IJCRT.ORG ISSN : 2320-2882

INTERNATIONAL JOURNAL OF CREATIVE RESEARCH THOUGHTS (IJCRT)
An International Open Access, Peer-reviewed, Refereed Journal

Some Identities of the Generalized Lucas Numbers

[1]Rima P. Patel, [2]Khushbu J. Das, [3]Devbhadra V. Shah

[1]Lecturer, [2]Assistant Professor, [3]Associate Professor

[1]H & S S Department, Bhagwan Mahavir Polytechnic, Surat, India

[2]H & S S Department, Shree Swami Atmanand Saraswati Institute of Technology, Surat, India

[3]Department of Mathematics, Veer Narmad South Gujarat University, Surat, India

Abstract: In this article, we consider a Generalized Lucas sequence $\{GL_n\}$ defined by the recurrence relation $GL_n = aL_{n-2} + bL_{n-1}$ *; for all* $n \geq 2$*;* where $L_0 = 2, L_1 = 1$. We derive its recursive formula using simple explicit form of GL_n. We also arise some interesting identities for this sequence.

Keywords **- Lucas sequence, Fibonacci sequence, generalized Lucas sequence, Binet formula**

I. INTRODUCTION

The well-known Fibonacci sequence and the Lucas sequence are the two outstanding leads in the huge range of integer sequences. Both these sequences are famous for having great and wonderful properties and have been studied over several years. In the theory of numbers, Fibonacci sequence has always productive the ground for mathematicians. Both these sequences have been generalized in several different ways.

In Recent years, various papers have been published related to different types of generalizations of Fibonacci sequence. One can refer [1,2,4,6,7] and the Fibonacci sequence is a source of many identities as appears in research works. One can refer [3,8].

The Fibonacci sequence $\{F_n\}$, named after Leonardo Pisano Fibonacci (1170−1250), is defined recursively by the relation $F_n = F_{n-1} + F_{n-2}$, for $n \geq 2$, where $F_0 = 0$, $F_1 = 1$. This gives the sequence 0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144 … Also the sequence of Lucas numbers $\{L_n\}$ is defined by $L_n = L_{n-1} + L_{n-2}$, for all $n \geq 2$ with initial conditions $L_0 = 2$ and $L_1 = 1$. The Binet formula for Fibonacci sequence and Lucas sequence are respectively given by $F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} = \frac{1}{\sqrt{5}}\left\{\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right\}$ and $L_n = \alpha^n + \beta^n = \left\{\left(\frac{1+\sqrt{5}}{2}\right)^n + \left(\frac{1-\sqrt{5}}{2}\right)^n\right\}$, where $\alpha = \left(\frac{1+\sqrt{5}}{2}\right)$ is famously referred as '*golden ratio*'. one can refer [5,9,10]

Here we consider the *generalized Lucas sequence* $\{GL_n\}$ defined by the recurrence relation $GL_n = aL_{n-2} + bL_{n-1}$; for all $n \geq 2$; with $L_0 = 2, L_1 = 1$ and a, b are nonzero real numbers. First few terms of this generalized Lucas sequence $\{GL_n\}$ are:

$-a + 2b\,, 2a + b\,, a + 3b\,, 3a + 4b\,, \dots$.

II. A FEW SUMMATIONS FORMULA FOR THE NUMBERS OF GENERALIZED RECURSIVE SEQUENCE

We remark the following interesting pattern for the generalized Lucas numbers:

$GL_1 = -a + 2b = (L_1 - 2)a - (L_2 - 1)b$

$GL_1 + GL_2 = a + 3b = (L_2 - 2)a - (L_3 - 1)b$

$GL_1 + GL_2 + GL_3 = 2a + 6b = (L_3 - 2)a - (L_4 - 1)b$

$GL_1 + GL_2 + GL_3 + GL_4 = 5a + 10b = (L_4 - 2)a - (L_5 - 1)b.$

From this pattern, we conclude that $\sum_{i=1}^{n} GL_n = (L_n - 2)a - (L_{n+1} - 1)b$. Now, we use the principle of Mathematical Induction (PMI) for verifying this result.

Lemma 2.1: $\sum_{i=1}^{n} GL_n = (L_n - 2)a - (L_{n+1} - 1)b$.

