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MATHEMATICAL ANALYSIS OF SIR DEMOGRAPHY MODEL USING TAYLOR SERIES METHOD

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Abstract: This paper focuses on the analysis of a first-order non-linear differential equation for SIR demography model. The mathematical model is constructed with two distinct forms of compartmentalization: without demography and with demography. This study is specifically centered around analytically solving the model that incorporates with demographic aspects, employing the Taylor Series Method (TSM). The TSM offers a straightforward way to determine the components of the series. As the order of approximation increases, the accuracy of the method also improves. Additionally, the influence of various parameters on the susceptible, infected, and recovered for the demography is also showcased.

Keywords: SIR demography model, Taylor Series Method (TSM), First order non-linear differential equation, Mathematical modelling.

Introduction:

The SIR model, an abbreviation for Susceptible-Infectious-Recovered model, serves as a fundamental epidemiological framework employed to depict the propagation of infectious diseases throughout a population [2, 3, 8]. It delineates the population into three distinct compartments:

- Susceptible (S): This category comprises individuals who are vulnerable to the disease but have yet to contract it.
- Infectious (I): This group encompasses individuals who are presently infected and possess the ability to transmit the disease to susceptible individuals.
- Recovered (R): This compartment includes individuals who have recuperated from the disease and have developed immunity, either through surviving the infection or undergoing treatment.

The dynamics of the SIR model are governed by a set of differential equations that clarify the evolution of the three compartments over time, particularly in the context of the demography model. In this framework, the transmission of the disease from susceptible individuals to infectious ones is typically represented by the parameter β (beta), and the recovery rate is symbolized by the parameter γ (gamma). Furthermore, the parameter μ (mu) is used to denote the mortality rate [6].

An analytical solution for the SIR model without the inclusion of a demography model has already been derived in a prior work [1, 5, 7].

The primary aim of this study is to employ analytical methods to address the SIR model, incorporating a demography component. Analytical solutions for the SIR model with demographic factors are derived through the effective application of the Taylor Series Method (TSM). The results obtained through this analytical approach are then meticulously compared with numerical simulations, contributing to a comprehensive insight into the system's dynamics.

Problem Description:

The most straightforward and commonly adopted method for introducing demography into the SIR model is by assuming a natural lifespan for the host population, often represented as 1/μ years. As a result, the parameter μ determines the rate at which individuals in any epidemiological compartment experience natural mortality. It's important to emphasize that this parameter is unrelated to the disease and doesn't reflect the infectious agent's pathogenicity. Traditionally, μ has also been considered to represent the overall population's crude birth rate, thus maintaining the stability of the total population size over time.

$$\frac{dS}{dt} = \mu - \beta SI - \mu S \quad (1)$$

$$\frac{dI}{dt} = \beta SI - \gamma I - \mu I \quad (2)$$

$$\frac{dR}{dt} = \gamma I - \mu R \quad (3)$$

With the corresponding initial conditions:

$$S(0) = a, I(0) = b, R(0) = c \quad (4)$$

where S(t) represents the susceptible population at time t, I(t) represents the infected population at time t, and R(t) represents the recovered population at time t. β stands for the transmission rate, γ stands for the recovery rate, μ stands for the mortality rate. These parameter definitions collectively construct a mathematical foundation for understanding disease transmission and progression within the SIR model.

Analytical expressing using TSM:

In 1715, Brooke Taylor introduced the Taylor Series Method (TSM), which has been instrumental in solving numerous differential problems throughout history. The TSM has been applied to address a range of equations, including the Lane-Emden equation [4], certain second-kind integral equations, and Fractal Bratu type equations. It has been a preferred choice for many scholars seeking analytical solutions to nonlinear equations in various fields due to its high accuracy [9]-[11].

In this study, we employ the Taylor series approach to solve the first-order differential equation of the SIR demography model (1-3) with the provided initial conditions (4). The detailed application of this method to the given model is elaborated in Appendix 1.

