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The Global Dynamical Behavior Of Mathematical Model Of Two Type Fish Species With Variable Effort Rate In Coastal Aquaculture

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1. Abstract

The dynamics of two ecologically independent species which are being harvested with variable effort have been discussed. The dynamics of effort is considered separately. The local and global dynamics of the system is studied. The co-existence of species in the form of stable equilibrium point is possible.

Key Word: Mathematical Model, Ordinary differential equation, Different parameters, Positive solution nonexistence of periodic solution etc.

2. Introduction

Exploitation of biological resources as practiced in fishery, and forestry has strong impact on dynamic evolution of biological population. The over exploitation of resources may lead to extinction of species which adversely affects the ecosystem. However, reasonable and controlled harvesting is beneficial from economic and ecological point of view. The research on harvesting in predator-prey systems has been of interest to economists, ecologists and natural resource management for some time now.

The optimal management of renewable resources and their solution analytically has been extensively studied by many authors [1, 2, 3, 7, 8, 12-21]. The mathematical aspects of management of renewable resources have been discussed by [10]. He had investigated the optimum harvesting of logistically growing species. The problem of combined harvesting of two ecologically independent species has been studied [10, 13]. The effects of harvesting on the dynamics of interacting species have been studied Measterton-Gibbons [14], Chaudhuri et.al. [6-9] with constant harvesting, the prey predator model is found to have interesting dynamical behavior including stability, Hopf bifurcation and limit cycle [4, 5, 11, 15].

The multi species food web models have found to have rich dynamical behavior [16, 18]. S Kumar et. al. [17] have investigated the harvesting of predator species predating over two preys.

In this paper the dynamics of two ecologically independent species which are being harvested have been discussed when the dynamics of effort is considered separately.

3. The Mathematical Model

Consider two independent biological species with densities X_1 and X_2 with logistic growth. The Mathematical model of two harvesting prey species with effort rate is given by the following system of ordinary differential equations:

$$\frac{dX_1}{dT} = r_1X_1\left(1-\frac{X_1}{K_1}\right) - \frac{A_1q_1X_1E}{1+B_1X_1+B_2X_2} = X_1f_1(X_1, X_2, E)$$

$$\frac{dX_2}{dT} = r_2X_2\left(1-\frac{X_2}{K_2}\right) - \frac{A_2q_2EX_2}{1+B_1X_1+B_2X_2} = X_2f_2(X_1, X_2, E)$$

$$\frac{dE}{dT} = E(h(X_1, X_2) - C) = Ek\left(\frac{p_1q_1A_1X_1+p_2q_2A_2X_2}{1+B_1X_1+B_2X_2} - C\right) = Ef_3(X_1, X_2) \qquad (1)$$

The logistic growth is considered for the two preys. The model does not consider any direct competition between the two populations. The constants K_i, r_i, A_i, and B_i, are model parameters assuming only positive values. The effort E is applied to harvest both the species and C is total cost of fishing. The harvesting is proportional to the product of effort E and the fish population density X_i. The catch-ability coefficients q_i are assumed to be different for the two species. In the model, the third equation considers the dynamics of effort E. The constants p_1 and p_2 are the price of the per unit prey species. The last equation of (1) implies that the rate of increase of the effort is proportional to the rate of net economic revenue. The constant k is the proportionality constant.