Proof: We use PMI to prove this result. For $n = 1$ it is clear that

$-a + 2b = GL_1 = (L_1 - 2)a - (L_2 - 1)b = (1 - 2)a - (3 - 1)b$.

We assume that the result holds for some positive integer not exceeding k and we show that it also holds for $n = k + 1$. Now,

$$\begin{aligned}\sum_{i=1}^{k+1} GL_i &= \sum_{i=1}^{k} GL_i + GL_{k+1} = (L_k - 2)a - (L_{k+1} - 1)b + aL_{k-1} + bL_k \\ &= a(L_k + L_{k-1}) + b(L_{k+1} + L_k) - 2a - b \\ &= aL_{k+1} + bL_{k+2} - 2a - b.\end{aligned}$$

Thus, $\sum_{i=1}^{k+1} GL_i = (L_{k+1} - 2)a - (L_{k+2} - 1)b$, which proves the result by PMI.

We now derive a formula for the sum of first n terms of sequence $\{GL_n\}$ with odd subscripts, by two different techniques.

Lemma 2.2: $\sum_{i=1}^{n} GL_{2i-1} = GL_{2n} - GL_0 = 3a + b$.

Proof: [First Method] By the definition of generalized Lucas recurrence numbers, we have

$GL_1 = GL_2 - GL_0$
$GL_3 = GL_4 - GL_2$
$GL_5 = GL_6 - GL_4$
$\vdots$
$GL_{2n-3} = GL_{2n-2} - GL_{2n-4}$
$GL_{2n-1} = GL_{2n} - GL_{2n-2}$
Adding all these results we get $\sum_{i=1}^{n} GL_{2i-1} = GL_{2n} - GL_0$.

Now we prove the same result by PMI.

Lemma 2.2: $\sum_{i=1}^{n} GL_{2i-1} = GL_{2n} - GL_0 = 3a + b$.

Proof: [Second method] For $n = 1$ it is clear that

$$-a + 2b = GL_1 = GL_2 - GL_0 = 2a + b - (3a - b).$$

We assume that the result holds for some positive integer not exceeding k and we show that it also holds for $n = k + 1$. Now,

$$\sum_{i=1}^{k+1} GL_{2i-1} = \sum_{i=1}^{k} 2i - 1 + GL_{2(k+1)-1} = GL_{2k} - GL_0 + GL_{2k+1} = GL_{2k+2} - GL_0.$$

Thus $\sum_{i=1}^{k+1} GL_{2i-1} = GL_{2(k+1)} - GL_0$, which proves the result for every positive integers n by PMI.

We next obtain similar result for the even subscripts.

Lemma 2.3: $\sum_{i=1}^{n} GL_{2i} = GL_{2n+1} - GL_1$, $GL_1 = -a + 2b$.

Proof: We use PMI to prove the result.
For $n = 1$ it is clear that $2a + b = GL_2 = GL_3 - GL_1 = (a + 3b) - (-a + 2b)$.
Assume that the result holds for some positive integer not exceeding k and we show that it also holds for$n = k + 1$. Now,

$$\sum_{i=1}^{k+1} GL_{2i} = \sum_{i=1}^{k} GL_{2i} + GL_{2(k+1)} = GL_{2k+1} - GL_1 + GL_{2k+1} = GL_{2k+3} - GL_1 .$$

Thus $\sum_{i=1}^{k+1} GL_{2i} = GL_{2(k+1)+1} - GL_1$, which proves the result by PMI.

Next we prove an important result for GL_n.

Lemma 2.4: $\sum_{i=1}^{n} (GL_i)^2 = GL_n GL_{n+1} + (3a^2 - 7ab + 2b^2)$.

