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AN OSCILLATORY CRITERIA FOR LINEAR SECOND-ORDER ORDINARY DIFFERENTIAL EQUATIONS WITH DELAY

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ABSTRACT:

We study the Oscillatory Criteria for second order ordinary differential equation with delay of the form $x''(t) + q(t)\, x(\tau(t)) = 0$, on the half line $R = [0,\infty)$. where τ is a constant delay, it is also a continuous functions, and $q : R \to R$ is an integrable function. So that $\tau(t) \le t$ for every $t \ge 0$ and $\lim_{t\to\infty} \tau(t) = +\infty$. The linear second-order ordinary differential equation of oscillatory theory is a well-established and widely studied branch of the general theory of the differential equations. The well-known Leighton- Nehari and Wintner type oscillatory solutions for ordinary differential equations also hold for delay differential equation (1). The differential equations with argumental deviations and related systems, may derive a similar set of oscillation conditions. The well-known Riccati approach for ordinary differential equations is used in this paper to provide oscillation criteria. We can infer from the equality (1) applied for the formula $h(t) = x'(t)/x(t)$ for sufficiently large t, the non-oscillatory solution x is accurate. $h'(t) = -\left(q(t)\, \frac{x(\tau(t))}{x(t)} + h^2(t)\right)$ for large t. And applying the Riccati technique to differential equations with argument deviations requires to find suitable lower and upper bounds for the quantity $\frac{x(\tau(t))}{x(t)}$, which is in the case of ordinary differential equations is equal to 1.

Keywords: Second order linear delay differential equations, Oscillation Criteria, Riccati technique.

1. INTRODUCTION

The second-order linear delay differential equation on half-line $R_+ = [0, +\infty)$ is

$$x''(t) + q(t)\, x(\tau(t)) = 0 \qquad (1)$$

where τ is a constant delay and it is also a continuous functions and $q : R \to R$ is an integrable function, so, that

$$\tau(t) \le t \quad \text{for every } t \ge 0 \text{ and } \lim_{t\to\infty} \tau(t) = +\infty. \qquad (2)$$

The linear second-order ordinary differential equation of oscillatory theory is a well-established and widely studied branch of the general theory of the differential equations. We should explicitly mention the works of

Zdenek Oplustil as they pertain to the discoveries that are sealed in regard to the results of this, see [1]. The classical conclusions have been successfully extended by Q. Yang for general equations, such as p-Laplacian, difference equations, or time scale equations; see [2]. The well-known Leighton-Wintner and Nehari type oscillations for ordinary difference equations and also hold for equation (1). The differential equations with argumental deviations and related systems, may derive a similar set of oscillation conditions for more information, see [3 - 13].

The below definations are introducing the proper solutions of equation (1). i..e oscillatory and non-oscillatory solutions.

Definition - 1: " Let $t_0 \in R$ and $u_0 = \inf\{ \tau(t): t_0 \le t \}$. A continuous function x: $[u_0, +\infty) \to R$ is referred to as a proper solution of the equation (1) on the interval $[t_0, +\infty)$, if it is absolutely continuous along with its first derivative on every compact interval in $[t_0, +\infty)$, satisfies the equality (1) almost everywhere in $[t_0, +\infty)$, and has the property that $\sup\{|x(s)|: s \ \ge \ t\} > 0$ for all $t \ge t_0$".

Definition – 2: "If a nontrivial solution to (1) has arbitrarily large zeros, it is said to be oscillatory; otherwise, it is non-oscillatory. If all of the solutions to equation (1) are non-oscillatory, then the equation is non-oscillatory".

The well-known Riccati approach for ordinary differential equations is used in this paper to provide oscillation criteria. We can infer from the equality (1) that when we apply the formula $h(t) = x'(t)/x(t)$ for sufficiently large t, the non-oscillatory solution x is accurate.

$$h'(t) = -\left(q(t)\,\frac{x(\tau(t))}{x(t)} + h^2(t)\right) \quad \text{for all t.}$$

Applying the Riccati technique to the differential equation with argumental deviations and requires to find suitable upper and lower bounds for the quantity $\frac{x(\tau(t))}{x(t)}$. which is in the case, the ordinary differential equations is equal to 1 and shown in Lemma-1.