Let the constant M_0 is the reference value of E. Introduce the following dimensionless transformations:

$$t = r_1T, y_i = \frac{X_i}{K_i}(i = 1,2), x = \frac{E}{M_0}, w_1 = \frac{A_1q_1E_0}{r_1}, w_2 = B_1K_1, w_3 = B_2K_2$$

$$w_4 = \frac{r_2}{r_1}, w_5 = \frac{A_2q_2M_0}{r_1}, w_6 = \frac{kK_1}{M_0}, w_7 = \frac{kK_2}{M_0};$$

The dimensionless nonlinear system is obtained as:

$$\frac{dy_1}{dt} = y_1\left(1 - y_1 - \frac{w_1x}{1+w_2y_1+w_3y_2}\right) = y_1f_1(y_1, y_2, x).$$

$$\frac{dy_2}{dt} = y_2\left((1-y_2)w_4 - \frac{w_5x}{1+w_2y_1+w_3y_2}\right) = y_2f_2(y_1, y_2, x)$$

$$\frac{dx}{dt} = x\left(\frac{p_1w_1w_6y_1+p_2w_5w_7y_2}{1+w_2y_1+w_3y_2} - C\right) = xf_3(y_1, y_2) \qquad (2)$$

Theorem 3.1(Positivity of the Solution of the Mathematical Model): The solution (y_1, y_2, x) is positive for all t greater than and equal to zero.

Proof: We have from the mathematical model equations

$$\frac{dy_1}{dt} \geq -\frac{w_1xy_1}{1+w_2y_1+w_3y_2} \Rightarrow \frac{dy_1}{y_1} \geq -Mdt \Longrightarrow y_1(t) \geq 0. Where\ max\left(\frac{w_1x}{1+w_2y_1+w_3y_2}\right) = M$$

$$\frac{dy_2}{dt} \geq -\frac{w_5x\,y_2}{1+w_2y_1+w_3y_2} \Longrightarrow \frac{dy_2}{y_2} \geq -Ndt \Rightarrow y_2(t) \geq 0. \quad where \max\left(\frac{w_5x\,y_2}{1+w_2y_1+w_3y_2}\right) = N$$

$$\frac{dx}{dt} \geq -Cx \Rightarrow x(t) \geq 0$$

Therefore, the solution (y_1, y_2, x) is positive for all t greater than and equal to zero.

4. Existence of Equilibrium Points

Since $0 \le y_i \le 1;\ i = 1,2$, the underlying non-linear model (2) is bounded and has a unique solution. There are at most seven possible equilibrium points of the nonlinear harvesting model:

$$E_0 = (0,0,0),\ E_1 = (1,0,0),\ E_2 = (0,1,0),\ E_3 = (1,1,0),$$

$$E_4 = (y_1^*, 0, x^*),\quad x^* = \frac{(1-y_1^*)(1+w_2 y_1^*)}{w_1};\quad y_1^* = \frac{C}{(w_1 p_1 w_6 - C w_2)}$$

$$E_5 = (0, y_2^*, x^*),\quad x^* = \frac{(w_4(1-y_2^*)(1+w_3 y_2^*))}{w_5};\quad y_2^* = \frac{C}{(w_5 p_2 w_7 - C w_3),}$$

$$E_6 = (y_1^*, y_2^*, x^*).$$

Theorem 4.1 The equilibrium point $E_4 = (y_1^*, 0, x^*)$ is feasible only when

$$C < \frac{w_1 p_1 w_6}{(1+w_2)} \qquad (3)$$

Theorem 4.2 The equilibrium point $E_5 = (0, y_2^*, x^*)$ is feasible only when

$$C < \frac{w_5 p_2 w_7}{(1+w_3)} \qquad (4)$$

The proofs of the two theorems are straightforward as $0 < y_i^* < 1;\ i = 1,2$.

Theorem 4.3 The positive non-zero equilibrium E_6 of nonlinear harvesting model (2) exists provided the following conditions are satisfied:

$$C < \frac{w_1 p_1 w_6}{w_2};\quad C < \frac{w_5 p_2 w_7}{w_3} \qquad (5)$$

Proof: The isoclines of harvesting model (2) are given by

$$f_1(y_1, y_2, x) = 0;\ f_2(y_1, y_2, x) = 0;\ f_1(y_1, y_2) = 0 \qquad (6)$$

Using first of (6) we get $x^* = \frac{(1-y_1^*)(1+w_2 y_1^* + w_3 y_2^*)}{w_1}$

Solving first and second of (6) we get

$$(p_1 w_1 w_6 - w_2 C) y_1^* + (p_2 w_5 w_7 - w_3 C) y_2^* = C \qquad (7)$$

Using third of (6) we get

$$y_1^* - \left(\frac{w_4 w_1}{w_5}\right) y_2^* = \frac{(w_5 - w_4 w_1)}{w_5} \qquad (8)$$