The TSM derivatives according to the SIR demography model can be written as:

$$S(t) = \sum_{n=0}^{\infty} \left.\frac{d^n S}{dt^n}\right|_{t=0} \frac{t^n}{n!} = S(0) + \frac{S'(0)}{1!}t + \frac{S(0)}{2!}t^2 + \cdots \quad (5)$$

$$I(t) = \sum_{n=0}^{\infty} \left.\frac{d^n I}{dt^n}\right|_{t=0} \frac{t^n}{n!} = I(0) + \frac{I'(0)}{1!}t + \frac{I''(0)}{2!}t^2 + \cdots \quad (6)$$

$$R(t) = \sum_{n=0}^{\infty} \left.\frac{d^n R}{dt^n}\right|_{t=0} \frac{t^n}{n!} = R(0) + \frac{R'(0)}{1!}t + \frac{R''(0)}{2!}t^2 + \cdots \quad (7)$$

By employing the TSM, we derive the subsequent analytical solutions as follows:

$$S(t) = a + (\mu - \beta ab - a\mu)t + (-\beta(-ab\beta - a\mu + \mu)b - \beta a(\beta ab - (\gamma + \mu)b - \mu(-ab\beta - a\mu + \mu)))\frac{t^2}{2!} \quad (8)$$

$I(t) = b + (\beta ab - (\gamma + \mu)b)t + (\beta(-ab\beta - a\mu + \mu)b + \beta a(\beta ab - (\gamma + \mu)b) - (\gamma + \mu)(\beta ab - (\gamma + \mu)b))\frac{t^2}{2!}$ (9)

$R(t) = c + (\gamma b - \mu c)t + (\gamma(\beta ab - (\gamma + \mu)b) - \mu(\gamma b - \mu c))\frac{t^2}{2!}$ (10)

Numerical Simulation:

To solve the system of first-order nonlinear differential equations, we employed the fourth-order Runge-Kutta technique. The MATLAB software program was used for obtaining the numerical solution. To evaluate the solution's accuracy, we conducted a comparison between these numerical results and the analytical solutions obtained through the Taylor Series Method. Graphical representations of the analytical concentrations S, I, and R were generated alongside their corresponding numerical results for specific parameter values. This comparative analysis demonstrated a favorable agreement between our approximate analytical solution and the obtained numerical results.

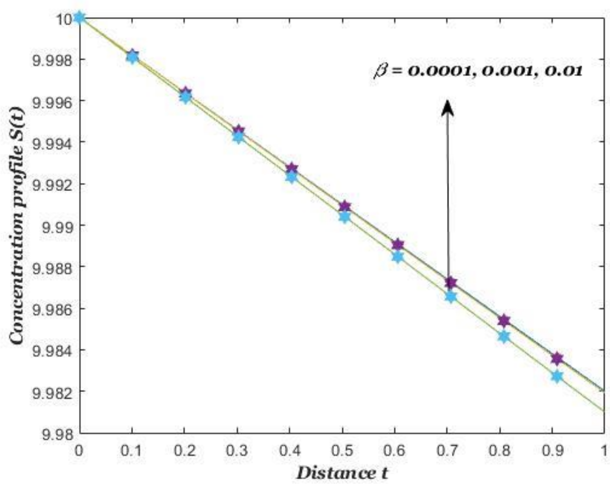

Figure 1: Comparison of analytical expression obtained by LADM eqn.(8) and numerical simulation for the concentration S(t) when $a = 10, b = 0.01, c = 0.05, \mu = 0.002, \gamma = 0.045$. Solid line represents numerical simulation and *** represents eqn.(8).