2. Main Results:

"If $\int_0^{+\infty} \tau(s)q(s)\,ds$ is convergent, then the equation (1) has non-oscillatory solution", see [7]. Consequently, we'll assume in the sequel that is

$$\int_0^{+\infty} \tau(s)q(s)\,ds = +\infty. \tag{3}$$

Theorem -1: Assume that the condition (3) satisfies, and

$$H^* = \limsup_{t\to\infty} \left\{\frac{1}{t}\int_0^t s\tau(s)\,q(s)ds\right\} > 1. \tag{4}$$

Consequently, all the solutions of equation (1) is oscillatory.

The condition (4) is a specific application of the oscillation criterion see in [1],

Now let us take

$$H_* = \liminf_{t\to+\infty} \left\{\frac{1}{t}\int_0^t s\tau(s)\,q(s)ds\right\} \le 1. \tag{5}$$

The following theorem presents a Wintner type criteria.

Theorem - 2: Assume that the conditions (3) and (5) be satisfied,

Let

$$\lim_{t\to+\infty} \frac{\tau(t)}{t} > 0. \quad \text{for } t \ge 0 \tag{6}$$

Here we have an ordinaty delay $\tau(t) \equiv t$ for every $t \ge 0$. Let us assume that $\mu < 1$(μ is a lowest value), such that

$$\int_0^{+\infty} s^{\mu} \left(\frac{\tau(s)}{s}\right)^{1-H_*} q(s)\,ds = +\infty. \qquad (7)$$

All solutions of equation (1) is oscillatory.

Remark - 2: It is clear that the condition (6) is fulfilled and condition (7) coincide with the well-established conclusions. (For $\mu = 0$, see [3 - 5]). Finally, we provide an oscillatory critetia to generalize a result of E. Muller-Pfeiffer to demonstrate for ordinary differential equations in the publication [6].

Theorem-3: Assume that the conditions (3), (5), and (6) hold, and let

$$\lim_{t\to+\infty} sup \left\{\frac{1}{In\,t} \int_0^t s \left(\frac{\tau(s)}{s}\right)^{1-H_*} q(s)\,ds\right\} > \frac{1}{4} \qquad (8)$$

Where In t denotes the natural logarithm of t (> 0). Then every proper solution is oscillatory for equation (1).

Example -: In particular, if τ is a proportional delay, then the equation (1) has in the form

$x''(t) + q(t)\,x(\alpha t) = 0$ with $0 < \alpha \le 1$, and the theorems 2 and 3 is satisfied for condition (6).

3. Auxilliary Statements for equation (1):

The lemma provides that certain priori estimations of non-oscillatory solutions of equation (1) and essential in the proof of main theorems.

Lemma -1: Assuming that (3) holds and let x be the solution to equation (1) then

there exists $t_x > 0$ such that $x(t) > 0$ for some $t \ge t_x$ (9)

Then

$$\lim_{t\to+\infty} sup \left\{\frac{1}{t} \int_0^t s\tau(s)q(s)ds\right\} \le 1. \qquad (10)$$

In addition, if the inequatlity (6) holds then

$$\lim_{t\to+\infty} inf \left\{\left(\frac{t}{\tau(t)}\right)^{1-H_*} \frac{x(\tau(t))}{x(t)}\right\} \ge 1 \qquad (11)$$

where the relation $\tau(t) = t$ for all $t \ge 0$.

Proof: For the sufficient large t, it is simple to prove that the inequality $x'(t) \ge 0$ holds. Since equation (1) is homogeneous second order delay differential equation and we can assume that $x(t) \ge 1$ for sufficient large t and without losing generality to show the result, In view, the assumption (2), also exists t_0 ($\ge t_x$). Therefore

$$x'(t) \ge 0, \quad x(\tau(t)) \ge 1, \ \tau(t) = t \quad \text{for all } t \ge t_0. \qquad (12)$$

and

$$(t\,x'(t) - x(t))' = -\,t\,q(t)\,x(\tau(t)) \quad \text{for all } t \ge 0. \qquad (13)$$

By integrating the inequality from t_0 to t we get,

$$t\,x'(t) - x(t) = \Delta - \int_{t_0}^t s\,q(s)x(\tau(s))\,ds \quad \text{for all } t \ge t_0. \qquad (14)$$

here $\Delta = t_0\,x'(t_0) - x(t_0)$.

