum of convex-convex ratios problem. The proposed algorithm is based on reformulating the problem as a monotonic optimization problem. The equivalent transformation (**ref.** Shen and Wang (2006)) and a new linear relaxation technique have been used by Wang *et al.*(2008) to develop a modified *B & B* algorithm for globally solving the sum of ratios linear FPPs.In another investigation, Shen and Jin(2010) have given the conical partition algorithm for the same FPP with convex feasible region. Recently, an extended global optimization algorithm is proposed by Shen *et al.*(2011) for generalized polynomial sum of rations problem via monotonic optimization. By utilizing exponent transformation and new three-level linear relaxation method, a global optimization algorithm has been proposed by Jiao *et al.*(2013) for sum of generalized polynomial ratios problem.Jain *et. al.*(2018) have developed an algorithm for Quadratic Fractional Integer Programming Problems with bounded variables (QFIPBV) which involves ranking and scanning of the set of integer feasible solutions. An efficient formulation and a computational approach have been successfully constructed in order to solve a general class of linear FPP on arbitrary time scales by Salih and Bohner(2018). A novel iterative algorithm depending on usual concept of continuity for a linear FPP has been developed by Ozkok(2020). Mitlif(2022) has prescribed an efficient algorithm for fuzzy linear FPP via ranking function. Fathy *et. al.*(2023) have studied a fully intuitionistic fuzzy multi-level linear FPP and solve it.

3. Basics of Interval Mathematics

Interval numbers are the generalization of real numbers and interval arithmetic is the generalization of real arithmetic in larger sense. According to Moore(1979), an interval number A is nothing but a closed interval and thus can be defined as $A = \{x: a_L \le x \le a_R\}$ and denoted by $[a_L, a_R]$. Moore(1979) not only gave the concepts of interval numbers and interval vectors but also proposed extensive interval arithmetic and various interval analytical processes. Ratschek and Rokne(1988), Kearfott(1996), Jaulin*et al.*(2001), Hansen and Walster(2004) have also developed modern interval arithmetic very rigorously. Interval arithmetic is used to calculate the interval inclusion functions which specify the rigorous bounds of the objective functions in global optimization solution techniques. General mathematical operations such as addition, subtraction, scalar multiplication, multiplication between two interval numbers, division etc. can be defined very easily in the existing literature.

An interval number can also be expressed in the form of centre and radius of the interval as $A = \langle a_C, a_W\rangle = \{x: a_C - a_W \le x \le a_C + a_W,\ x \in \mathbf{R}\}$, where $a_C = (a_L + a_R)/2$ = centre and $a_W = (a_R - a_L)/2$ = radius of the interval. In this form, the interval arithmetic can be depicted for $A = \langle a_C, a_W\rangle$ and $B = \langle b_C, b_W\rangle$ as,

$$A + B = \langle a_C,a_W\rangle + \langle b_C,b_W\rangle = \langle a_C + b_C,\ a_W + b_W\rangle$$
$$A - B = \langle a_C,a_W\rangle - \langle b_C,b_W\rangle = \langle a_C ,a_W\rangle + \langle -b_C,b_W\rangle = \langle a_C - b_C,\ a_W + b_W\rangle$$
$$\lambda A = \lambda\langle a_C,a_W\rangle = \langle \lambda a_C, |\lambda|\, a_W\rangle.$$

Let $A = [a_L,a_R]$ be an interval and n be any non-negative integer, then the nth power of A is defined by

$$A^n = \begin{cases} [1,\ 1] & \text{if } n = 0 \\ [a_L^n,\ a_R^n] & \text{if } a_L \ge 0 \text{ or if } n \text{ is odd} \\ [a_R^n,\ a_L^n] & \text{if } a_R \le 0 \text{ and } n \text{ is even} \\ [0,\ \max(a_L^n,\ a_R^n)] & \text{if } a_L \le 0 \le a_R \text{ and } n(>0) \text{ is even.} \end{cases}$$

Karmakar*et al.*(2009) defined the *n*th root, any rational power and modulus of an interval as follows:

The *n*throot of an interval $A = [a_L,a_R]$, n being any positive integer, is defined as

$$(A)^{\frac{1}{n}} = [a_L,\ a_R]^{\frac{1}{n}} = \sqrt[n]{[a_L,\ a_R]} = [\sqrt[n]{a_L},\ \sqrt[n]{a_R}] \quad \text{if } a_L \ge 0 \text{ or if } n \text{ is odd,}$$
$$= [0,\ \sqrt[n]{a_R}] \quad \text{if } a_L \le 0,\ a_R \ge 0 \text{ and } n \text{ is even,}$$
$$= \phi \quad \text{if } a_R < 0 \text{ and } n \text{ is even.}$$

where ϕ is an empty interval.

Again, by applying the definitions of power and different roots of an interval, the rational power of an interval $A = [a_L,a_R]$ is defined as

$$A^{\frac{p}{q}} = \left(A^p\right)^{\frac{1}{q}}.$$

The modulus of an interval can be defined as follows:

$$|A| = \left|[a_L,\ a_R]\right| = [a_L,\ a_R] \quad \text{if } a_L \ge 0,$$
$$= \left[|a_R|,\ |a_L|\right] \quad \text{if } a_R \le 0,$$
$$= \left[0,\ |a_L|\right] \quad \text{if } a_L < 0,\ a_R > 0,\ |a_L| \ge |a_R|,$$
$$= \left[0,\ |a_R|\right] \quad \text{if } a_L < 0,\ a_R > 0,\ |a_L| < |a_R|.$$

According to Moore (1979), an interval function means an interval-valued function of interval arguments. Let $f: \mathbf{R^n} \to \mathbf{R}$be a real valued function of real variables $x_1, x_2,\ldots, x_n$ and $F: \mathbf{I^n} \to \mathbf{I}$be an interval valued function of interval variables $X_1, X_2,\ldots, X_n$. The function F is said to be an interval extension of f if $F(x_1, x_2,\ldots, x_n) = f(x_1, x_2,\ldots, x_n)$ for all x_i $(i = 1, 2,\ldots, n)$. Again, an interval function F is said to be inclusion monotonic if $X_i \subseteq Y_i$ $(i = 1, 2,\ldots, n)$ implies $F(X_1, X_2,\ldots, X_n) \subset F(Y_1, Y_2,\ldots, Y_n)$. Now we shall state the fundamental theorem of interval analysis which is also the most important for any interval-orientedmethod (Hansen and Walster(2004)).