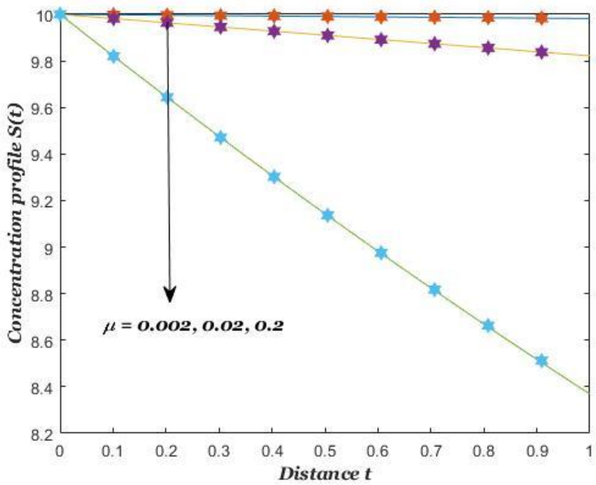

Figure 2: Comparison of analytical expression obtained by LADM eqn.(8) and numerical simulation for the concentration S(t) when $a = 10, b = 0.01, c = 0.05, \beta = 0.0001, \gamma = 0.045$. Solid line represents numerical simulation and *** represents eqn.(8).

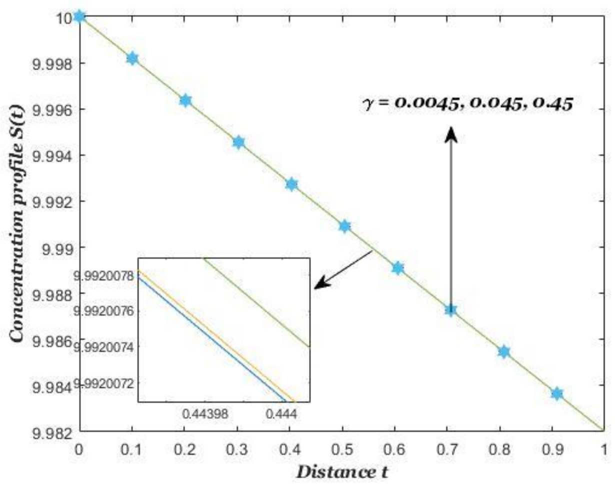

Figure 3: Comparison of analytical expression obtained by LADM eqn.(8) and numerical simulation for the concentration S(t) when $a = 10, b = 0.01, c = 0.05, \mu = 0.002, \beta = 0.0001$. Solid line represents numerical simulation and *** represents eqn.(8).

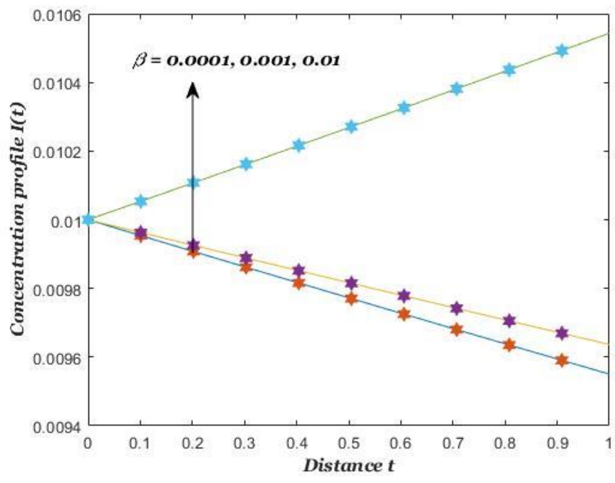

Figure 4: Comparison of analytical expression obtained by LADM eqn.(9) and numerical simulation for the concentration I(t) when $a = 10, b = 0.01, c = 0.05, \mu = 0.002, \gamma = 0.045$. Solid line represents numerical simulation and *** represents eqn.(9).