Proof: We prove this result by the principle of mathematical induction. For $n = 1$ it is clear that

$$(-a + 2b)^2 = (GL_1)^2 = GL_1 GL_2 + (3a^2 - 7ab + 2b^2)$$
$$= (-a + 2b)(2a + b) + (3a^2 - 7ab + 2b^2).$$

Now let us assume that the result holds for some positive integer not exceeding k and we show that it also holds for $n = k + 1$. Now,

$$\sum_{i=1}^{k+1} (GL_i)^2 = \sum_{i=1}^{k} (GL_i)^2 + (GL_{k+1})^2$$
$$= GL_k GL_{k+1} + (3a^2 - 7ab + 2b^2) + (GL_{k+1})^2$$
$$= GL_{k+1}(GL_k + GL_{k+1}) + (3a^2 - 7ab + 2b^2)$$
$$\therefore \sum_{i=1}^{k+1} (GL_i)^2 = GL_{k+1} GL_{k+2} + (3a^2 - 7ab + 2b^2).$$

Thus, by PMI this result is true for every positive integer n.

The following interesting result follows easily from this lemma.

Corollary 2.5:$(GL_n)^2 = GL_n GL_{n+1} - GL_{n-1} GL_n$; for every positive integer n.

Proof: Using the lemma 2.5, we have
$GL_n GL_{n+1} = (GL_n)^2 + (GL_{n-1})^2 + \cdots + (GL_1)^2 - (3a^2 - 7ab + 2b^2)$ and
$GL_{n-1} GL_n = (GL_{n-1})^2 + (GL_{n-2})^2 + \cdots + (GL_1)^2 - (3a^2 - 7ab + 2b^2)$.
On subtraction, we get $(GL_n)^2 = GL_n GL_{n+1} - GL_{n-1} GL_n$; $n \geq 1$, as required.

III. EXTENDED BINET'S FORMULA FOR THE GENERALIZED LUCAS NUMBERS

Almost all of these properties can be derived from Binet's formula. A main objective of this paper is to prove that many of the properties of the Fibonacci sequence can be stated and verified for a much larger class of sequences, namely the generalized Fibonacci sequence. We now develop extended Binet's formula for the generalized Lucas numbers.

Theorem 3.1: [*Extended Binet's formula*]
The terms of the generalized Lucas sequence $\{GL_n\}$ are given by $GL_n = c\alpha^n + d\beta^n$; where $c = a + (a - b)\beta$, $d = a + (a - b)\alpha$ and $\alpha = \left(\frac{1+\sqrt{5}}{2}\right), \beta = \left(\frac{1-\sqrt{5}}{2}\right)$.

Proof: We have $GL_n = aL_{n-2} + bL_{n-1}$

$$= a(\alpha^{n-2} + \beta^{n-2}) + b(\alpha^{n-1} + \beta^{n-1})$$
$$= \alpha^n \left(\frac{a}{\alpha^2} + \frac{b}{\alpha}\right) + \beta^n \left(\frac{a}{\beta^2} + \frac{b}{\beta}\right)$$
$$= \alpha^n \{a\beta^2 + b(\alpha\beta)\beta\} + \beta^n \{a\alpha^2 + b(\alpha\beta)\alpha\}.$$

Here it can be observed that $\alpha\beta = -1, \alpha^2 = 1 + \alpha$ and $\beta^2 = 1 + \beta$. Thus we have

$$GL_n = \alpha^n \{a(1 + \beta) - b\beta\} + \beta^n \{a(1 + \alpha) - b\alpha\}$$
$$= \alpha^n \{a + (a - b)\beta\} + \beta^n \{a + (a - b)\alpha\}.$$

Hence, $GL_n = c\alpha^n + d\beta^n$, when $c = a + (a - b)\beta$ and $d = a + (a - b)\alpha$.

Remark: $cd = \{a + (a - b)\beta\}\{a + (a - b)\alpha\}$

$$= a^2 + (a - b)^2 \alpha\beta + a(a - b)\alpha + a(a - b)\beta$$
$$= a^2 + (a - b)^2(-1) + a(a - b)(\alpha + \beta)$$
$$= a^2 + ab - b^2.$$

This constant occurs in many of the formulas for generalized Lucas numbers. We call it the *characteristic* of the generalized Lucas sequence. We write $\mu = cd = a^2 + ab - b^2$.

We now use this extended Binet's formula to prove some interesting identities for this sequence. In the following theorem, we derive the *extended Cassini's identity* for the generalized Lucas numbers which connects three consecutives GL_n's together.

