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HORIZONTAL LIFTS OF THE METALLIC STRUCTURES FROM MANIFOLD ONTO TANGENT BUNDLE

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Abstract: The present paper deal with the study of complete and horizontal lifts of the general quadratic structure on tangent bundles. Integrability conditions for complete and horizontal lifts of this structure are investi-gated. Also, the prolongation of the general quadratic structure in the third tangent bundle is studied.

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1 Introduction

The notion of polynomial structures of degree n on manifold introduced by Goldberg and Yano [7]. The polynomial structure of degree 2 satifying

$$F^2 + \alpha F + \beta I = 0 \tag{1.1}$$

where α, β are integers, is called general quadratic structure on differentiable manifold M .
The theory of the tangent bundle of geometric structure is an important topic in differential geometry. The complete and horizontal lifts generalized of geometric structures on any manifold M to its tangent bundles have been studied by numerous investigators, for example; Yano and Ishihara [20], Om- ran at el [1], Khan [8]. The complete, vertical, horizontal lifts of tensor field and connections on any manifold M to its tangent bundle TM has been obtained by Yano and Ishihara [20]. Khan [4] studied complete and horizontal lifts of metallic structure and investigated integrability conditions. Das and the Khan [2] studied almost product structure by means of the complex, ver- tical and horizontal lifts of an almost r-contact structure. Studies of complete, horizontal and vertical lifts and integrability conditions. The aim of this paper is to study the general quadratic structure on tangent bundles and establish integrability conditions of various structures includes [5, 6, 9, 10, 11, 12, 13] and others.
Let M be n-dimensional differentiable manifold of class C^∞. A tensor field F of type (1,1) is called the general quadratic structure on M if F satisfies the equation [7]

$$F^2 + \alpha F + \beta I = 0, \tag{1.2}$$

where α, β are positive integers and I is the unit vector field on M and F is of constant rank r everywhere in M.
Let l and m be operators define as

(a) $l = -\dfrac{F^2 + \alpha F}{\beta}$

(b) $m = I \dfrac{F^2 + \alpha F}{\beta}$ (1.3)

and satisfy the following conditions:

$$l + m = 0$$
$$l^2 = l, \quad m^2 = m, \quad lm = ml = 0 \tag{1.4}$$
$$Fl = lF = F, \quad Fm = mF = 0.$$

Thus, there exist two complementary distributions D_l and D_m corresponding to the projection tensors l and m respectively in M. If the rank odd F is r, then D_l is r-dimensional and D_m is $(n - r)$-dimensional, where dim$M = r$.

2 THE COMPLETE LIFT OF F IN THE TANGENT BUNDLE $T(M)$

Let M be an n-dimensional differentiable manifold of class C^∞ and $T_p(M)$ the tangent space at a point p of M then $T(M) = \cup_{p \in M} T_p M$ is a tangent bundle over the manifold M. The tangent bundle TM of M is a differentiable manifold of dimension $2n$. Let $\mathfrak{s}^r_s$ denote the set of tensor field of class C^∞ and type (r, s) in M and $\mathfrak{s}^r_s(T(M))$ denote the corresponding set of tensor fields in $T(M)$ [3, 8].

Let F, tt be elements of $\mathfrak{s}^1_1(M)$. Then we have [20]

$$(Ftt)^C = F^C tt^C. \tag{2.1}$$

Putting $F = tt$ in equation (2.3), we obtain

$$(F^2)^C = (F^C)^2. \tag{2.2}$$

Also,

$$(F + tt)^C = F^C + tt^C. \tag{2.3}$$

Operating the complete lifts of both sides of equation (1.2), we get

$$(F^2 + \alpha F + \beta I)^C = 0 \ (F^2)^C + (\alpha F)^C + \beta I^C = 0$$

In the view of (2.2) and $I^C = I$, we get

$$(F^C)^2 + \alpha F^C + \beta I = 0 \tag{2.4}$$

In the view of equations (1.2), (2.4) and [20], we can easily say that the rank of F^C is $2r$ if and only if the rank of F is r. Therefore , we have the following theorems:

Theorem 2.1 *Let $F \in \Im^1_1$ be a general quadratic structure in M , then its complete lift F^C is also general quadratic structure in TM [17, 16].*

Theorem 2.2 *The general quadratic structure F of rank r in M if and only if its complete lift F^C is of rank $2r$ in TM .*

Let F be a general quadratic structure of rank r in M. Then the complete lift l^C of l and m^C of m are complementary projection tensors in TM. Thus there exist two complementary distributions D_{l^C} and D_{m^C} determined by l^C and m^C respectively in TM. The distributions D_{l^C} and D_{m^C} are respectively the complete lifts of D_l^C and D_m^C of D_l and D_m [2].

3 INTEGRABILITY CONDITIONS OF GENERAL QUADRATIC STRUCTURE IN THE TANGENT BUNDLE

Let F be the general quadratic structure that is $F^2 + \alpha F + \beta I = 0$. Then the Nijenhuis tensor N of F is a tensor of type (1,2) given by [20]

$$N(X,Y) = [FX,FY] - F[FX,Y] - F[X,FY] + F^2[X,Y]. \tag{3.1}$$

Let N^C be the Nijenhuis tensor of F^C in TM , then we have

$$N^C(X^C,Y^C) = [F^C X^C, F^C Y^C] - F^C[F^C X^C, Y^C] - F^C[X^C, F^C Y^C] + (F^C)^2[X^C, Y^C]. \tag{3.2}$$

Let $X, Y \in \Im^1_0(M)$ and $F \in \Im^1_1(M)$, we have

$$[X^C, Y^C] = [X, Y]^C$$

$$(X+Y)^C = X^C + Y^C \tag{3.3}$$

$$F^C X^C = (FX)^C.$$

In the viw of equations (1.4) and (3.5), we get

$$F^C l^C = (Fl)^C = F^C$$

$$F^C m^C = (Fm)^C = 0 \tag{3.4}$$

Theorem 3.1 *The following identities holds:*

$$N^C(m^C X^C, m^C Y^C) = (F^C)^C[m^C X^C, m^C Y^C], \tag{3.5}$$

$$m^C N^C(X^C, Y^C) = m^C[F^C X^C, F^C Y^C], \tag{3.6}$$

$$m^C(l^C X^C, l^C Y^C) = m^C[F^C X^C, F^C Y^C], \tag{3.7}$$

$$m^C N^C((F^2 - \alpha F)^C X^C, (F^2 - \alpha F)^C Y^C) = \beta^2 m^C N^C(l^C X^C, l^C Y^C). \tag{3.8}$$

Proof: The proof of equations (3.5) to (3.8) follow by virtue of equations (1.4), (3.4) and (3.1).

Theorem 3.2 *Let* $X, Y \in \mathfrak{s}^1(M)$, *the following conditions are equivalent*

$m^C N^C(X^C, Y^C) = 0$
$m^C N^C(l^C X^C, l^C Y^C) = 0$
$m^C N^C((F^2 - \alpha F)^C X^C, (F^2 - \alpha F)^C Y^C) = 0.$

Proof: In consequence of equation (3.8), we have

$$N^C(l^C X^C, l^C Y^C) = 0 \leftrightarrow N^C((F^2 - \alpha F)^C X^C, (F^2 - \alpha F)^C Y^C) = 0$$

Now the right sides of the equations (3.6), (3.7) are equal which in view of the last equation shows that conditions (a), (b), and (c) are equivalent.

Theorem 3.3 *The complete lift* D_m^C *in* TM *of a distribution* D_m *in* M *is integral if* D_m *is integrable in* M.

Proof: The distribution D_m is integral if and only if [20]

$$l[mX, mY] = 0 \tag{3.9}$$

for all $X, Y \in \mathfrak{s}(M)$, where $l = I - m$. Operating complete lift of both sides and using (3.5), we get

$$l^C[m^C X^C, m^C Y^C] = 0 \tag{3.10}$$

for all $X, Y \in \mathfrak{s}(M)$, where $l^C = (I + m)^C = I + m^C$ is the projection tensor complementary to m. Thus the condition (3.9) implies (3.10).