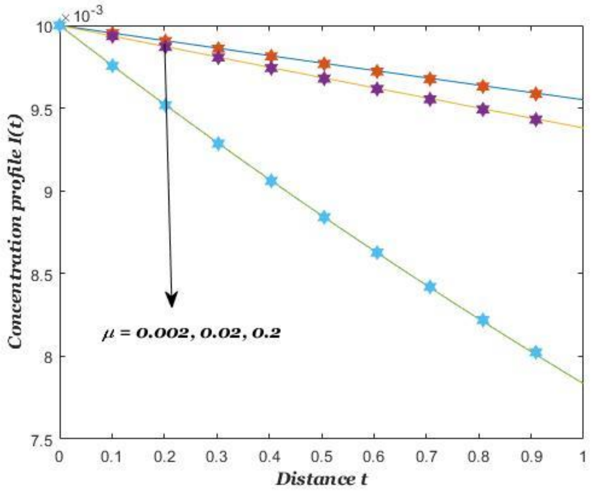

Figure 5: Comparison of analytical expression obtained by LADM eqn.(9) and numerical simulation for the concentration I(t) when $a = 10, b = 0.01, c = 0.05, \beta = 0.0001, \gamma = 0.045$. Solid line represents numerical simulation and *** represents eqn.(9).

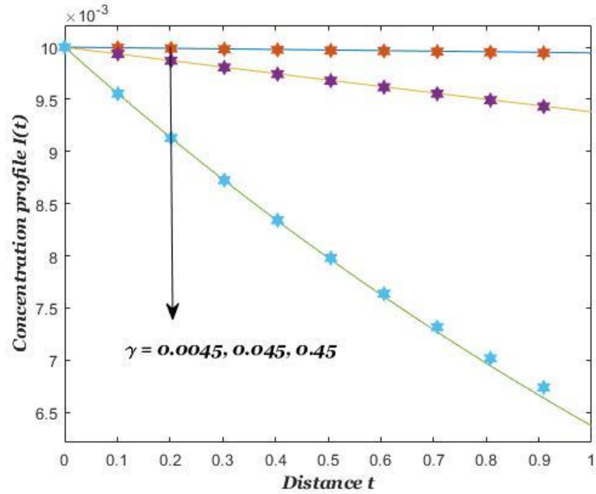

Figure 6: Comparison of analytical expression obtained by LADM eqn.(9) and numerical simulation for the concentration I(t) when $a = 10, b = 0.01, c = 0.05, \mu = 0.002, \beta = 0.0001$. Solid line represents numerical simulation and *** represents eqn.(9).

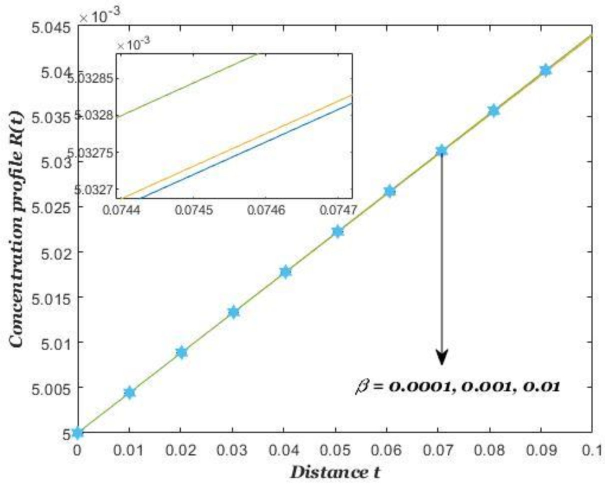

Figure 7: Comparison of analytical expression obtained by LADM eqn.(10) and numerical simulation for the concentration R(t) when $a = 10, b = 0.01, c = 0.05, \mu = 0.002, \gamma = 0.045$. Solid line represents numerical simulation and *** represents eqn.(10).