Let $\delta \in (0,1)$ be an integral and has arbitrary constant values. The view of assumption (3) exists that $t_0 \le t_1(\delta)$ is

$$\Delta \le \frac{\delta}{2} \int_{t_o}^t s\,q(s)x(\tau(s))\,ds \qquad \text{for every } t \ge t_1(\delta). \qquad (15)$$

Thus, the relation (14) implies

$$tx'(t) - x(t) \leq \left(\frac{\delta}{2} - 1\right) \int_{t_o}^{t} s\, q(s) x(\tau(s))\, ds \leq 0 \quad \text{for every } t \geq t_1(\delta). \tag{16}$$

Therefore,

$$\left(\frac{x(t)}{t}\right)' = \frac{1}{t^2}\left(tx'(t) - x(t)\right) \leq 0 \qquad \text{for all } t \geq t_1(\delta). \tag{17}$$

We can determine that the existence of $t_1(\delta) \leq t_2(\delta)$ sby applying this inequality and assumption (2) in the formula (16).

$$\begin{aligned} tx'(t) - x(t) &\leq \left(\frac{\delta}{2} - 1\right) \int_{t_2(\delta)}^{t} s\tau(s) q(s) \frac{x(\tau(s))}{x(s)}\, ds \\ &\leq \left(\frac{\delta}{2} - 1\right) \frac{x(t)}{t} \int_{t_2(\delta)}^{t} s\tau(s) q(s) ds \quad \text{for all } t \geq t_2(\delta). \end{aligned} \tag{18}$$

According to the above statements, we can conclude that the inequality is

$$\begin{aligned} tx'(t) &\leq x(t) + \left[\left(\frac{\delta}{2} - 1\right) \frac{x(t)}{t} \int_{t_2(\delta)}^{t} s\tau(s) q(s) ds\right] \\ &\leq x(t)\left[1 + \left(\frac{\delta}{2} - 1\right) \frac{1}{t} \int_{t_2(\delta)}^{t} s\tau(s) q(s) ds\right] \quad \text{for all } t \geq t_2(\delta) \end{aligned} \tag{19}$$

Hence, we obtain by (9) and (12),

$$0 \leq 1 + \left(\frac{\delta}{2} - 1\right) \frac{1}{t} \int_{t_2(\delta)}^{t} s\tau(s) q(s) ds$$

$$- \left(\frac{\delta}{2} - 1\right) \frac{1}{t} \int_{t_2(\delta)}^{t} s\tau(s) q(s) ds \leq 1$$

$$\left(1 - \frac{\delta}{2}\right) \frac{1}{t} \int_{t_2(\delta)}^{t} s\tau(s) q(s) ds \leq 1$$

That is

$$\frac{1}{t} \int_{t_2(\delta)}^{t} s\tau(s) q(s) ds \leq \frac{2}{2-\delta} \qquad \text{for all } t \geq t_2(\delta)$$

and therefore

$$\lim_{t\to+\infty} \sup \left\{\frac{1}{t} \int_{0}^{t} s\tau(s) q(s) ds\right\} \leq \frac{2}{2-\delta}$$

Since $\delta \in [0,1]$ was arbitraty, Inequality (10) as expected, is valid.

The inequality (11) follows from equation (6) and exists that $t_2(\delta) \leq t_3(\delta)$, such that

$$\frac{1}{t} \int_{t_3(\delta)}^{t} s\tau(s) q(s) ds \geq \left(1 - \frac{\delta}{2}\right) H_* \quad \text{for all } t \geq t_3(\delta).$$

Multiplying both sides of the above inequality by $\left(\frac{\delta}{2} - 1\right) x(t) (< 0)$ and using (18), we obtain

$$tx'(t) - x(t) \leq \left(\frac{\delta}{2} - 1\right) x(t) \left(1 - \frac{\delta}{2}\right) H_* \leq (1 - \delta) x(t) H_* \quad \text{for } t \geq t_3(\delta),$$

and hence we have

$$\left(\frac{x(t)}{t}\right)' = \frac{1}{t^2}\left(tx'(t) - x(t)\right) \leq \frac{(1-\delta)H_*}{t} \frac{x(t)}{t} \quad \text{for } t \geq t_3(\delta). \tag{20}$$

Observe that, in view of (2), there exists $t_3(\delta) \leq t_4(\delta)$ such that $\tau(t) \geq t_3(\delta)$ for all $t \geq t_4(\delta)$. Consequently, from the inequality (20) we get

$$\ln \frac{x(t)/t}{x(\tau(t))/\tau(t)} \leq (1 - \delta) H_* \ln \frac{t}{\tau(t)} \quad \text{for } t \geq t_4(\delta).$$

Conversely, based on the assumption (6), there is $t_4(\delta) \leq t_5(\delta)$ such that $0 < \beta \leq \tau(t)/t$ for all $t_5(\delta) \leq t$.