Theorem 2.1 *Let $F(X_1, X_2,\ldots, X_n)$ be an inclusion monotonic interval extension of a real function $f(x_1, x_2,\ldots, x_n)$. Then $F(X_1, X_2,\ldots, X_n)$ contains the range of values of $f(x_1, x_2,\ldots, x_n)$ for all $x_i \in X_i$ $(i = 1, 2,\ldots, n)$.*

4. Order Relations of Interval Numbers

Interval order is a type of binary relation that is often used in the context of ordered sets or partially ordered sets. Inmathematics, an ordered set is a set equipped with a binary relation that is reflexive, antisymmetric, and transitive. The concept of an interval order is particularly useful when dealing with intervals on a real line or in a more general ordered set. In this section, we discuss the previous developments of order relations of interval numbers for both maximization and minimization problems. Let $A = [a_L,a_R] = \langle a_C,a_W\rangle$ and $B = [b_L,b_R] = \langle b_C,b_W\rangle$ be any two closed intervals. A details of interval ordering definitions are reviewed in Karmakar and Bhunia(2012). These intervals may be of the following types:

Type I: Non-overlapping intervals i.e., when $a_L \geq b_R$ or $b_L \geq a_R$.
Type II: Partially overlapping intervals i.e., when $b_L \leq a_L < b_R \leq a_R$ or $a_L < b_L \leq a_R < b_R$.
Type III: Fully overlapping intervals i.e., when $a_L < b_L < b_R \leq a_R$ or $b_L \leq a_L < a_R < b_R$.

Definition 4.1 For any two intervals A and B, Moore (1979) gave the first transitive order relation '<' as

$$A < B \text{ iff } a_R < b_L.$$

Other transitive order relation for intervals is

$$A \subseteq B \text{ iff } b_L \leq a_L \text{ and } a_R \leq b_R.$$

These two order relations cannot order two partially or fully overlapping intervals. Second transitive order relation which is the extension of set inclusion property cannot order A and B in terms of value. It describes the condition that the interval A is nested in B.

Noticing the drawbacks of Moore's (1979) approach, Ishibuchi and Tanaka (1990) defined the order relations for minimization problems of two closed intervals A and B in the following way:

Definition 4.2 (i) $A \leq_{LR} B$ iff $a_L \leq b_L$ and $a_R \leq b_R$

$$A <_{LR} B \text{ iff } A \leq_{LR} B \text{ and } A \neq B.$$

(ii) $A \leq_{CW} B$ iff $a_C \leq b_C$ and $a_W \leq b_W$

$$A <_{CW} B \text{ iff } A \leq_{CW} B \text{ and } A \neq B.$$

These order relations are reflexive, transitive and anti-symmetric i.e., it is a partial order. Clearly, for minimization problems, a decision maker will prefer the interval A.

Generalizing the definitions of Ishibuchi and Tanaka (1990), Chanas and Kuchta(1996) proposed the concept of t_0t_1 – cut of an interval for the ranking of interval numbers.

Definition 4.3 Let $A = [a_L,a_R]$ be any interval and t_0 and t_1 be any two fixed numbers such that $0 \leq t_0 < t_1 \leq 1$. Then the t_0,t_1 – cut of the interval is given by

$$A/_{[t_0,t_1]} = \left[a_L + t_0\left(a_R - a_L\right),\ a_L + t_1\left(a_R - a_L\right)\right].$$

Using this definition on interval numbers, Chanas and Kuchta(1996) modified the interval ranking definitions of Ishibuchi and Tanaka (1990). For minimization problems, they considered the **Definition 4.2** and redefined as follows:

Definition 4.4 (i) $A \leq_{LR} /_{[t_0,t_1]} B \Leftrightarrow A/_{[t_0,t_1]} \leq_{LR} B/_{[t_0,t_1]}$,

$$A <_{LR} /_{[t_0,t_1]} B \Leftrightarrow A/_{[t_0,t_1]} <_{LR} B/_{[t_0,t_1]}.$$

(ii) $A \leq_{CW} /_{[t_0,t_1]} B \Leftrightarrow A/_{[t_0,t_1]} \leq_{CW} B/_{[t_0,t_1]}$,

$$A <_{CW} /_{[t_0,t_1]} B \Leftrightarrow A/_{[t_0,t_1]} <_{CW} B/_{[t_0,t_1]}.$$

For $t_0 = 0$ and $t_1 = 1$, the definitions (i), and (ii) certainly lead to the order relations $\leq_{LR}$, and $\leq_{CW}$ proposed by Ishibuchi and Tanaka (1990)respectively.

Regarding interval ranking a remarkable work was done by Sengupta and Pal (2000). They defined their first definition of order relation with respect to the decision makers' point of view using acceptability function.

Definition 4.6 The acceptability function (or acceptability index or value judgment index) χ: $\mathbf{I}\times\mathbf{I}\rightarrow [0, \infty)$ for the intervals A and B is defined as

$$\chi(A,B) = \frac{b_C - a_C}{b_W + a_W}, \text{ where } b_W + a_W \neq 0.$$

$\chi(A, B)$ may be regarded as a grade of acceptability of the 'first interval to be inferior to the second'. If $\chi(A, B) = 0$ then for a minimization problem, the interval A can't be accepted as smaller. If $0 < \chi(A,B) < 1$, A can be accepted as such with the grade of acceptability $\frac{b_C - a_C}{b_W + a_W}$. Finally, if $\chi(A, B) = 1$, A is accepted fully.

According to them, the acceptability index is only a value based ranking index and it can be applied partially to select the best alternative from the pessimistic point of view of the decision maker. So, only the optimistic decision maker can use it completely.

For a pessimistic decision maker, Sengupta and Pal (2000) introduced the fuzzy preference ordering for ranking of a pair of interval numbers on the real line. The fuzzy preference method was actually described for maximizing the profit interval; however, the method is equally applicable for minimizing the cost/time intervals. Therefore, they have assumed that *A* and *B* are two profit intervals and the problem is to choose the maximum profit interval form among them. Thereafter they considered the fuzzy set "Rejection of an interval *A* in comparison to the interval *B*" or "Acceptance of *B* in comparison to *A*".