Lemma 3.2: [*Extended Cassini's identity*]
$GL_{n+1}GL_{n-1} - (GL_n)^2 = 5\mu(-1)^{n-1}$.
Proof: We have

$$\begin{aligned} GL_{n+1}GL_{n-1} - (GL_n)^2 &= (c\alpha^{n+1} + d\beta^{n+1})(c\alpha^{n-1} + d\beta^{n-1}) - (c\alpha^n + d\beta^n)^2 \\ &= cd(\alpha^{n+1}\beta^{n-1} + \alpha^{n-1}\beta^{n+1}) - 2cd(-1)^n \\ &= \mu(\alpha\beta)^{n-1}(\alpha^2 + \beta^2) - 2\mu(-1)^n \quad . \end{aligned}$$

Since $L_n = \alpha^n + \beta^n$ and $L_2 = 3$, we have

$$GL_{n+1}GL_{n-1} - (GL_n)^2 = \mu(-1)^{n-1}(3) - 2\mu(-1)^n = -3\mu(-1)^n - 2\mu(-1)^n.$$

Hence, $GL_{n+1}GL_{n-1} - (GL_n)^2 = -5\mu(-1)^n = 5\mu(-1)^{n-1}$, as required.

We next prove more generalized form of extended Catalan's identity which connects three consecutives GL_n's with suffixes in arithmetic progression for fixed n.

Lemma 3.3: [*Extended Catalan's identity*]
$(GL_n)^2 - GL_{n+r}GL_{n-r} = cd(-1)^n(2 + L_{2r})$.
Proof: Using extended Binet's theorem for generalized Lucas numbers, we have

$$\begin{aligned} &(GL_n)^2 - GL_{n+r}GL_{n-r} \\ &= (c\alpha^n + d\beta^n)^2 - (c\alpha^{n+r} + d\beta^{n+r})(c\alpha^{n-r} + d\beta^{n-r}) \\ &= c^2\alpha^{2n} + 2cd(\alpha\beta)^n + d^2\beta^{2n} - c^2\alpha^{2n} - cd\alpha^{n+r}\beta^{n-r} - cd\alpha^{n-r}\beta^{n+r} - d^2\beta^{2n} \\ &= 2cd(\alpha\beta)^n - cd(\alpha\beta)^n\alpha^r(-\alpha)^r - cd(\alpha\beta)^n\beta^r(-\beta)^r \\ &= 2cd(-1)^n - cd(-1)^n\alpha^r(-\alpha)^r - cd(-1)^n\beta^r(-\beta)^r \\ &= cd(-1)^n(2 + \alpha^{2r} + \beta^{2r}). \end{aligned}$$

Hence, $(GL_n)^2 - GL_{n+r}GL_{n-r} = cd(-1)^n(2 + L_{2r})$.

We now express GL_n explicitly in terms of powers of α.

Theorem 3.4: $GL_n = d\left[\frac{c}{d}\alpha^n\right]$, where [x] represents the integer part of *x*.

Proof: Using the extended Binet's formula for GL_n we have $GL_n = c\alpha^n + d\beta^n$, where $c = a + (a-b)\beta$ and $d = a + (a-b)\alpha$. This gives $\frac{GL_n - c\alpha^n}{d} = \beta^n$. Then,

$$\left|\frac{GL_n - c\alpha^n}{d}\right| = |\beta^n| = |\beta|^n = \left|\frac{1-\sqrt{5}}{2}\right|^n = (0.618)^n < 1, \text{ as } n \to \infty.$$

This gives $GL_n = d\left[\frac{c}{d}\alpha^n\right]$, as required.

The next result shows how extended Binet's formula is used to express GL_{n+1} in terms of GL_n.

Lemma 3.5: $GL_{n+1} = \alpha GL_n - d\sqrt{5}\beta^n$, where $c = a + (a-b)\beta$ and $d = a + (a-b)\alpha$.
Proof: We have $GL_n = c\alpha^n + d\beta^n$. Then,

$$\begin{aligned} \alpha GL_n &= c\alpha^{n+1} + d\alpha\beta^n \\ &= c\alpha^{n+1} + d(\alpha\beta)\beta^{n-1} \\ &= c\alpha^{n+1} - d\beta^{n-1} \\ &= (c\alpha^{n+1} + d\beta^{n+1}) - d(\beta^{n-1} + \beta^{n+1}) \\ &= GL_{n+1} - d\beta^{n-1}(\beta^2 + 1) \\ &= GL_{n+1} - d\beta^{n-1}(-\sqrt{5}\beta) \\ &= GL_n + 1 + d\sqrt{5}\beta^n. \end{aligned}$$

Thus, $GL_{n+1} = \alpha GL_n - d\sqrt{5}\beta^n$.