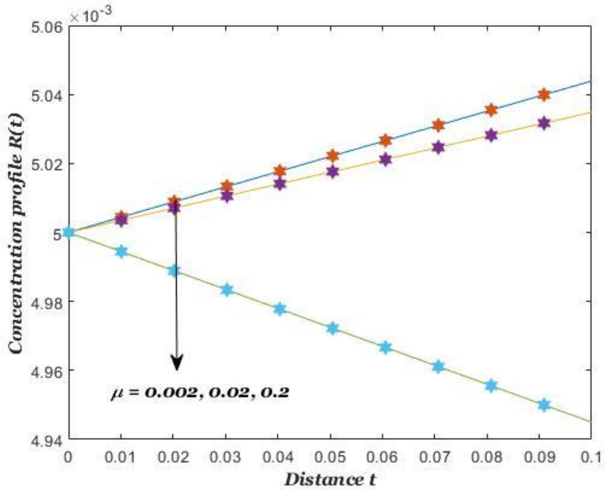

Figure 8: Comparison of analytical expression obtained by LADM eqn.(10) and numerical simulation for the concentration R(t) when $a = 10, b = 0.01, c = 0.05, \beta = 0.0001, \gamma = 0.045$. Solid line represents numerical simulation and *** represents eqn.(10).

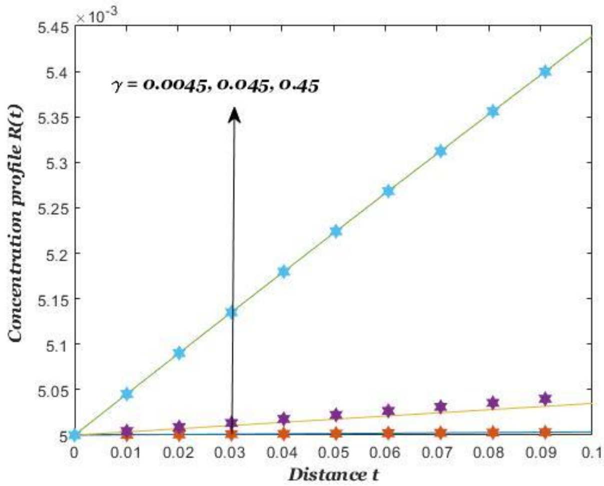

Figure 9: Comparison of analytical expression obtained by LADM eqn.(10) and numerical simulation for the concentration R(t) when $a = 10, b = 0.01, c = 0.05, \mu = 0.002, \beta = 0.0001$. Solid line represents numerical simulation and *** represents eqn.(10).

Table 1: Comparison of analytical and numerical simulation for the SIR demography model when $a = 10, b = 0.01, c = 0.05, \mu = 0.002, \beta = 0.0001, \gamma = 0.045$.

	T(t)			I(t)			V(t)		
t	Num.	TSM	Error %	Num.	TSM	Error %	Num.	TSM	Error %
0	10.0000	10.0000	0.0000	0.0100	0.0100	0.0000	0.0050	0.0050	0.0000
0.2	9.9640	9.9640	0.0000	0.0098	0.0098	0.0000	0.0050	0.0050	0.0000
0.4	9.9282	9.9282	0.0000	0.0097	0.0097	0.0000	0.0051	0.0051	0.0000
0.6	9.8926	9.8926	0.0000	0.0096	0.0096	0.0000	0.0052	0.0052	0.0000
0.8	9.8571	9.8571	0.0000	0.0095	0.0095	0.0000	0.0052	0.0052	0.0000
1	9.8217	9.8217	0.0000	0.0093	0.0093	0.0000	0.0053	0.0053	0.0000
Average error %			0.0000	Average error %		0.0000	Average error %		0.0000

Discussion:

We utilized numerical simulations to juxtapose against the analytical solutions acquired through the Laplace Adomian Decomposition Method. The comparison of these two datasets is visually presented in Figures 1 to 9 across a range of parameter values.

Figures 1, 4, and 7 provide insights into the impact of alterations in the transmission rate (β) on the susceptible population (S), infected population (I), and recovered population (R). As the parameter β increases, the susceptible population (S(t)) decreases, while the infected population (I(t)) and the recovered population (R(t)) increase, underscoring the significance of this parameter.