Now solving (7) and (8) we get

$$y_1^* = \frac{(w_5 - w_1 w_4)(w_7 p_2 w_5 - C w_3) + C w_1 w_4}{w_5 (w_7 p_2 w_5 - C w_3) + w_1 w_4 (w_1 p_1 w_6 - C w_2)}$$

$$y_2^* = \frac{C w_5 - (w_5 - w_1 w_4)(w_1 p_1 w_6 - C w_2)}{w_5 (w_7 p_2 w_5 - C w_3) + w_1 w_4 (w_1 p_1 w_6 - C w_2)}$$

The positive non-zero biological equilibrium $E_6 = (y_1^*, y_2^*, x^*)$ exists provided the conditions (5) are satisfied.

It may further be observed that conditions (3) and (4) imply (5), that is if $E_4 = (y_1^*, 0, x^*)$ and $E_5 = (0, y_2^*, x^*)$ exists then E_6 will also exists. However, the existence of E_6 is possible irrespective of E_4 and E_5 provided the condition (5) is satisfied.

Theorem 4.4 The flow of nonlinear harvesting model (2) contracts volume uniformly for positive non-zero equilibrium E_6 provided the following condition is satisfied [15]:

$$x^* < \frac{(y_1^*+w_4y_2^*)(1+w_2y_1^*+w_3y_2^*)^2}{w_1w_2+w_3w_4}$$

Proof: Because the divergence of the vector field for the positive non-zero equilibrium E_6 is

$$\frac{\partial}{\partial y_1}\frac{dy_1}{dt}+\frac{\partial}{\partial y_2}\frac{dy_2}{dt}+\frac{\partial}{\partial x}\frac{dx}{dt}=y_1\left(-1+\frac{w_1xw_2}{(1+w_2y_1+w_3y_2)^2}\right)+y_2\left(-w_4+\frac{w_3xw_5}{(1+w_2y_1+w_3y_2)^2}\right)<0$$

When

$$x^* < \frac{(y_1^*+w_4y_2^*)(1+w_2y_1^*+w_3y_2^*)^2}{w_1w_2+w_3w_4}$$

Hence the result.

Theorem 4.5: Periodic Solution does not exist at non zero equilibrium point.

Proof: Let $n = y_1i + y_2j + xk$ *and* $F = F_1i + F_2j + F_3k$ be the vectors. Let $F_1 = y_1f_1(y_1, y_2, x)$, $F_2 = y_2f_2(y_1, y_2, x)$, $F_3 = xf_3(y_1, y_2)$ be the scalar field in mathematical model (2). Let $N = N_1i + N_2j + N_3k$ be the vector then consider the vector field $n \times F = N_1i + N_2j + N_3k$, Here $f_1(y_1, y_2, x) = 0$, $f_2(y_1, y_2, x) = 0$, $f_3(y_1, y_2) = 0$ are the isocline of the mathematical model for nonzero positive equilibrium point. Then we have Curl N=0. Thus, Periodic Solution does not exist at non zero equilibrium point.