Definition 4.7 The membership function of this fuzzy set is given by

$$\mu(B,A)=\begin{cases}1 & b_C=a_C,\\ \max\left\{0,\dfrac{b_C-a_L-b_W}{a_C-a_L-b_W}\right\} & a_L+b_W\le b_C\le a_C,\\ 0 & otherwise.\end{cases}$$

This non-linear membership function lies in the interval [0, 1]. When the values of this membership function lies within the interval [0.333, 0666], this definition fails to find the order relations.

Recently, Mahato and Bhunia(2006) proposed two types of definitions of interval order relations in the context of optimistic and pessimistic decision making. As usual, let the intervals $A=[a_L,a_R]=\langle a_C,a_W\rangle$ and $B=[b_L,b_R]=\langle b_C,b_W\rangle$ represent the interval numbers.

4.1 Optimistic decision-making

In this decision making, a decision is taken by ignoring the uncertainty and the interval containing the lowest cost/ time for minimization problems and the highest profit for maximization problem are accepted.

Definition 4.8 For minimization problems, the order relation $\le_{omin}$ between the intervals *A* and *B*is defined as

$$A\le_{\text{omin}}B \text{ iff } a_L\le b_L, \text{ and}$$
$$A<_{omin}B \text{ iff } A\le_{omin}B \text{ and } A\neq B.$$

This implies that *A* is superior to *B* and *A* is accepted. This order relation is obviously not symmetric.

Definition 4.9 For maximization problems, the order relation $\ge_{omax}$ between the intervals *A* and *B* is

$$A\ge_{omax}B \text{ iff } a_R\ge b_R, \text{ and}$$
$$A>_{omax}B \text{ iff } A\ge_{omax}B \text{ and } A\neq B.$$

This implies that *A* is superior to *B* and the optimistic decision maker accepts the profit interval *A*. Here, again the order relation $\ge_{omax}$ is not symmetric.

4.2 Pessimistic decision-making

In this case, the decision maker expects the minimum cost/time for minimization problems and the maximum profit for maximization problems according to the principle "Less uncertainty is better than more uncertainty".

Definition 4.10 For minimization problems, the order relation $<_{pmin}$ between the intervals $A=[a_L,a_R]=\langle a_C,a_W\rangle$ and $B=[b_L,b_w]=\langle b_C,b_W\rangle$ for a pessimistic decision maker is defined as

(i) $A<_{pmin}B$ iff $a_C<b_C$, for Type - I and Type – II intervals, and

(ii) $A<_{pmin}B$ iff $a_C\le b_C$ and $a_W<b_W$, for Type – III intervals.

However, for Type – III intervals with $a_C<b_C$ and $a_W>b_W$, a pessimistic decision cannot be taken. Here, the optimistic decision is considered.

Definition 4.11 For maximization problems, the order relation $>_{pmax}$ between the intervals $A=[a_L,a_R]=\langle a_C,a_W\rangle$ and $B=[b_L,b_R]=\langle b_C,b_W\rangle$ for a pessimistic decision maker is defined as

(i) $A>_{pmax}B$ iff $a_C>b_C$, for type - I and Type – II intervals

(ii) $A>_{pmin}B$ iff $a_C\ge b_C$ and $a_W<b_W$, for Type – III intervals.

Again, for Type – III intervals with $a_C>b_C$ and $a_W>b_W$, a pessimistic decision cannot be taken. Here, the optimistic decision is taken.

Some important interval ordering definitions with different mathematical backgrounds have been mentioned here. A details comparative discussion is given by Karmakar and Bhunia(2012).

Generalised Ordering Definition

The interval ranking definitions in terms of pessimistic decision making given byMahato and Bhunia(2006) is insufficient for some of the pair of Type - III intervals.According to their suggestion those inapplicable cases may be handled by consideringoptimistic decision making. To tackle those situations, some more general orderingdefinitions are suggested by Sahoo *et. al.*(2012) which are as follows:

Definition 4.12 For any pair of intervals A and B, the order relations for maximizationproblems denoted by $>_{max}$ is defined as

(i) $A >_{max} B \Leftrightarrow a_C > b_C$ for Type - I and Type - II intervals

(ii) $A >_{max} B \Leftrightarrow$ either $a_C \geq b_C \wedge a_W < b_W$ or $a_C \geq b_C \wedge a_L > b_L$ for Type - III intervals.

Definition 4.13 For the same pair of intervals A and B, the order relation forminimization problems denoted by $<_{min}$ is defined as

(i) $A <_{min} B \Leftrightarrow a_C < b_C$ for Type - I and Type - II intervals

(ii) $A <_{min} B \Leftrightarrow$ either $a_C \leq b_C \wedge a_W < b_W$ or $a_C \leq b_C \wedge a_L < b_L$ for Type - III intervals.

Clearly, the order relations defined above are all partially ordered.

5. Statement of the Problem and Solution Procedure

Let us consider a constrained optimization problem with fractional type of objective:

$$\text{Optimize} \quad f(x),$$
$$\text{subject to} \; g_k(x) \leq 0, \quad k = 1, 2, \ldots, p;$$
$$h_k(x) = 0, \quad k = 1, 2, \ldots, q;$$
$$l \leq x \leq u, \qquad \ldots \qquad (5.1)$$

where $f(x)$ be the objective-function of the form described in the definition of FPP **(P1)**,$x = (x_1, x_2, \ldots, x_n)$, $l = (l_1, l_2, \ldots, l_n)$, $u = (u_1, u_2, \ldots, u_n)$, n, p and q represent the number of decision variables, the number of inequality constraints and the number of equality constraints respectively. The decision variable $x_j (j=1, 2, \ldots, n)$ lies in the prescribed interval $[l_j, u_j]$. Hence, the search region of the above problem is as follows:

$$\mathbf{D} = \{x \in \mathbf{R}^n : l_j \leq x_j \leq u_j \,,\, j = 1, 2, \ldots, n\}.$$

Solution procedure

The search region of the problem (5.1) is as follows:$\mathbf{D} = \{x : l_j \leq x_j \leq u_j, j = 1, 2 \ldots n\}$.
Now, our objective is to split the accepted region (for the first time, either it is the prescribed search region or assumed if it is not prescribed) into a number of distinct equal subregions $R_1, R_2, \ldots, R_\lambda (\lambda = m^n$, with each direction of the variable x_j of the hyper rectangular search region is divided into m sections simultaneously).

