IJCRT.ORG ISSN : 2320-2882

INTERNATIONAL JOURNAL OF CREATIVE RESEARCH THOUGHTS (IJCRT)

An International Open Access, Peer-reviewed, Refereed Journal

The Number Of Ith Smallest Parts Of R-Mi$_i$ Partitions Of N

K. Janakamma

PG Department of Mathematics

S.K. Arts & H. S. K. Science Institute, Hubli-580031

Karnataka, India

Abstract:

In this paper, we derive the generating functions for the number of **ith** smallest parts of**r-MI $_i$**partitions of n. We obtained results for even and odd partitions.

Mathematics Subject Classification 2020: 11P81.

Keywords: M1 partition; M2 partition; r-partition;over partitions; generating function .

1. Introduction

Ahlgren, Bnringmann and lovejoy [1] defined M2spt(n) to be the number of smallest parts in the partitions of n without repeating the odd numbers and smallest part even. Hanumareddy defined [3] ith smallest part and derived the relation between ithsmallest parts and ith greatest parts of partitions and over partitions of n in general form. In this paper derivation of generating functions for the number of ith smallest parts of r-MI$_i$ partitons of n. For more details on partitions see [2,4,5,6,7].

We first obtain a formula for the generating function of r-partitions of n whose ith smallest part is the first part.

Definition and notations:

- A r -jthpartition over of n whose th ith smallest parts are of the form a^{k-1} is denoted by r-jth partition.
  - A $M1$ partition of n is a partition with unrepeated even number and smallest part odd number.
  - A $M2$ partition of n is a partition with unrepeated odd number and with even smallest parts.
  - A $M1_i$ partition of n is a partition with unrepeated odd numbers and i^{th} smallest part even.
  - A $M2_i$ partition of n is one with unrepeated odd numbers and i^{th} smallest part even. As usual $M2_i\xi(n)$ stands for the set of such partitions and $M2_ip(n)$, for the cardinality of this set.
  - A $M1_i\xi(n)$ stand for the set of $M1_i$ partitions and $M1_ip(n)$ for the cardinality of this set.

For r-partitions, the corresponding $MJ_i\xi_r(n)$, $MJ_ip_r(n)$ can be defined similarly for $J = 1$ and $J = 2$.

If $J \in \{1,2\}$, $MJ_ispt_j(n)$ denotes the number of j^{th} smallest parts including repetitions in all MJ_i partitions of n and sum $MJ_ispt_j(n)$ denotes the sum of the j^{th} smallest parts.

If $1 \le r \le n$ and $I \in \{1,2\}$ write $f_{r,s}(I, N)$ for the number of $r - MI$ partitions of n with least part s.

Let us write $r - \xi_e(n)$ for the set of all r-partitions of n with unrepeated even numbers as parts and $p_{r,e}(n)$ for the cardinality of this set. Similarly let $r - \xi_o(n)$ stand for the set of all r-partitions of n with unrepeated odd numbers as parts and $p_{r,o}(n)$ for the cardinality of this set. Also, write $r - spt\xi_e(n)$ for the set of all r-partitions of n with smallest part even and $r - spt\xi_o(n)$ for that of all r-partitions of n with smallest part odd. Then

$$r - M2\xi(n) = r - M2\xi_o(n) \cap r - spt\xi_o(n)$$

and $$r - M1\xi(n) = r - M2\xi_e(n) \cap r - spt\xi_e(n)$$

Examples

(i) (7,5,4,4,3,2,2) is 7-M2 partition of 27

this is $7 - M2_3$, partition but not $7 - M2_4$ partition nor $7 - M2_5$ partition.

(ii) (6,5,5,4,3,3) is 6-M1 partition, which is also $6 - M1_i$, partition for $i = 1$ and 3 but not for $i = 2$ and 4.

The sets $r - MI\xi(n)$, I=1,2:

2. Main Results

Theorem 2.1 If $k \in N$ and $1 \leq k \leq \left[\frac{n}{2r}\right]$ then the number $f_{r,2}(2k, n)$ of $r - M2$ partitions of n with least part 2k is

$$f_{r,2}(2k, n) = p_{r-1,0}(n - 2kr).$$

Proof. let $n = (\lambda_1, \lambda_2, \dots, \lambda_r)$ be a $r - M2$ partition of n with least part $2k$
i.e, $n = (\lambda_1, \lambda_2, \dots, \lambda_{r-1}, \lambda_{2k})$