Thus,

$$\left(\frac{t}{\tau(t)}\right)^{1-H_*} \frac{x(\tau(t))}{x(t)} \geq \beta^{\delta H_*} \qquad \text{for all } t \geq t_5(\delta).$$

Consequently, we have

$$\lim_{t\to+\infty} inf \left\{\left(\frac{t}{\tau(t)}\right)^{1-H_*} \frac{x(\tau(t))}{x(t)}\right\} \geq \beta^{\delta H_*}, \tag{21}$$

The arbitrariness of $\delta \in (0,1)$ gives the desired equality (11).

Lemma - 2: Let x be a non-oscillatory solution to equation (1). Then the following limit is finite.

$$\lim_{t\to+\infty} \int_{t_x}^{t} s^{\mu} \frac{x(\tau(s))}{x(s)} \, q(s)\, ds < \infty$$

for any μ (< 1). Futhermore,

$$\lim_{t\to+\infty} \sup \left\{\frac{1}{In\, t} \int_{t_x}^{t} s \frac{x(\tau(s))}{x(s)} \, q(s)\, ds\right\} \leq \frac{1}{4}. \tag{22}$$

Proof: For $t \geq t_x$, let's put $\mu < 1$ and enter $h(t) = x'(t)/x(t)$, therefore (1) yields the result that,

$$h'(t) = -\, q(t) \frac{x(\tau(s))}{x(s)} - h^2(t) \quad \text{for all } t \geq t_x .$$

The result is obtained by integrating it from t_x to t and multiplying t^{μ} on both sides of this equality.

$$t^{\mu-1}\left[t\, h(t) - \frac{\mu}{2}\right] = \Delta_1 - \left[\frac{\mu\,(2-\mu)}{4(1-\mu)} \frac{1}{t^{1-\mu}} + \int_{t_x}^{t} s^{\mu} \frac{x(\tau(s))}{x(s)} \, q(s)\, ds + \int_{t_x}^{t} s^{\mu-2}\left[sh(s) - \frac{\mu}{2}\right]^2 ds\right] \quad \text{for all } t \geq t_x \tag{23}$$

Where $\Delta_1 = t_x^{\mu}\, h(t_x) + \frac{1}{4}\, \mu^2(1-\mu)^{-1} t_x^{\mu-1}$.

We first show that

$$\int_{t_x}^{+\infty} s^{\mu-2}\left[sh(s) - \frac{\mu}{2}\right]^2 ds < +\infty. \tag{24}$$

Alternatively, suppose that the integral in (24) is divergent. The inequality exists for some $t_1 \geq t_x$ due to relation (23).

$$t\, h(t) - \frac{\mu}{2} \leq -\frac{1}{2} t^{1-\mu} \int_{t_x}^{t} s^{\mu-2}\left[s\, h(s) - \frac{\mu}{2}\right]^2 ds < 0 \quad \text{for all } t_1 \leq t. \tag{25}$$

Satisfies.

Let us denote that

$$u(t) = \int_{t_x}^{t} s^{\mu-2}\left[s\, h(s) - \frac{\mu}{2}\right]^2 ds \quad \text{for all } t_1 \leq t .$$

Using the relation (24), we obtain

$$u'(t) = t^{\mu-2}\left[t\, h(t) - \frac{\mu}{2}\right]^2 \geq \frac{1}{4t^{\mu}} u^2(t) \quad \text{for all } t_1 \leq t.$$