First of all, we check whether in each subregion the given set of constraints $g_i(x) \leq 0$ (or $= 0$) are satisfied or not. If they are satisfied then the corresponding subregion is called feasible and the value of the objective function will be calculated. Otherwise, the subregion will be discarded. Let us explain the constraint satisfaction (here the constraints are non-interval-valued) rule by interval method.

Constraint satisfaction rules

The concept of interval extension of real valued functions due to Moore (1979) has been used in these rules. Calculate the interval inclusion function $G_i(R_\gamma) = [\underline{g}_i, \bar{g}_i]$ of the real valued function $g_i(x)$ in each of the subregion R_γ ($\gamma = 1, 2, \ldots, \lambda$). An inequality constraint$g_i(x) \leq 0$ is satisfied if $\underline{g}_i \leq 0$ whereas an equality constraint$g_i(x) = 0$ is satisfied when $\underline{g}_i \leq 0$ and $\bar{g}_i \geq 0$. According to the fundamental theorem of interval analysis and the concept of interval extension and inclusion function theory, a constraint (inequality or equality) satisfying the conditions of the above constraint satisfaction rules in any subregion does not mean the necessarily satisfaction by all the real points of that particular subregion; however, the subregion is called feasible. The case is quite different for infeasible subregions because, in that case no real points of the infeasible subregion can be found by which the constraints are satisfied. This is quite compatible with our intension.

For illustration, let us consider an inequality constraint of the form $2x - 2y \geq 3$ or $-2x + 2y + 3 \leq 0$. Now we want to check whether this constraint is satisfied or not in the subregions created by dividing the box $\mathbf{D}^* = \{(x, y) : 0 \leq x \leq 5 \text{ and } 0 \leq y \leq 5\}$ for $m = 5$. This is shown in **Figure 1**. The region ABC is satisfied by the constraint in $\mathbf{D}^*$. According to the stated constraint satisfaction rule, the subregion will be feasible if any part of the satisfied region is included in that subregion. Here, the subregions marked by ✯ are not feasible and clearly the rest are feasible. Verifications:

$$R_1 = \{(x, y) : 1 \leq x \leq 2 \text{ and } 0 \leq y \leq 1\} \Rightarrow -2\,[1, 2] + 2\,[0, 1] + 3 = [-1, 3] \Rightarrow R_1 \text{ is feasible.}$$
$$R_2 = \{(x, y) : 4 \leq x \leq 5 \text{ and } 0 \leq y \leq 1\} \Rightarrow -2\,[4, 5] + 2\,[0, 1] + 3 = [-7, -5] \Rightarrow R_2 \text{ is feasible.}$$
$$R_3 = \{(x, y) : 3 \leq x \leq 4 \text{ and } 2 \leq y \leq 3\} \Rightarrow -2\,[3, 4] + 2\,[2, 3] + 3 = [-1, 3] \Rightarrow R_3 \text{ is feasible.}$$
$$R_4 = \{(x, y) : 0 \leq x \leq 1 \text{ and } 0 \leq y \leq 1\} \Rightarrow -2\,[0, 1] + 2\,[0, 1] + 3 = [1, 3] \Rightarrow R_4 \text{ is infeasible.}$$
$$R_5 = \{(x, y) : 0 \leq x \leq 1 \text{ and } 4 \leq y \leq 5\} \Rightarrow -2\,[0, 1] + 2\,[4, 5] + 3 = [9, 13] \Rightarrow R_5 \text{ is infeasible.}$$

Similarly, we can verify the rule for other subregions.

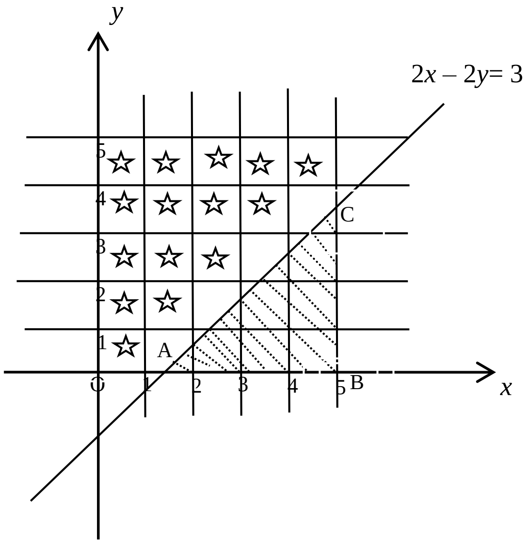

Figure 1. Linear inequality constraint satisfaction.

The rule is equally applicable for a nonlinear constraint, say, $g(x, y) \leq 0$. The curve of $g(x, y) = 0$ and the corresponding region satisfied by the constraint $g(x, y) \leq 0$ is shown in **Figure 2**. In this case, the subregions indicated by the symbol 'o' are feasible. For equality constraint, the feasible subregions are those through which the real curve passes. The equality constraint case is graphically shown in **Figure 3**.

Now in each of the feasible subregion, the interval inclusion function values of the fractional objective function have been calculated with the help of basic interval arithmetic operations. We know from Moore's (1979) discussion that the interval inclusion function F: $\mathbf{I}^n \rightarrow \mathbf{I}$ of $f(x)$ is a function having the property $f(x) \in F(X)$ whenever $X \in \mathbf{I}^n$. Let $F(R_\gamma) = \left[\underline{f}_\gamma, \overline{f}_\gamma\right]$ be the interval inclusion of the objective function $f(x)$ in the γ-th subregion, R_γ, where $\underline{f}_\gamma$ and $\overline{f}_\gamma$ denote the lower and upper bounds of $F(R_\gamma)$ in R_γ, computed by applying interval arithmetic. Now, comparing the objective function values calculated in the feasible subregions with the help of interval order relations, the subregion containing the best objective function value is accepted. Again, this accepted subregion is further subdivided into smaller disjoint subregions R_γ ($\gamma = 1, 2, \ldots, m^n$) by the aforesaid process. Then applying the same constrained satisfaction procedure and the acceptance criteria of subregion we obtain a further reduced subregion. This process is terminated after reaching the desired degree of accuracy and finally, we get the best value of the objective function in interval form and also the corresponding values of each decision variable in the form of closed interval with negligible width.