Following result follows immediately from this result by taking $n \to \infty$ and keeping in mind that $|\beta| < 1$.

Corollary 3.6: $GL_{n+1} = \alpha GL_n$.

We next show that limiting ratio of any two consecutive terms of this sequence converge to a fixed real number.

Lemma 3.7: $\lim_{n\to\infty} \frac{GL_{n+1}}{GL_n} = \alpha$.

Proof: We have $GL_n = d\left[\frac{c}{d}\alpha^n\right]$. Now we know that for any arbitrary real number x, we have $[x] = x - \theta$, where $0 < \theta < 1$. Then,

$$\lim_{n\to\infty} \frac{GL_{n+1}}{GL_n} = \lim_{n\to\infty} \frac{d\left[\frac{c}{d}\alpha^{n+1}\right]}{d\left[\frac{c}{d}\alpha^n\right]} = \lim_{n\to\infty} \frac{\frac{c}{d}\alpha^{n+1} - \theta_1}{\frac{c}{d}\alpha^n - \theta_2}.$$

Since $0 < \theta_1, \theta_2 < 1$, we have $\lim_{n\to\infty} \frac{GL_{n+1}}{GL_n} = \alpha$.

IV. Generalized Lucas sequence with negative subscripts

We now extend the generalized Lucas sequence backward with negative subscripts. In fact, if we try to extend the generalized Lucas sequence backwards still keeping to the rule that any generalized Lucas sequence is the sum of the two numbers on its left, we get the following:

n	GL_{-n}
$\vdots$	$\vdots$
-3	$-11a + 7b$
-2	$7a - 4b$
-1	$-4a + 3b$
0	$3a - b$
1	$-a + 2b$
2	$2a + b$
3	$a + 3b$
$\vdots$	$\vdots$

We can now consider GL_n being defined for all integer values of n, both positive and negative and the generalized Lucas sequence extending infinitely in both the positive and negative directions. We observe here that $GL_{-n} = (-1)^n(aL_{n+2} - bL_{n+1})$. We prove this result in the following theorem.

Theorem 4.1: $GL_{-n} = (-1)^n(aL_{n+2} - bL_{n+1})$.

Proof: We have $GL_n = c\alpha^n + d\beta^n$. Now considering $-n$ in place of n, we get $GL_{-n} = c\alpha^{-n} + d\beta^{-n}$. Since $\alpha\beta = -1$, we have

$$\begin{aligned} GL_{-n} &= c\alpha^{-n} + d\beta^{-n} \\ &= c(-\beta)^n + d(-\alpha)^n \\ &= (-1)^n(c\beta^n + d\alpha^n) \\ &= (-1)^n(\{a + (a-b)\beta\}\beta^n + \{a + (a-b)\alpha\}\alpha^n) \\ &= (-1)^n(a\beta^n + (a-b)\beta^{n+1} + a\alpha^n + (a-b)\alpha^{n+1}) \\ &= (-1)^n\big(a(\alpha^n + \beta^n) + (a-b)(\alpha^{n+1} + \beta^{n+1})\big) \\ &= (-1)^n(aL_n + (a-b)L_{n+1}) \\ &= (-1)^n(aL_n + aL_{n+1} - baL_{n+1}) \\ &= (-1)^n(a(L_n + L_{n+1}) - bL_{n+1}) \end{aligned}$$

Hence, $GL_{-n} = (-1)^n(aL_{n+2} - bL_{n+1})$.

We next obtain two beautiful results which show connection between the Fibonacci numbers, Lucas numbers and generalized Lucas numbers.

Theorem 4.2: $GL_{n+m} = F_{n+1}GL_m + F_nGL_{m-1} = F_mGL_{n+1} + F_{m-1}GL_n$.

Proof: We shall prove this result by induction over n keeping m fixed.