By reducing each part by $2k - 2$ we get

$n - (2k - 2)r = (\lambda_1 - (2k - 2), \lambda_2 - (2k - 2), \dots, \lambda_{r-1} - (2k - 2), 2)$ and
$n - (2k - 2)r - 2 = (\lambda_1 - (2k - 2), \lambda_2 - (2k - 2), \dots, \lambda_{r-1} - (2k - 2))$ is $(r - 1) - M2$ partition of $n - (2k - 2)r - 2$ with unrepeated odd parts hence, $\in (r - 1)\xi_o(n)$

In this way we get a $(r - 1) - M2$ partition of $n - (2k - 2)r - 2$ from a $(r - 1) - M2$ partition of n.

coversely let $(\mu_1, \mu_2, \dots \mu_{r-1}) \in (r - 1)\xi_0(n - 2kr)$

Then, $(\mu_1 + 2k, \mu_2 + 2k, \dots \mu_{r-1} + 2k, 2k) \in r - M2\xi(n)$

The correspondence

$$n = (\lambda_1, \lambda_2, \dots, \lambda_{r-1}, \lambda_{2k}) \Leftarrow (\lambda_1 - 2k, \dots, \lambda_{r-1} - 2k)$$

is one - one and onto from $r - M2\xi(n)$ to $(r - 1)\xi_0(n - 2kr)$

Hence $f_{r,2}(2k, n) = p_{r-1,0}(n - 2kr)$.

In a similar way we can prove Theorem 2.2.

Theorem 2.2. If $k \in N$ and $1 \leq k \leq \left[\frac{n}{2r-1}\right]$ then the number $f_{r,1}(2k - 1, n)$ of n with least part $2k - 1$is $f_{r,1}(2k - 1, n) = p_{r,e}(n - (2k - 1)r)$.

Proof. The correspondance

$$(\lambda_1, \dots \lambda_{r-1}, \lambda_r = 2k-1) \Leftarrow (\lambda_1 - (2k-1), \dots \lambda_{r-1} - (2k-1))$$

can easily be verified as above, to be one-one and onto between $r - M1\xi(n)$ and $(r-1) - \xi_e(n-(2k-1)r)$.

The following theorem is well known. However, we present the proof for completeness.

Theorem 2.3. The generating function for the number of divisors of n is $\sum_{r=1}^{\infty} \frac{q^r}{1-q^r}$.

Proof. Since $\frac{n}{r} = t \Leftarrow n = tr, d(n) =$ the number of partition of n with equal parts, so the generating function is **(ref: mathword.wolfform.com)**

$$= \sum_{r=1}^{\infty} \sum_{t=1}^{\infty} q^{tr}$$

$$= \sum_{r=1}^{\infty} [q^r + q^{2r} + q^{3r} + \cdots]$$

$$= \sum_{r=1}^{\infty} [q^r(1 + q^r + q^{2r} + q^{3r} + \cdots)]$$

$$= \sum_{r=1}^{\infty} \left[\frac{q^r}{1-q^r}\right]$$

Corollary 2.4. If $k \in N, 1 \leq k \leq n$ and $\frac{n-a}{r} = 1$, then $\sum q^n = \sum q^{a+tr}$

Proof. Since $\frac{n-a}{r} = n - a = tr = a + tr$

Therefore $\sum q^n = \sum q^{a+tr}$.

Proposition 2.5. There is a one one correspondence between r-partition of n and r-partition of n+r with smallest part$\geq$ 2. Under this correspondence $r - M2$, partition of n correspond to $r - M1$, partition of n+r.

Proof. Associate with each r-partition $(\lambda_1, \dots \lambda_r)$ of n the r-partition $(\mu_1, \dots \mu_r)$ where $\mu_i = \lambda_i + 1 \forall i$. This correspondence is one one and onto between the sets mentioned. Since λ_i is even(odd) iff $\lambda_i + 1$ is odd(even) this stands $r - M2$ partions onto $r - M1$ partitions and $r - M2_i$ partitions onto $r - M_i$ partitions and vice-versa.

Theorem:2.6: Given n, $r \le n$ and $\alpha_1, \alpha_2, ...\alpha_l$ there is a one-one correspondence between decreasing l-tuples $(\mu_1, ...\mu_l)$ such that $\alpha_1\mu_1 +, ... + \alpha_l\mu_l = n$ *and* $l-tuples$ $(a_1, ...a_l)$ such that $\mu_{j-1} = \mu_j + a_{t-j}$ and $(\mu_1^{\alpha_1}, \mu_2^{\alpha_2} ....\mu_l^{\alpha_l})$ can be reduced to $(a_1^{\alpha_1})$ by successive subtraction method.