5. Stability Analysis

The variational matrix about the point E_0 is given by

$$J_0 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & w_4 & 0 \\ 0 & 0 & -C \end{bmatrix}$$

From the above variational matrix, it is seen that there are two unstable manifolds along both$X$,Y axis and one stable manifold along Zaxis. Therefore, the point$E_0$is a saddle point, that is, at very small densities of species the effort decreases and tends to zero, while for small efforts the densities of harvesting species will start increasing,

The variational matrices about the axial point $E_1 = (1,0,0)$ and $E_2 = (0,1,0)$are given by

$$J_1 = \begin{bmatrix} -1 & 0 & -1/(1+w_2) \\ 0 & w_4 & 0 \\ 0 & 0 & \frac{w_1p_1w_6}{1+w_2} - C \end{bmatrix} \text{ and } J_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -w_4 & -w_5/(1+w_3) \\ 0 & 0 & (\frac{w_5p_2w_7}{1+w_3} - C) \end{bmatrix} \text{respectively.}$$

From the matrixJ_1, it is seen that there exists a stable manifold along$X$ axis and an unstable manifold along Z axis. Stable manifold along Y axis exists provided$w_1p_1w_6 - C(1 + w_2) < 0$. Observe that this condition violates the existence of $E_4 = (y_1^*, 0, x^*)$. The point E_1 is a saddle point.

Similarly, from the matrixJ_2, it is seen that there exists a stable manifold along Y axis and an unstable manifold along X axis. Stable manifold along Z axis exists provided$w_1p_1w_6 - C(1 + w_2) < 0$. This condition excludes the existence of equilibrium point $E_5 = (0, y_2^*, x^*)$. The point E_2 is a saddle point.

The variational matrix about the point $E_3 = (1,1,0)$ is given by

$$J_3 = \begin{bmatrix} -1 & 0 & -\frac{w_1}{(1+w_2+w_3)} \\ 0 & -w_4 & -\frac{w_5}{(1+w_2+w_3)} \\ a_{31} & a_{32} & \frac{w_1p_1w_6+w_5p_2w_7}{(1+w_2+w_3)} - C \end{bmatrix}$$

Thus, the equilibrium point $E_3 = (1,1,0)$ is stable provided the following condition is satisfied:

$$\frac{w_1 p_1 w_6 + w_5 p_2 w_7}{(1+w_2+w_3)} - C < 0 \tag{9}$$

Theorem 5.1: The equilibrium point $E_4 = (y_1^*, 0, x^*)$is locally asymptotically stable provided

$$\frac{(w_2-1)}{2w_2} < y_1^* < \frac{(w_1-w_5)}{w_5} < 1 \tag{10}$$

Proof. The variational matrix about the point $E_4 = (y_1^*, 0, x^*)$is given by

$$J_4 = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{23} \end{bmatrix}$$

$$a_{11} = y_1^* \left[-1 + \frac{w_1 w_2 x^*}{(1+w_2 y_1^*)^2}\right], \quad a_{12} = \frac{w_1 w_3 y_1^* x^*}{(1+w_2 y_1^*)^2}, \quad a_{13} = -\frac{w_1 y_1^*}{(1+w_2 y_1^*)};$$

$$a_{22} = \left[w_4 - \frac{w_5 x^*}{(1+w_2 y_1^*)}\right], \qquad a_{21} = a_{23} = a_{33} = 0$$

$$a_{31} = \frac{x^* w_1 w_6 p_1}{(1+w_2 y_1^*)^2}; \qquad a_{32} = \frac{x^*\left(w_7 w_5 p_2 + y_1^*(w_2 w_7 w_5 p_2 - w_3 w_1 w_6 p_1)\right)}{(1+w_2 y_1^*)^2};$$

The equilibrium point$E_4$is locally stable if the following conditions are satisfied:

$$w_1 x^* > w_2(1-y_1^*)^2; \text{ and } \quad y_1^* < \frac{(w_1-w_5)}{w_5};$$

Substitution for x^* and simplification yields the stability conditions as

$$\left(\frac{(w_2-1)}{2w_2}\right) < y_1^* < \left(\frac{(w_4 w_1 - w_5)}{w_1 w_4}\right) < 1$$

The equilibrium$E_4$is unstable when the condition (10) is violated.