Therefore , by integrating the final inequality from t_1 to t, i..e

$$4(1-\mu)/u(t_1) + t_1^{1-\mu} \geq t^{1-\mu} \text{ satisfies for all } t_1 \leq t .$$

which contradicts itself. The resultant contradiction establishes the equality's validity (24). The equality (23) may now be rewritten as follows:

$$\int_{t_x}^{t} s^{\mu} \frac{x(\tau(s))}{x(s)} q(s)\, ds = \Delta_2 - t^{\mu} h(t) - \left[\frac{\mu^2}{4(1-\mu)} \frac{1}{t^{1-\mu}} - \int_{t}^{+\infty} s^{\mu-2} \left[s\, h(s) - \frac{\mu}{2}\right]^2 ds\right]$$

For all $t \geq t_x$, (26)

where $\Delta_2 = \Delta_1 - \int_{t_x}^{+\infty} s^{\mu-2} \left[s\, h(s) - \frac{\mu}{2}\right]^2 ds$. Consequently, we obtain that

$$-\infty < \lim_{t\to+\infty} \int_{t_x}^{t} s^{\mu} \frac{x(\tau(s))}{x(s)} q(s)\, ds = \Delta_2 < +\infty \quad (27)$$

Since, the inequality $h(t) \leq 1/t$ satisfies for large t in view of condition (15).

The validity of the relationship needs to be demonstrated (20). Using the previously established relation (27), we may obtain by multiplying $t^{-\mu}$ on both sides of the equality (26), integrating it from t_x to t by parts,

$$\int_{t_x}^{t} s \frac{x(\tau(s))}{x(s)} q(s)\, ds \leq \Delta_3 + \frac{\mu(2-\mu)}{4} \ln t$$

$$+ \int_{t_x}^{t} \frac{1}{s} \left(s\, h(s) - \frac{\mu}{2}\right)\left(1 - \mu - \left[s\, h(s) - \frac{\mu}{2}\right]\right) ds \quad \text{for } t \geq t_x, \quad (28)$$

where Δ_3 is some appropriate constant. Let $u = s\, h(s) - \frac{\mu}{2}$ ($\in R$). Then using the fact that the function $f(u) = 4u(1-\mu-u)$ takes the maximum value $(1-\mu)^2$ when $u = \frac{1-\mu}{2}$ for all $u \in R$, so the following holds:

$$\int_{t_x}^{t} s \frac{x(\tau(s))}{x(s)} q(s)\, ds \leq \Delta_3 + \frac{1}{4} \ln t \quad \text{for all } t \geq t_x, \quad (29)$$

Consequently, the required condition (20) is satisfied.

4. THE PROOFS OF MAIN RESULTS:

Proof of Theorem -1: Let us say the assertion of the theorem is not hold. There exist a solution x of the equation (1) holds with the condition (9). Lemma -1 states that the relation (10), which defies the assumption (4), holds.

Proof of Theorem-2: Consider a scenario in which the theorem's assertion is not hold. Then, the equation (1) has a solution x that satisfies (9). Define $\delta \in (0,1)$ as arbitrarily fixed. There exists t_0 ($\geq t_x$) such that, according to Lemma -1, We find that

$$\left(\frac{t}{\tau(t)}\right)^{1-H_*} \frac{x(\tau(t))}{x(t)} \geq (1-\delta) \quad \text{for all } t \geq t_0, \quad (30)$$

Thus we have from (28) that the following inequality

$$\left(\frac{\tau(t)}{t}\right)^{1-H_*} \leq \frac{1}{1-\delta} \frac{x(\tau(t))}{x(t)} \quad \text{for all } t \geq t_0(\delta), \quad (31)$$

Multiplying both sides of (31) by s^{μ} q(s) and integrating over the interval $[t_x, t]$, we obtain

$$\int_{0}^{t} s^{\mu} \left(\frac{\tau(s)}{s}\right)^{1-H_*} q(s)\, ds$$

$$\leq \int_{0}^{t_0} s^{\mu} \left(\frac{\tau(s)}{s}\right)^{1-H_*} q(s)\, ds + \frac{1}{1-\delta} \int_{t_x}^{t} s^{\mu} \frac{x(\tau(s))}{x(s)} q(s)\, ds \quad \text{for } t \geq t_0.$$

Therefore, Lemma - 2 implies that

$$\int_{0}^{+\infty} s^{\mu} \left(\frac{\tau(s)}{s}\right)^{1-H_*} q(s)\, ds < +\infty, \quad (32)$$

Which differ the assumption (7).