Given $\mu_1 > \mu_2 > ... > \mu_l$ *write* $\mu_j = \mu_{j+1} + a_{j+1}$ *for* $j < l$ *and* $a_l = \mu_l$

apply successive subtraction method to $(\mu_1^{\alpha_1}, \mu_2^{\alpha_2} ....\mu_l^{\alpha_l})$. Subtract μ_l from each part. Since zerocannot be a part we get r_1 partition $(\mu_1^{(1)\alpha_1}, ...\mu_{l-1}^{(1)\alpha_{l-1}})$ of n_1 where

$r_1 = r - \alpha_1, n_1 = n - \alpha_l\mu_l$ *and* $\mu_j^{(1)} = \mu_j - \mu_l$. Subtract $\mu_{l-1}^{(1)}$ from each part of this partition and get r_2 partition

$n_2 = (\mu^{(2)\alpha_1}_{1}, ...\mu^{(2)\alpha_{l-2}}_{l-2})$ *where*

$\mu_j^2 = \mu_j^{(1)} - \mu_{j+1}^{(1)} = \mu_j - \mu_{j+1} \quad \forall j,$

$r_2 = r_1 - \alpha_2 = r - (\alpha_1 + \alpha_2)$ *and* $n_2 = n_1 - \alpha_{l-1}\mu_{l-1}^{(1)} = n_1 - \alpha_{l-1}(\mu_{l-1} - \mu_l).$

Repeating this process we get a finite sequence of $(l-k)$ partitions

$(\mu^{(k)\alpha_1}_{1}, ...\mu^{(k)\alpha_{l-2}}_{l-2})$ where

$r_k = r_{k-1} - \alpha_k = r - (\alpha_1 + ... + \alpha_k)$ and $n_k = n_{k-1} - \alpha_{l-k+1}\mu_{l-k+1}^{(k-1)}$ and $\mu_j^{(k)} = \mu_j^{(k-1)} - \mu_{l-k+1}^{(k-1)}$

when $k = l-1$ we get $r_{l-1} = (\alpha_1)$ partition of *n*. $n_{l-1} = (\mu_l^{(L\_!)\alpha_1})$ where

$n_{l-1} = n_{l-2} - \alpha_2\mu_2^{(l-2)} = \mu_1 - \mu_2.$

Thus the given r-partition of n reduces to the partition

(a^{α_1}). write $a_1 = \mu_1 - \mu_2, a_2 = \mu_2 - \mu_3 ....a_{l-1} = \mu_{l-1} - \mu_l$ and $a_1 = \mu_l$

$a_1 + a_2 + ... + a_l = \mu_l.$

$$(a_1, a_2, ...a_l) \in N^l \text{ and } \sum \alpha_i (a_l + a_{l-1} + ...a_{l-i}) = n$$

Conversely assume that $\mu_l = a_l$ and $\mu_{l-j} = \mu_{l-j+1} + a_{l-j} = a_{l-j} + a_{l-(j-1)} + ...a_l$ write $r_0 = r = \sum \alpha_i$

$$n_0 = n, \mu_j^0 = \mu_i, \mu_k^j = \mu_j^{(k-1)} - \mu_{l-k+1}^{(n-1)} \quad 1 \le j \le k-1, \ n_j = n_{j-1} - \alpha_{l-j+1}\mu_{l-j+1}^{(j-1)}$$

We apply the successive subtraction method for $(\mu_1^{\alpha_1}, \mu_2^{\alpha_2} ....\mu_l^{\alpha_l})$ using the above notation and finally get the α_1 partition $((\mu_1 - \mu_2)^{\alpha_1}) = (a_1^{\alpha_1})$.

2.7. Corollary: The number of $r - M2$ partitions of n with i^{th} smallest part coincidingwith the first is equal to the number of $\alpha -$ *partitions of* m with equal parts α being the frequency of the smallest part of the r-partition of n.