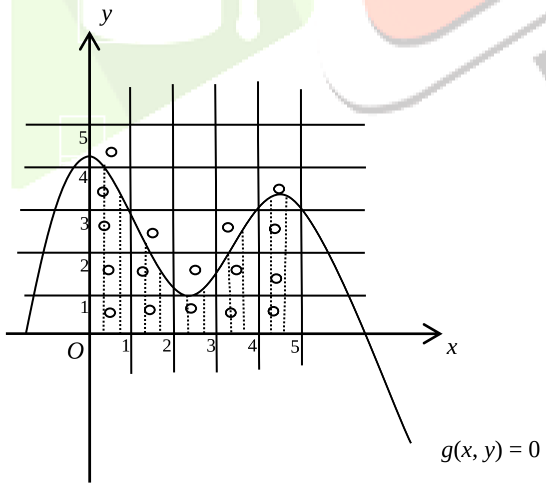

Figure 2. Nonlinear inequality constraint satisfaction.

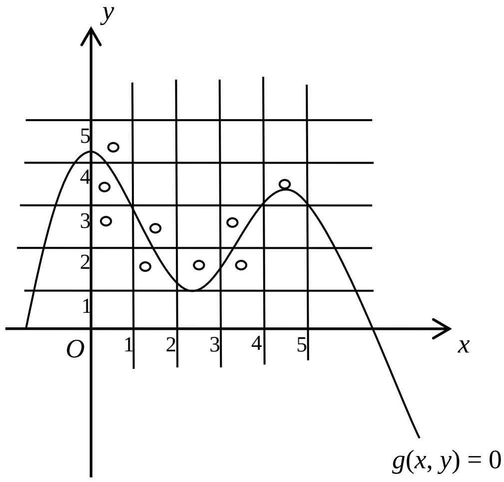

Figure 3. Nonlinear equality constraint satisfaction.

The above procedure is applied for multi-section division criterion of the accepted region developed recently by Karmakar*et al.*(2009). Since in our proposed technique no other vital information like the inclusion of the gradient or the inclusion of the Hessian regarding the interval inclusion function are available which are very important in some cases to fix the subdivision direction selection rule, so the multi-section technique is the best to use here. In this technique, all the directions of the decision variables are multi sectioned simultaneously. The idea of multi-section comes out from the concept of multiple-bisection, where several bisections are done at a single iteration cycle. For three-dimensional case, the accepted region is a rectangular parallelepiped (3D Box) that can be multi-sectioned into 23 = 8 (in case of triple bisection) sub-boxes. The pictorial representation is given in **Figure 4** for $m = 2$, $n = 3$.

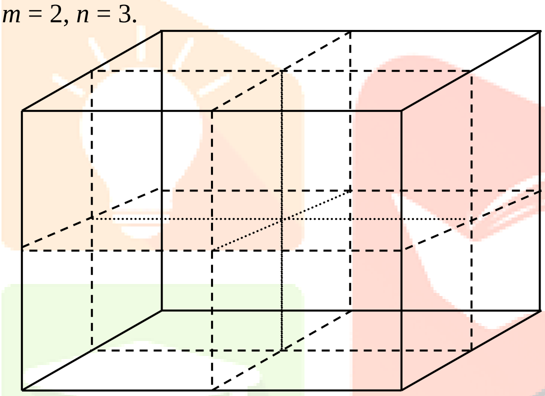

Figure 4. Multi-section method.

Table I: Computational results of FPP $\mathbf{F_1} – \mathbf{F_5}$

Test Problems	**Dimension (*n*)**	***m***	**Objective function value**	**No. of Function Evaluations**	**Computational time of CPU**
$\mathbf{F_1}$	3	5	[3.718639, 3.718639]	338	0.030
		10	[3.710924, 3.710924]	2544	0.080
$\mathbf{F_2}$	3	5	[2.862442, 2.862442]	66	0.005
		10	[2.861905, 2.861905]	52	0.008
$\mathbf{F_3}$	3	10	[3.000349, 3.000349]	15112	0.050
		20	[3.002924, 3.002924]	666	0.080
$\mathbf{F_4}$	3	5	[4.090703, 4.090703]	704	0.050
		10	[4.090703, 4.090703]	106	0.080
$\mathbf{F_5}$	3	5	[-1.900000, -1.900000]	84	0.007
		10	[-1.900000, -1.900000]	94	0.010

Table II: Computational results of FPP $\mathbf{F_6} – \mathbf{F_{10}}$

Test Problems	**Dimension (*n*)**	***m***	**Objective function value**	**No. of Function Evaluations**	**Computational time of CPU**
$\mathbf{F_6}$	2	5	[4.932127, 4.932127]	38	0.005
		10	[4.736084, 4.736084]	54	0.008
$\mathbf{F_7}$	2	10	[4.430933, 4.430933]	84	0.005
		100	[4.608277, 4.608277]	368	0.010
$\mathbf{F_8}$	2	5	[1.347222, 1.347222]	1098	0.005
		10	[1.347222, 1.347222]	2980	0.008
$\mathbf{F_9}$	2	5	[-147.666667, -147.666667]	3220	0.030

		10	[-147.666667, -147.666667]	17624	0.080
F_{10}	2	10	[0.818565, 0.818565]	14150	0.030
		20	[0.818565, 0.818565]	80886	0.170

Table III: Computational results of FPP **$F_{11} – F_{14}$**

Test Problems	**Dimension (*n*)**	***m***	**Objective function value**	**No. of Function Evaluations**	**Computational time of CPU**
F_{11}	2	10	[2.328547, 2.328547]	452	0.030
		20	[2.328407, 2.328407]	1282	0.080
F_{12}	2	10	[1.883333, 1.883333]	882	0.030
		20	[1.883333, 1.883333]	1310	0.085
F_{13}	3	5	[-0.534314, -0.534314]	2750	0.08
		10	[-0.534314, -0.534314]	15758	0.15
F_{14}	3	5	[-0.380556, -0.380556]	3000	0.095
		10	[-0.380556, -0.380556]	18000	0.22