Proof of Theorem-3: Let's say that the theorem's assertion not hold. Then, the equation (1) has a solution x that satisfies (9). Define $\delta \in (0,1)$ as arbitrarily fixed. There exists t_0 ($\geq t_x$) such that the relation (31) holds, according to Lemma-1. Multiplying both sides of (31) by sq(s), integrating over $[t_x, t]$, and dividing both sides by In t, we obtain

$$\frac{1}{In\,t}\int_0^t s\left(\frac{\tau(s)}{s}\right)^{1-H_*} q(s)\,ds$$

$$\leq \frac{1}{In\,t}\int_0^{t_0} s\left(\frac{\tau(s)}{s}\right)^{1-H_*} q(s)\,ds + \frac{1}{(1-\delta)\,In\,t}\int_{t_x}^t s\,\frac{x(\tau(s))}{x(s)}\,q(s)\,ds \qquad \text{for all } t \geq t_0.$$

Using the condition (22) from Lemma - 2, we obtain

$$\lim_{t\to+\infty} sup\left\{\frac{1}{In\,t}\int_0^t s\left(\frac{\tau(s)}{s}\right)^{1-H_*} q(s)\,ds\right\} \leq \frac{1}{4(1-\delta)}, \qquad (33)$$

Which, given the arbitrary nature of $\delta \in [0, 1]$, contradicts the supposition (8).

Conclusion:

In this study, we have taken the second order ordinary linear delay differential equations into consideration to find oscillatory criteria. Applying the Riccati technique, which is a well-established one in the context of ordinary differential equations by computing lim sup and lim inf to determine the oscillation or non-oscillation solutions. The primary benefit of these oscillatory criteria is that they are significantly out form of all related oscillation conditions and time scale equations in the literature and half-line equations of second order linear delay differential equations is solved for delay.

References:

[1]. Zdenek Oplustil, Jiri Sremr, Brno., “Some Oscillation criteria for second-order linear differential Equations”. Vol. No.2, 195 – 204, (2011).

[2]. Q. Yang, Z. Xu, “Oscillation criteria for second order quasilinear neutral delay differential equations on time scales”, Comp. Math. Appl., vol. no. 62. 3682-3691, (2011).

[3]. A.Wintner “On the non-existence of conjugate points.” Amer. J. Math. 73, 368 – 380, (1951).

[4]. E. Hille: “Non-oscillation theorems”. Trans. Amer. Math. Soc. Vol. no. 64, 234 – 252, (1948),.

[5]. W. B. Fite: “Concerning the zeros of the solutions of certain differential equations”. Trans. Amer. Math. Soc. Vol. no.19, 341 – 352, (1918).

[6]. E. Muller-Pfeiffer: “Oscillation criteria of Nehari-type for Schrödinger equation”. Math. Nachr. Vol. no. 96,185 – 194, (1980).

[7]. N. Partsvania: “On oscillatory and monote solutions of two-dimensional differential systems with deviated arguments”. Ph.D. Thesis, A. Razmadze Mathematical Institute of the Georgian Academy of Sciences, Tbilisi, 1999.

[8]. P. Rehak: “Half-linear dynamic equations on time scales”: IVP and oscillatory properties. Nonlinear Funct. Anal. Appl. Vol. no.7, 361 – 403, (2002).

[9]. N. Kandelaki, A. Lomtatidze, D. Ugulava: “On oscillation and nonoscillation of a second order half-linear equation”. Georgian Math. J. 7 (2000), 329 – 346.

[10].O. Dosly, P. Rehak: “Half-linear Differential Equations”. North-Holland Mathematics Studies 202, Elsevier, Amsterdam, 2005.

[11]. Baculikova, B.Bzurina, J. “Oscillation theorems for second order neutral differential equations Computer, Math. Appl.61 94 – 99(2011).

[12]. Z. Oplustil, Z. Pospíšil: “An oscillation criterion for a dynamic Sturm-Liouville equation”, New progress in difference equations. Proceedings of the 6th International Conference on Difference Equations, 2004, pp. 317 – 324.

[13]. P. Rehak: Hartman-Wintner type lemma, “oscillation, and conjugacy criteria for halflinear difference equations”. J. Math. Anal. Appl. 252 (2000), 813 – 827.