Similarly, Figures 2, 5, and 8 illustrate the consequences of variations in the mobility rate (μ) on the susceptible (S), infected (I), and recovered (R) populations. An increase in the parameter μ results in a decline in the susceptible population (S(t)), along with reductions in the infected population (I(t)) and the recovered population (R(t)), underscoring the relevance of this parameter.

Likewise, Figures 3, 6, and 9 portray the effects of changes in the recovery rate (γ) on the susceptible (S), infected (I), and recovered (R) populations. An increase in the parameter γ leads to a reduction in the infected population (I(t)), while the susceptible population (S(t)) and the recovered population (R(t)) experience increments, emphasizing the implications of this parameter.

Table 1 provides an estimation of the error between the Taylor series approach and numerical simulations. It indicates a notable agreement between the Taylor series method and numerical simulations.

Conclusion:

The primary focus of this study is to obtain an analytical solution for the SIR model with demography, utilizing the Taylor Series Method. A thorough comparative analysis between the obtained analytical solution and numerical simulations has been conducted. The findings reveal a high level of concurrence across a wide range of parameter values. Furthermore, this research underscores the efficiency of the Taylor series method in tackling nonlinear equations through the use of graphical representations.

Appendix 1: Taylor series method is used for solving eqn.(1-3). We assume that $t = 0$ in eqn.(1-3) and using initial conditions (4) we get,

$$S'(0) = \mu - \beta ab - \mu a \quad (11)$$
$$I'(0) = \beta ab - (\gamma + \mu)b \quad (12)$$
$$R'(0) = \gamma b - \mu c \quad (13)$$

Now differentiate eqn.(1-3) with respect to t we get,

$$S''(t) = -\beta S'I - \beta SI' - \mu S' \quad (14)$$
$$I''(t) = \beta S'I + \beta SI' - (\gamma + \mu)I' \quad (15)$$
$$R''(t) = \gamma I' - \mu R' \quad (16)$$

Setting up $t = 0$ in eqn.(14-16) and substituting necessary values we get,

$$S''(0) = -\beta(-ab\beta - a\mu + \mu)b - \beta a(\beta ab - (\gamma + \mu)b - \mu(-ab\beta - a\mu + \mu)) \quad (17)$$
$$I''(0) = \beta(-ab\beta - a\mu + \mu)b + \beta a(\beta ab - (\gamma + \mu)b) - (\gamma + \mu)(\beta ab - (\gamma + \mu)b) \quad (18)$$
$$R''(0) = \gamma(\beta ab - (\gamma + \mu)b) - \mu(\gamma b - \mu c) \quad (19)$$

Proceeding like this, one can obtain the Taylor series derivatives as follows:

$$S(t) = S(0) + S'(0)t + S''(0)\frac{t^2}{2!} + \cdots$$
$$I(t) = I(0) + I'(0)t + I''(0)\frac{t^2}{2!} + \cdots$$
$$R(t) = R(0) + R'(0)t + R''(0)\frac{t^2}{2!} + \cdots$$

Appendix 2: MATLAB program for Numerical simulation eqn.(1-3) with initial condition (4)

```
function demography
options= odeset ('RelTol',1e-6,'Stats','on');
%initial conditions
Xo= [1; 0.01; 0.005];
tspan = [0,1];
tic
[t,X] = ode45(@TestFunction,tspan,Xo,options);
%-----------------------------------------------------------------
figure
hold on
plot(t, X(:,1),'-')
plot(t, X(:,2),'-')
plot(t, X(:,3),'-')
legend('x1','x2','x3')
ylabel('x')
xlabel('t')
return
%-----------------------------------------------------------------
function [dx_dt]= TestFunction(t,x)
beta=0.0001; gamma=0.045; mu=0.002;
dx_dt(1)=mu-beta*x(1)*x(2)-mu*x(1);
dx_dt(2)=beta*x(1)*x(2)-gamma*x(2)-mu*x(2);
```

```
dx_dt(3)=gamma*x(2)-mu*x(3);
dx_dt = dx_dt';
return
```

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