Table IV: Summary of comparison of the computational results of FPP**$F_1 – F_9$**

T.P.	**Shen and Wang (2006)**	**Jiao *et al.* (2006)**	**Wang and Shen (2008)**	**Shen and Wang (2008)**	**ICCGO**
F_1	…	…	3.888365 (2.968694 s)	…	3.710924 (0.08 s)
F_2	…	…	3.002924 (8.566259 s)	…	2.861904 (0.008 s)
F_3	3.002924 (0.03 s)	…	…	3.002924 (0.01548 s)	3.002924 (0.08 s)
F_4	4.090700 (0 s)	…	…	…	4.090703 (0.08 s)
F_5	-1.900000 (0 s)	…	…	…	-1.900000 (0.010 s)
F_6	…	…	5.000000 (1.089285 s)	…	4.736084 (0.008 s)
F_7	3.291667 (0 s)	…	…	3.291667 (0.01651 s)	4.608277 (0.01 s)
F_8	…	1.347222 (< 2 s)	…	…	1.347222 (0.008 s)
F_9	…	-83.250249 (< 1 s)	…	…	-147.666667 (0.08 s)

The stepwise solution procedure is presented in the following algorithm:

Algorithm

Step-1: *Initialize m, the number of divided subregions in each direction and n, the number of decision variables.*
Step-2: *Initialize the lower and upper bounds l_j and u_j (j =1, 2,…, n) of all the decision variables. Set R^f = D.*
Step-3: *Divide the accepted region (initially it is the prescribed region of the problem or assumed region if it is not given) into m^n equal distinct subregions R_i (i=1, 2,…,m^n) such that* $\bigcup_{i=1}^{m^n} R_i = R^f$.
Step-4: *Calculate only the lower bounds* $\underline{g}_k$ *for all the constraints of the form $g_k(x) \le 0$ and both lower and upper bounds* $\underline{h}_k$ *and* $\overline{h}_k$ *respectively for all the equality constraints and check whether the constraint are satisfied or not. An inequality constraint $g_k(x) \le 0$ is satisfied if* $\underline{g}_k \le 0$ *whereas an equality constraint $h_k(x) = 0$ is satisfied when* $\underline{h}_k \le 0$ *and* $\overline{h}_k \ge 0$.

Step-5: *Applying finite interval arithmetic, compute an interval value* $F(R_i) = [\underline{f_i}, \overline{f_i}]$ *of the objective function in the feasible subregions R_i (i=1, 2,…,m^n).*

Step-6: *Select the feasible subregion R^f among R_i (i=1, 2,…,m^n) which has the best objective function value by comparing the interval valued numbers $F(R_i)$, i=1, 2,…, m^n to each other with the help of the pessimistic order relations between any two interval numbers.*

Step-7: *Compute the widths of R^f, $w_j = u_j – l_j$, j =1, 2,…, n.*

Step-8: *If* $w_j<\varepsilon$*, a pre-assigned very small positive number, for j =1, 2,…, n, go to* Step-9*; otherwise, go to* Step-3*.*

Step-9: *Print the values of the decision variables and the objective function in the form of closed intervals with negligible width.*

Step-10: *Stop.*

Table V: Summary of comparison of the computational results of FPP**F₁₀ – F₁₄**

T.P.	**Jiao *et al.* (2006)**	**Qu *et al.* (2007)**	**Shen and Wang (2008)**	**Jiao *et al.* (2008)**	**Shen *et al.* (2009)**	**Wang *et al.* (2010)**	**ICCGO**
$\mathbf{F_{10}}$	…	0.902000 (…)	…	…	0.833900 (…)	0.833333 (1.89713 s)	0.818565 (0.03 s)
$\mathbf{F_{11}}$	…	…	…	…	…	3.714240 (2.912117 s)	2.328407 (0.08 s)
$\mathbf{F_{12}}$	…	1.886829 (…)	…	…	1.883616 (…)	…	1.883333 (0.03 s)
$\mathbf{F_{13}}$	…	…	…	-0.534314 (6.0 s)	…	…	-0.534314 (0.15 s)
$\mathbf{F_{14}}$	…	…	…	-0.380556 (< 3.0 s)	…	…	-0.380556 (0.22 s)

6. Numerical Experiments and Comparative Discussions

To verify the performance of our intervalbranch and bound constraint handling technique ICCGO for a wide class of sum of ratios FPPs, a set of 14 test problems (T.P.) have been considered form the existing literature. This class of FPPs includes problems such as: linear and nonlinear sum of ratios problems, nonlinear product of ratios problems, polynomial ratios problems etc – over convex/nonconvex feasible region. The test problems are given in the **Appendix**. The problems $\mathbf{F_1}$ – $\mathbf{F_9}$, $\mathbf{F_{13}}$ and $\mathbf{F_{14}}$ are with linear sum of ratios objectives. The objective functions of the problems $\mathbf{F_{10}}$ **–** $\mathbf{F_{12}}$ are nonlinear sum of ratios.For the problems $\mathbf{F_{13}}$ and $\mathbf{F_{14}}$, the constraints are also in the form of linear sum of ratios. Each problem has been solved by the proposed technique taking suitable values of m and error tolerance $\varepsilon = 10^{-6}$. **Tables** – **I** to **III** contain a summary of the execution results including suitable values of m, optimum objective function values, number of function evaluations and computational time (CPU time) in second. The algorithm is coded in C programming language and implemented on a PC with INTEL® CORE ™ 2 Duo CPU @ 2.00 GHz and 1 GB RAM in LINUX Operating System.

We have compared our computational results including optimal objective function values and computational times taken by the PC to solve each problem with the same of eight existing methods which have been reported from the literature. The comparative results are shown in **Table IV** and **V**, from whichit is clear that the proposed interval oriented constrained handling technique ICCGO gives better solutions. In addition, for some problems (*e.g.*, $\mathbf{F_6}$, $\mathbf{F_7}$, $\mathbf{F_9}$ and $\mathbf{F_{11}}$) our solutions are far better than the solutions obtained by the previous techniques. It is noteworthy that the elapsed time to solve each problem by ICCGO is significantly less than the computational time taken by the other existing methods.