Theorem 3.4 *The complete lift* D_m^C *in* TM *of a distribution* D_m *in* M *is integral if* $l^C N^C(m^C X^C, m^C Y^C) = 0$, *or equivalently* $N^C(m^C X^C, m^C Y^C) = 0$, *for all* $X, Y \in \mathfrak{s}(M)$.

Proof: The distribution D_m is integral in M if and only if [20]

$N(mX, mY) = 0$

for all $X, Y \in \mathfrak{s}(M)$. By virtue of condition (3.5), we have

$N^C(m^C X^C, m^C Y^C) = (F^2)^C(m^C X^C, m^C Y^C)$

Multiplying throughout by l^C, we get

$l^C N^C(m^C X^C, m^C Y^C) = (F^2)^C l^C(m^C X^C, m^C Y^C)$

In view of (3.10), the above relation becomes

$$l^C N^C(m^C X^C, m^C Y^C) = 0 \tag{3.11}$$

Also, we have

$$m^C N^C(m^C X^C, m^C Y^C) = 0 \tag{3.12}$$

Adding equations (3.11) and (3.12), we get

$$(l^C + m^C)N^C(m^C X^C, m^C Y^C) = 0$$

Since $l^C + m^C = I^C = I$, we have

$$N^C(m^C X^C, m^C Y^C) = 0$$

.

Theorem 3.5 *Let the distribution D_l be integrable in M, that is $mN(X,Y) = 0$ for all $X, Y \in \mathfrak{s}_0^1(M)$. Then the distribution D^C is integrable in TM if and only if the one of the conditions of Theorem (3.2) is satisfied.*

Proof: The distribution D_l is integral in M if and only if

$mN(lX, lY) = 0$

Thus distribution D^C_l is integrable in TM if and only if

$m^C N^C(l^C X^C, l^C Y^C) = 0,$

Thus the theorem follows by making use of equation (3.8).

Theorem 3.6 *Let complete lift F^C of a general quadratic structure F in M is partially integrable in TM if and only if F is partially integrable in M.*

Proof: The general quadratic structure F in M is partially integrable if and only if
$N(lX, lY) = 0, \forall X, Y \in \mathfrak{s}_0^1(M)$. (3.13)

In view of the equations (1.4) and (3.1), we obtain

$N^C(l^C X^C, l^C Y^C) = (N(lX, lY))^C$

which implies

$$N^C(l^C X^C, l^C Y^C) = 0 \Leftrightarrow N(lX, lY) = 0$$

Also from Theorem (3.2), $N^C(l^C X^C, l^C Y^C) = 0$ is equivalent to

$$N^C((F^2 + \alpha F)^C, (F^2 + \alpha F)^C X^C, (F^2 + \alpha F)^C, (F^2 + \alpha F)^C Y^C = 0.$$

Theorem 3.7 *Let complete lift F^C of a general quadratic structure F in M is partially integrable in TM if and only if F is partially integrable in M.*

Proof: A necessary and sufficient condition for a general quadratic structure in M to be integrable is that

$$(N(X, Y)) = 0 \qquad (3.14)$$

for all $X, Y \in \mathfrak{s}^1(M)$.

In view of equation (3.1), we get

$$N^C(X^C, Y^C) = (N(X, Y))^C.$$

Therefore, with the help of equation (3.14) we obtain the result.