2.8. Example:

List of $6I_3$ *: partitions of* $20:(34)$

$(\alpha_1+\alpha_2+\alpha_3=6)$

$\mu_1^{\alpha_1}$	$\mu_2^{\alpha_2}$	$\mu_3^{\alpha_3}$	$\mu_1^{\alpha_1}$	$\mu_2^{\alpha_2}$	$\mu_3^{\alpha_3}$	$\mu_1^{\alpha_1}$	$\mu_2^{\alpha_2}$	$\mu_3^{\alpha_3}$	$\mu_1^{\alpha_1}$	$\mu_2^{\alpha_2}$	$\mu_3^{\alpha_3}$	$\mu_1^{\alpha_1}$	$\mu_2^{\alpha_2}$	$\mu_3^{\alpha_3}$
14	2	1^4	11^1	5^1	1^4	7^2	3^1	1^3	8^1	4^1	2^4	9^1	3^1	2^4
6^2	4	1^4	9^1	7^1	1^4	9^1	3^3	1^2	8^1	3^2	2^3	7^1	5^1	2^4
10	6	1^4	13^1	2^2	1^3	5^2	4^2	1^2	6^1	4^2	2^3	5^2	4^1	2^3
6^2	3^2	1^2	11^1	3^2	1^3	5^3	3^1	1^2	4^3	3^2	2^1	7^1	3^3	2^2
10^1	2^3	1^2	9^1	4^2	1^3	11	2^4	1^1	5^2	3^3	1^1	5^2	3^2	2^2
4^4	3^1	1^1	7^1	5^2	1^3	7	3^4	1^1	13	3^1	1^4	7^2	4^1	2^1
5^3	4^1	1^1	7^2	2^2	1^2	5^3	2^2	1^1	6	4^3	1^2			

List *of* $6-M1_3$ partitions: (12)

odd	even	even	odd	odd	odd	odd	even	odd	odd	odd	even
7^2	4^1	2^1	13^1	3^1	1^4	5^3	4^1	1^1		nil	
			11^1	5^1	1^4						
			9^1	7^1	1^4						
			11^1	3^2	1^3						
			7^1	5^2	1^3						
			7^2	3^1	1^3						
			9^1	3^3	1^2						
			5^3	3^1	1^2						
			7^1	3^4	1^1						
			5^2	3^3	1^1						

List of $6-M2_3$ *partitions*: (3)

even	*even*	*even*	*even*	odd	odd	*even*	*even*	odd	*even*	odd	*even*
6^1	4^2	2^3	4^4	3^1	1^1		nil			nil	
8^1	4^1	2^4									

number of 6 partitions that are not $6-MI_3$ partitions; $34-15=19$

The re**maining 19 partitions are liste**d below**:**

List of remaining $6I_3$: partitions of 20 :

$\mu_1^{\alpha_1}$	$\mu_2^{\alpha_2}$	$\mu_3^{\alpha_3}$	$\mu_1^{\alpha_1}$	$\mu_2^{\alpha_2}$	$\mu_3^{\alpha_3}$	$\mu_1^{\alpha_1}$	$\mu_2^{\alpha_2}$	$\mu_3^{\alpha_3}$	$\mu_1^{\alpha_1}$	$\mu_2^{\alpha_2}$	$\mu_3^{\alpha_3}$	$\mu_1^{\alpha_1}$	$\mu_2^{\alpha_2}$	$\mu_3^{\alpha_3}$
14	2	1^4										9^1	3^1	2^4
6^2	4	1^4							8^1	3^2	2^3	7^1	5^1	2^4
10	6	1^4	13^1	2^2	1^3	5^2	4^2	1^2				5^2	4^1	2^3
6^2	3^2	1^2							4^3	3^2	2^1	7^1	3^3	2^2
10^1	2^3	1^2	9^1	4^2	1^3	11	2^4	1^1				5^2	3^2	2^2
			7^2	2^2	1^2	5^3	2^2	1^1	6	4^3	1^2			

References

[1] Ahlgren S, Bringmann K and Lovejoy J, l-adic properties of smallest parts functions, Adv. Math., 228(1) (2011), 629-645.

[2] Andrews G. E, The theory of partitions , Vol 2, encycl. Of Math. And Appl. Addison –Wesley, Reading. (Reprinted: Cambridge University Press, 1998).

[3] Hanuma reddy K. A study of r-partitions, thesis submitted to Acharya Nagarjuna University for award of Ph. D in Mathematics.

[4] Hanuma reddy K, Manjushree A, The number of smallest parts of over partition of n, Internation research Journal of Mathematics, Engineering and IT, 2(3) (2015), 23-31.

[5] Ramanujan S, Some properties of p(n); the number of partitions of n , Proc. Cambridge Philos. Soc., 19 (1919), 207-2010.

[6] RamaBhadra Sarma I, Hanuma Reddy K and S, Rao Gunakala, relation between smallest and greatest parts of over partitions of n, International Journal of Mathematical research, 3(3) (2010), 195-205.

[7] Sylvester J. J, On the partition of numbers, quaterly Joournal of Mathematics 1 (1857), 141-152.