7. Conclusions

In this paper, an interval-oriented branch and bound constraint-handling global optimization technique has been deployed to solve a variety of FPPs. It is significant that the used algorithm can effectively be applicable for wide class of FPPs. It uses interval arithmetic and the interval order relations with respect to the decision makers' point of view developed by Sahoo et. al. (2012). This technique does not require any derivative information of the objective function. It is also different from any stochastic method or any heuristic or meta-heuristic methods. This method has global exploration as the feasible search space is reduced to $\frac{1}{m^n}$ th times of prescribed/accepted search space in each iteration and finally, the accepted subregion tends to a point which is the solution of the problem. From the numerical experimental results, it is clear that in most of the cases, the used algorithm gives far better solution compared to the same obtained from the existing methods. Especially, the computational time is notably very less than the same for the others.

However, for higher dimensional problems, sometimes the multi-section algorithm may not work with its expected level of efficiency. Also, the theoretical proof of convergence of the algorithm is not provided due to the unavailability of complete interval ordering definition and some limitations of existing interval mathematics. In future, further developments of this algorithm can be done by overcoming this couple of drawbacks.

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Appendix

List of Test Problems:

F_1 :Minimize $Z(x)=\frac{4x_1+3x_2+3x_3+50}{3x_2+3x_3+50}+\frac{3x_1+4x_3+50}{4x_1+4x_2+5x_3+50}+\frac{x_1+2x_2+4x_3+50}{x_1+5x_2+5x_3+50}+\frac{x_1+2x_2+4x_3+50}{5x_2+4x_3+50}$

(Wang and Shen (2008))

Subject to $g_1(x) = 2x_1 + x_2 + 5x_3 \le 10$
$g_2(x) = x_1 + 6x_2 + 2x_3 \le 10$
$g_3(x) = 9x_1 + 7x_2 + 3x_3 \ge 10$
Search region: $x_i \in [0, 2]$, $i = 1, 2, 3$.

F_2: Minimize $Z(x)=\frac{3x_1+5x_2+3x_3+50}{3x_1+4x_2+5x_3+50}+\frac{3x_1+4x_2+50}{4x_1+3x_2+2x_3+50}+\frac{4x_1+2x_2+4x_3+50}{5x_1+4x_2+3x_3+50}$

(Wang and Shen (2008))

Subject to $g_1(x) = 2x_1 + x_2 + 5x_3 \le 10$
$g_2(x) = x_1 + 6x_2 + 2x_3 \le 10$
$g_3(x) = 9x_1 + 7x_2 + 3x_3 \ge 10$
Search region: $x_i \in [0, 2]$, $i = 1, 2, 3$.

F_3: Maximize $Z(x)=\frac{3x_1+5x_2+3x_3+50}{3x_1+4x_2+5x_3+50}+\frac{3x_1+4x_2+50}{4x_1+3x_2+2x_3+50}+\frac{4x_1+2x_2+4x_3+50}{5x_1+4x_2+3x_3+50}$

(Shen and Wang (2008))

Subject to $g_1(x) = 6x_1 + 3x_2 + 3x_3 \le 10$
$g_2(x) = 10x_1 + 3x_2 + 8x_3 \le 10$
Search region: $x_i \in [0, 2]$, $i = 1, 2, 3$.

F_4 :Maximize $Z(x)=\frac{4x_1+3x_2+3x_3+50}{3x_2+3x_3+50}+\frac{3x_1+4x_3+50}{4x_1+4x_2+5x_3+50}+\frac{x_1+2x_2+4x_3+50}{x_1+5x_2+5x_3+50}+\frac{x_1+2x_2+4x_3+50}{5x_2+4x_3+50}$

(Shen and Wang (2006))

Subject to $g_1(x) = 2x_1 + x_2 + 5x_3 \le 10$
$g_2(x) = x_1 + 6x_2 + 3x_3 \le 10$
$g_3(x) = 5x_1 + 9x_2 + 2x_3 \le 10$
$g_4(x) = 9x_1 + 7x_2 + 3x_3 \le 10$
Search region: $x_i \in [0, 10]$, $i = 1, 2, 3$.

F_5 :Maximize $Z(x)=\frac{3x_1+4x_2+50}{3x_1+5x_2+4x_3+50}-\frac{3x_1+5x_2+4x_3+50}{5x_1+5x_2+4x_3+50}-\frac{x_1+2x_2+4x_3+50}{5x_2+4x_3+50}-\frac{4x_1+3x_2+3x_3+50}{3x_2+3x_3+50}$

(Shen and Wang(2006))

Subject to $g_1(x) = 6x_1 + 3x_2 + 3x_3 \le 10$
$g_2(x) = 10x_1 + 3x_2 + 8x_3 \le 10$
Search region: $x_i \in [0, 10]$, $i = 1, 2, 3$.

F_6: Minimize $Z(x)=\frac{37x_1+73x_2+13}{13x_1+13x_2+13}+\frac{63x_1-18x_2+39}{13x_1+26x_2+13}$ (Wang and Shen, 2006)

Subject to $g_1(x) = 5x_1 - 3x_2 = 3$
Search region: $x_i \in [0, 2]$, $i = 1, 2$.

F_7 : Maximize $Z(x)=\frac{37x_1+73x_2+13}{13x_1+13x_2+13}-\frac{63x_1-18x_2+39}{13x_1+26x_2+13}+\frac{13x_1+13x_2+13}{63x_1-18x_2+39}-\frac{13x_1+26x_2+13}{37x_1+73x_2+13}$

(Shen and Wang [35])

Subject to $g_1(x) = 5x_1 - 3x_2 = 3$
Search region: $x_i \in [0, 100]$, $i = 1, 2$.

F_8: Minimize $Z(x)=\frac{x_1+x_2+1}{x_1+x_2+2}+\frac{x_1+x_2+3}{x_1+x_2+4}+\frac{x_1+x_2+5}{x_1+x_2+6}+\frac{x_1+x_2+8}{x_1+x_2+9}$

(Jiao *et al.*(2006))

Subject to $g_1(x) = x_1x_2^2 + x_2x_1^2 \le 10$
Search region: $x_i \in [1, 2]$, $i = 1, 2$.