Now, We shall prove some theorems on horizontal lift of the general quadratic structure. Suppose that there are tensor fields S and $\nabla_\gamma S$ in M and TM re- spectively with affine connection ∇ are given by

$$S = S^{i\dots h}_{k\dots j} \frac{\partial}{\partial x^i} \otimes \dots \otimes \frac{\partial}{\partial x^h} \otimes dx^k \otimes \dots \otimes dx^j$$

$$\nabla_\gamma S = y^l \nabla_\gamma S^{i\dots h}_{k\dots j} \frac{\partial}{\partial x^i} \otimes \dots \otimes \frac{\partial}{\partial y^h} \otimes dx^k \otimes \dots \otimes dx^j$$

corresponding to the induced coordinates (x^h, y^h) in $\pi^{-1}(U)$[20].
Now, we define the horizontal lift S^H of a tensor field S in M to TM by

$$S^H = S^C - \nabla_\gamma S.$$

Theorem 3.8 *Let $F \in s^1$ be a general quadratic structure in M, then its horizontal lift F^H is also general quadratic structure in TM.*

Proof: If $P(t)$ is a polynomial in one variable t, then we have [20]

$$(P(F))^H = P(F^H) \tag{3.15}$$

for all $F \in s^1(M)$.

Operating the horizontal lifts of both sides of equation (1.2), we get

$(F^2 + \alpha F + \beta I)^H = 0$ $(F^2)^H + (\alpha F)^H + \beta I^H = 0$

In the view of (3.15) and $I^H = I$, we get

$$(F^H)^2 + \alpha F^H + \beta I = 0 \tag{3.16}$$

which shows that F^H is a general quadratic structure in TM [8]. In the view of equations (1.2) and (3.16), we can easily say that the rank of F^H is $2r$ if and only if the rank of F is r. Therefore , we have the following theorem: Also, let I be identity tensor field of type (1,1) in M. Then

Theorem 3.9 *The general quadratic structure F of rank r in M if and only if its complete lift F^H is of rank $2r$ in TM.*

Let m be a projection tensor field of type (1,1) in M defined by (1.4), then there exists in M a distribution D determined by m.Also

$$m^2 = m$$

In view of (3.15), we get

$$(m^H)^2 = m^H$$

Thus, m^H is also a projection in TM. Hence there exists in TM a dis- tribution D^H corresponding to m^H, which is called the horizontal lift of the distribution D.

4 Prolongation of a general quadratic structure in third tangent bundle T_3M

Let T_3M be the third order tangent bundle over M and let F^{III} be the third lift on F in T_3M. Then for any F, $tt \in s^1(M)$, we have [18, 19]

$$\begin{aligned}(tt^{III}F^{III})X^{III} &= (tt^{III}(F^{III}X^{III}) \\ &= (tt^{III}(FX)^{III}) \\ &= (tt(FX))^{III} \\ &= (ttF)^{III}X^{III} \qquad (4.1)\end{aligned}$$

for all $X \in s^1(M)$. Thus we have

$$tt^{III}F^{III} = (ttF)^{III}$$

If $P(t)$ is a polynomial in one variable t, then we have [20]

$$(P(F))^{III} = P(F^{III}) \qquad (4.2)$$

for all $F \in s^1(M)$.

Theorem 4.1 *Let $F^1(M)$ be a general quadratic structure in M, then the third lift F^{III} is also general quadratic structure in T_3M.*

Proof: If $P(t)$ is a polynomial in one variable t, then we have [20]

$$(P(F))^{III} = P(F^{III}) \qquad (4.3)$$

for all $F \in s^1(M)$. Operating the third lifts of both sides of equation (1.2), we get

$$(F^2 + \alpha F + \beta I)^{III} = 0 \; (F^2)^{III} + (\alpha F)^{III} + \beta I^{III} = 0$$

In the view of (4.3) and $I^{III} = I$, we get

$$(F^{III})^2 + \alpha F^{III} + \beta I = 0 \qquad (4.4)$$

which shows that F^{III} is a general quadratic structure in T_3M.

Theorem 4.2 *The third lift F^{III} is integrable in T_3M if and only if F is integrable in M.*

Proof: Let N^{III} and N be Nijenhuis tensors of F^{III} and F respectively. Then we have

$$N^{III}(X, Y) = (N(X, Y))^{III}. \qquad (4.5)$$

since general quadratic structure is integrable in M if and only if $N(X, Y) = 0$. then from (4.5), we get

$$N^{III}(X, Y) = 0. \qquad (4.6)$$

Thus F^{III} is integrable if and only if F is integrable in M.

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