For $n = 1$, we have $GL_{m+1} = F_2GL_m + F_1GL_{m-1} = GL_m + GL_{m-1}$.This proves the result for $n = 1$, since $F_1 = F_2 = 1$.

Now assume that result holds for some positive integer $n = k$. Then by assumption, $GL_{k+m} = F_{k+1}GL_m + F_kGL_{m-1}$ holds.

Now

$$\begin{aligned} GL_{k+1+m} &= GL_{m+k+1} = GL_{m+k} + GL_{m+k-1} \\ &= F_{k+1}GL_m + F_kGL_{m-1} + F_kGL_m + F_{k-1}GL_{m-1} \\ &= (F_{k+1} + F_k)GL_m + (F_k + F_{k-1})GL_{m-1} \\ &= F_{k+2}GL_m + F_{k+1}GL_{m-1}, \end{aligned}$$

which is precisely our identity when $n = k + 1$.

Thus, result is true for every positive integer n.

The second part of the result follows immediately by interchanging m and n.

We use this result to prove the analogous of *d'Ocagne's identity*.

Theorem 4.3: [Extended *d'Ocagne's identity*]

$GL_{n-m} = (-1)^m[F_{m+1}GL_m - F_mGL_{n+1}]$.

Proof:

$$\begin{aligned} GL_{n-m} &= F_{-m-1}GL_m + F_{-m}GL_{n+1} \\ &= (-1)^{m+2}F_{m+1}GL_m + (-1)^{m+1}F_mGL_{n+1} \\ &= (-1)^{m+1}[F_mGL_{n+1} - F_{m+1}GL_m] \end{aligned}$$

$\therefore GL_{n-m} = (-1)^m[F_{m+1}GL_m - F_mGL_{n+1}]$.

The following result follows from the above two reduction formulae.

Corollary 4.4: $GL_{n+m} + (-1)^mGL_{n-m} = GL_nL_m$.

Proof: By the above two lemmas, we have

$GL_{n+m} = F_mGL_{n+1} + F_{m-1}GL_n$ and $GL_{n-m} = (-1)^m[F_{m+1}GL_n - F_mGL_{n+1}]$.

Thus

$$\begin{aligned} GL_{n+m} + (-1)^mGL_{n-m} &= [F_mGL_{n+1} + F_{m-1}GL_n] + [F_{m+1}GL_n - F_mGL_{n+1}] \\ &= GL_n[F_{m-1} + F_{m+1}]. \end{aligned}$$

Since $F_{m-1} + F_{m+1} = L_m$, the required result follows immediately.

REFERENCES

[1] Arvadia Mansukh P., Shah Devbhadra V.,2015. Left k-Fibonacci sequence and related identities, Journal Club for Applied Sciences, Vol – 2, Issue 1, 20 – 26.

[2] Arvadia Mansukh P., Shah Devbhadra V., 2015. Right k-Fibonacci sequence and related identities, International Research Journal of Mathematics, Engineering and IT, Vol. 2, Issue-4, 25 – 39.

[3] Carlitz: "Some Fibonacci and Lucas identities", The Fibonacci Quarterly, 8 (1970), 61-73.

[4] Bilgici G.: New generalizations of Fibonacci and Lucas Sequences, Applied Mathematical Sciences, 8 (29), 2014, 1429 – 1437.

[5] Burton David, 2015. Elementary Number Theory, Mc Graw Hill Education (India) Private Limited, New Delhi, Seventh Edition.

[6] Edson M., Yayenie O.: A new generalization of Fibonacci sequence and Extended Binet's formula, Integers, 9, 2009, 639 – 654.

[7] Horadam A. F., 1961. A Generalized Fibonacci sequence, American Mathematical Monthly, 68, 455 – 459.

[8] Horadam A. F., 1965. Basic properties of certain generalized sequence of numbers, Fibonacci Quarterly, 3, 161 – 176.

[9] Knott, R.: Fibonacci numbers and the Golden Section, http://www.mcs.surrey.ac.uk/Personal/R.Knott/Fibonacci/

[10] Koshy, Thomas., 2001. Fibonacci and Lucas Numbers with applications, Pure and Applied mathematics, Wiley-Inter Science, New York, NY, USA.