F_9: Minimize $Z(x)=0.5\frac{x_1}{x_2}-x_1-\frac{5}{x_2}$ (Jiao *et al.*(2006))

Subject to $g_1(x)=0.01\frac{x_2}{x_3}+0.01x_2+0.0005x_1x_2\le 1$

Search region: $x_1\in[70, 150]$, $x_2\in[1, 30]$, $x_3\in[0.5, 21]$.

F₁₀: Minimize $Z(x)=\dfrac{x_1^2+x_2^2+2x_1x_2}{x_3^2+5x_1x_2}+\dfrac{x_1+1}{x_1^2-2x_1+x_2^2-8x_2+20}$ (Wang *et al.*(2008))

Subject to $g_1(x)=x_1^2+x_2^2+x_3\le 5$

$g_2(x)=(x_1-2)^2+x_2^2+x_1^2\le 5$

Search region: $x_1\in[1, 3]$, $x_2\in[1, 3]$, $x_3\in[1, 2]$.

F₁₁: Minimize $Z(x)=\dfrac{-x_1^2+3x_1-x_2^2-3x_2+3.5}{x_1+1}+\dfrac{x_2}{x_1^2+2x_1+x_2^2-8x_2+20}$

(Wang *et al.*(2008))

Subject to $g_1(x) = 2x_1 + x_2\le 6$

$g_2(x) = 3x_1 + x_2\le 8$

$g_3(x) = x_1 - x_2\le 1$

Search region: $x_i\in[0.1, 3]$, $i = 1, 2$.

F₁₂: Minimize $Z(x)=-\dfrac{x_1^2-3x_1+x_2^2-3x_2+10}{x_1+1}+\dfrac{x_2}{x_1^2+2x_1+x_2^2-8x_2+20}$

(Shen *et al.*(2009))

Subject to $g_1(x) = 2x_1 + x_2^2 - 6x_2\le 0$

$g_2(x) = 3x_1 + x_2\le 8$

$g_3(x) = x_1^2 - x_1 - x_2\le 0$

Search region: $x_i\in[1, 3]$, $i = 1, 2$.

F₁₃: Minimize $Z(x)=\dfrac{2x_1+x_2+x_3+1}{x_1+2x_2+x_3+2}+\dfrac{x_1+2x_2+x_3+2}{2x_1+2x_2+x_3+3}-\dfrac{x_1+x_2+3x_3+5}{x_1+2x_2+3x_3+4}-\dfrac{1.5x_1+x_2+x_3+6}{x_1+x_2+1.5x_3+5}$

(Jiao *et al.*(2013))

Subject to $g_1(x)=\dfrac{2x_1+x_2+x_3+2}{2x_1+x_2+x_3+3}-\dfrac{2x_1+2x_2+x_3+5}{x_1+2x_2+x_3+4}-\dfrac{2x_1+3x_2+x_3+6}{x_1+2x_2+2x_3+5}-\dfrac{1.5x_1+x_2+2x_3+7}{1.5x_1+2x_2+x_3+6}\le -2.4$

$g_2(x)=\dfrac{1.5x_1+x_2+x_3+3}{x_1+1.5x_2+x_3+4}+\dfrac{2x_1+x_2+x_3+4}{x_1+x_2+2x_3+5}+\dfrac{x_1+2x_2+x_3+5}{x_1+2x_2+x_3+6}+\dfrac{x_1+x_2+3x_3+6}{x_1+3x_2+x_3+7}\le 3.8$

$g_3(x)=\dfrac{x_1+x_2+x_3+4}{x_1+x_2+x_3+5}+\dfrac{x_1+x_2+x_3+5}{x_1+x_2+2x_3+6}+\dfrac{x_1+x_2+3x_3+6}{x_1+x_2+x_3+7}+\dfrac{x_1+x_2+x_3+7}{x_1+x_2+x_3+8}\le 3.9$

$g_4(x)=\dfrac{x_1+x_2+x_3+5}{x_1+x_2+x_3+6}+\dfrac{x_1+x_2+x_3+6}{x_1+x_2+2x_3+7}-\dfrac{x_1+x_2+x_3+9}{x_1+x_2+x_3+8}-\dfrac{x_1+x_2+x_3+10}{x_1+x_2+x_3+9}\le 0.1$

Search region: $x_i\in[1, 3]$, $i = 1, 2, 3$.

F₁₄: Minimize $Z(x)=\dfrac{x_1+x_2+x_3+1}{x_1+x_2+x_3+2}+\dfrac{x_1+x_2+x_3+2}{x_1+x_2+x_3+3}-\dfrac{x_1+x_2+x_3+5}{x_1+2x_2+2x_3+4}-\dfrac{x_1+x_2+x_3+6}{x_1+x_2+x_3+5}$

(Jiao *et al.*(2013))

Subject to $g_1(x)=\dfrac{x_1+x_2+x_3+2}{x_1+x_2+x_3+3}-\dfrac{x_1+x_2+x_3+5}{x_1+x_2+x_3+4}-\dfrac{x_1+x_2+x_3+6}{x_1+x_2+x_3+5}-\dfrac{x_1+x_2+x_3+7}{x_1+x_2+x_3+6}\le -2$

$g_2(x)=\dfrac{x_1+x_2+x_3+3}{x_1+x_2+x_3+4}+\dfrac{x_1+x_2+x_3+4}{x_1+x_2+x_3+5}+\dfrac{x_1+x_2+x_3+5}{x_1+x_2+x_3+6}+\dfrac{x_1+x_2+x_3+6}{x_1+x_2+x_3+7}\le 3.59$

$g_3(x)=\dfrac{x_1+x_2+x_3+4}{x_1+x_2+x_3+5}+\dfrac{x_1+x_2+x_3+5}{x_1+x_2+x_3+6}+\dfrac{x_1+x_2+x_3+6}{x_1+x_2+x_3+7}+\dfrac{x_1+x_2+x_3+7}{x_1+x_2+x_3+8}\le 3.6$

$g_4(x)=\dfrac{x_1+x_2+x_3+5}{x_1+x_2+x_3+6}+\dfrac{x_1+x_2+x_3+6}{x_1+x_2+x_3+7}+\dfrac{x_1+x_2+x_3+7}{x_1+x_2+x_3+8}+\dfrac{x_1+x_2+x_3+8}{x_1+x_2+x_3+9}\le 3.7$

Search region: $x_i\in[1, 3]$, $i = 1, 2, 3$.