Similarly, the stability conditions for the equilibrium E_5 are stated in the theorem 5.2. Its proof is omitted.

Theorem 5.2: The equilibrium point $E_5 = (0, y_2^*, x^*)$is locally asymptotically stable provided

$$\frac{(w_3-1)}{2w_3} < y_1^* < \frac{(w_1-w_5)}{w_5} < 1 \tag{11}$$

The equilibrium E_5 is unstable when the condition (11) is violated.

The following theorem gives the conditions for the locally stability of the nonzero positive equilibrium point$E_6 = (y_1^*, y_2^*, x^*)$.

Theorem 5.3: The positive non-zero biological feasible equilibrium$E_6 = (y_1^*, y_2^*, x^*)$is locally asymptotically stable if the following conditions are satisfied:

$$x^* > w_2(1-y_1^*)^2; \tag{12}$$

$$w_4 w_1^2 x^* > w_3 w_5 (1-y_1^*)^2 \tag{13}$$

$$w_1^2 x^* y_1^* > (w_1 w_2 y_1^* + w_3 w_5 y_2^*)(1-y_1^*)^2 \tag{14}$$

$$w_4 w_1^2 x^* y_2^* > (w_1 w_2 y_1^* + w_3 w_5 y_2^*)(1-y_1^*)^2 \tag{15}$$

Proof: Assume $y_1 = y_1^* + u, y_2 = y_2^* + v, y_3 = y_3^* + w$; where u, v, w are small perturbations. The coefficients f the variational matrix about $E_6 = (y_1^*, y_2^*, x^*)$ are given by

$$a_{11} = y_1^* \left[-1 + \frac{w_1 w_2 x^*}{(1+w_2 y_1^* + w_3 y_2^*)^2}\right]; \; a_{12} = \frac{w_1 w_3 y_1^* x^*}{(1+w_2 y_1^* + w_3 y_2^*)^2}; \; a_{13} = -\frac{w_1 y_1^*}{(1+w_2 y_1^* + w_3 y_2^*)};$$

$$a_{21} = \frac{w_2 w_5 y_2^* x^*}{(1+w_2 y_1^* + w_3 y_2^*)^2}; \; a_{22} = y_2^* \left[-w_4 + \frac{w_3 w_5 x^*}{(1+w_2 y_1^* + w_3 y_2^*)^2}\right]; \; a_{23} = -\frac{w_5 y_2^*}{(1+w_2 y_1^* + w_3 y_2^*)}$$

$$a_{31} = \frac{[w_1 w_6 p_1 + (w_3 w_6 w_1 p_1 - w_5 w_7 w_2 p_2) y_2^*] x^*}{(1+w_2 y_1^* + w_3 y_2^*)^2};$$

$a_{32} = \frac{[w_7 w_5 p_2 + (w_5 w_7 w_2 p_2 - w_1 w_6 w_3 p_1) y_1^*] x^*}{(1 + w_2 y_1^* + w_3 y_2^*)^2}; \quad a_{33} = 0;$

The corresponding characteristic equation is

$\lambda^3 + a_0 \lambda^2 + a_1 \lambda + a_2 = 0$ with $a_0 = -(a_{11} + a_{22}), a_1 = a_{11} a_{22} - a_{12} a_{21} - a_{32} a_{23} - a_{13} a_{31}$ and $a_2 = a_{11} a_{23} a_{32} + a_{13} a_{31} a_{22} - a_{12} a_{23} a_{31} - a_{13} a_{21} a_{32}$ (16)

According to Routh-Hurwitz criterion for stability, the conditions are

$$a_0 > 0, a_1 > 0, a_2 > 0 \text{and } a_0 a_1 - a_2 > 0.$$

The necessary condition for $a_0 > 0$gives stability conditions (12) and (13). The positive ness of a_0 ensures that $a_1 > 0, a_2 > 0$

Further the necessary condition for $a_0 a_1 - a_2 > 0$ gives the stability conditions (14) and (15).

Thus, the positive non-zero biological feasible equilibrium$E_6$is locally asymptotically stable if the conditions given by (12-15) are satisfied.

The following theorem gives the conditions for the global stability of the nonzero positive equilibrium point.

Theorem 5.4 Let the local stability conditions given by (12-15) hold. The positive non-zero biological feasible equilibrium$E_6 = (y_1^*, y_2^*, x^*)$is global stable if the following condition is satisfied:

$$(w_3 - \alpha w_2 w_4)^2 < 4\alpha w_4((1 + w_2 y_1^* + w_3 y_2^*) - w_2)((1 + w_2 y_1^* + w_3 y_2^*) - w_3)$$

$$\boldsymbol{\alpha = \frac{w_1(w_7 w_5 p_2 - w_3 C)}{w_5(w_1 w_6 p_1 - w_2 C)} > 0} \quad (17)$$

Proof: Assume $y_1 = y_1^* + u, y_2 = y_2^* + v, y_3 = y_3^* + w;$ where u, v, w are small perturbations.

Consider the following positive definite function for arbitrarily chosen positive constants D_1, D_2 and D_3:

$$V(t) = D_1\left(u - y_1^* \log\left(1 + \frac{u}{y_1^*}\right)\right) + D_2\left(v - y_2^* \log\left(1 + \frac{v}{y_2^*}\right)\right) + D_3\left(w - x^* \log\left(1 + \frac{w}{x^*}\right)\right)$$

Then

$$\frac{dV}{dt} = D_1 u\left[1 - y_1^* - u - \frac{w_1(x^* + w)}{1 + w_2 y_1 + w_3 y_2}\right] + D_2 v\left[w_4(1 - y_2^* - v) - \frac{w_5(x^* + w)}{1 + w_2 y_1 + w_3 y_2}\right]$$
$$+ D_3 w\left(\frac{p_1 w_1 w_6 y_1 + p_2 w_5 w_7 y_2}{1 + w_2 y_1 + w_3 y_2} - C\right)$$

Rearranging and choosing arbitrary constants $D_1$and D_2 as

$D_2 = \alpha D_1$; where $\alpha = \frac{w_1(w_7 w_5 p_2 - w_3 C)}{w_5(w_1 w_6 p_1 - w_2 C)} > 0$, we get

$$\frac{dV}{dt} = -\frac{D_2}{(1 + w_2 y_1 + w_3 y_2)}[m_1 u^2 + m_2 v^2 - C' uv]$$

$$where \ \ m_1 = ((1 + w_2 y_1^* + w_3 y_2^*) - w_2) > 0 ,$$

$$m_2 = w_4 \alpha((1 + w_2 y_1^* + w_3 y_2^*) - w_3) > 0, \ \ C' = (w_3 + \alpha w_2 w_4) > 0$$

Therefore $\frac{dV}{dt} < 0$ provided

$$(w_3 + \alpha w_2 w_4)^2 < 4\alpha w_4((1 + w_2 y_1^* + w_3 y_2^*) - w_2)((1 + w_2 y_1^* + w_3 y_2^*) - w_3).$$

Further simplification yields

$$(w_3 - \alpha w_2 w_4)^2 < 4\alpha w_4((1 + w_2 y_1^* + w_3 y_2^*) - w_2)((1 + w_2 y_1^* + w_3 y_2^*) - w_3)$$

Thus, $\frac{dV}{dt}$ is negative definite when the condition (17) is satisfied. Therefore, V is a Lyapunov function provided condition (17) is satisfied.

9. Conclusions

In this model, separate dynamics of harvesting effort is considered. The positivity of the species and effort rate is shown analytically. The solution of the system about non zero positive equilibrium contract volume uniformly is analyzed. The nonexistence of periodic solution analytically is carried out. Local and global persistence of the harvested preys has been analyzed.

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