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Adjacency Energy Of Cover Pebbling Graphs

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Abstract:

Given a distribution of pebbles onto the vertices of a connected graph G, a pebbling move is defined as the removal of two pebbles from some vertex v_i and the placement of one of those pebbles on an adjacent vertex v_j. After a sequence of pebbling moves, if we place a pebble on every vertex of the graph, then the graph is said to be **Cover Pebbled**. In this paper, we compute the energy for cover pebbling graphs. Also the lower bounds and upper bounds of energy levels of various graphs were found through cover pebbling concepts.

Keywords: cover pebbling, energy, lower bound, upper bound, adjacency matrix .
AMSCN(2010):05C12,05C25,05C38,05C76

1. Introduction

Graph pebbling was first introduced by Lagarias and Saks in order to find a more intuitive proof for the following number theoretic result of Lemke and Kleitman.

For any given integers $a_1, a_2,\ldots,a_n$ there is a nonempty subset $X \subseteq \{1,2,\ldots,n\}$ Such that $n \Big| \sum_{i\in X} a_i$ and $\sum_{i\in X} \gcd(a_i, n) \le n.$ Chung[1] successfully used this tool to prove the result and established other results concerning pebbling numbers.

By a graph, we mean a finite undirected graph without loops or multiple edges. Let G be a graph with n vertices and m edges. Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the Eigen values of the adjacency matrix of G. The spectrum of the graph G ,consisting of the numbers $\lambda_1, \lambda_2, \ldots, \lambda_n$ is the spectrum of its adjacency matrix.

The energy of the graph G is defined as $E(G) = \sum_{i=1}^{n} |\lambda_i|$

The Adjacency matrix A(G) is defined as $A(G) = a_{ij} = \begin{cases} 1 & \text{if } i \text{ and } j \text{ are adjacent} \\ o & otherwise \end{cases}$

2. Preliminaries

2.1 Pebbling graph[4]

In a simple graph, consisting of two vertices a pebbling move can be defined as the removal of two pebbles from the first vertex and placing one pebble on an adjacent vertex is called a pebbling graph.

2.2 Cover pebbling graph[2]

In a simple graph, consisting of two vertices, place a pebble on every vertex of the graph using sequence of pebbling moves is called a cover pebbling graph.

2.3. Adjacency matrix of cover pebbling graph

Adjacency matrix of cover pebbling graph G is the n x n matrix

$A_{cp}(G) = (a_{ij})$ where

$$a_{ij} = \begin{cases} 1 & \text{if } v_i v_j \in E(G) \text{ and a pebbling move occurs between } v_i \text{ and } v_j \\ o & \text{otherwise} \end{cases}$$

2.4 Notation

Consider $v_i(r,s) \xrightarrow{x} v_j(x)$

In $v_i(r,s)$, r denotes the number of pebbles initially at vertex v_i (i.e., before pebbling move) and s denotes the number of pebbles at vertex v_i after the pebbling move .Arrow mark $\rightarrow$ indicates that pebbling move is from v_i to v_j and x (above the arrow mark) indicates that x number of pebbling move occurs between v_i and v_j and $v_j(x)$ denotes the number of pebbles is x at vertex v_j after pebbling move.

2.5. Energy Graph[3]

Energy graph E(G) is the total of singular values of the eigen values of the adjacency matrix A_{cp} (G) and it is expressed as $E(G) = \sum_{i=1}^{n} |\lambda_i|$

2.4 Bounds

$$E(G) \le \sqrt{2n \sum_{i=1}^{n} |\lambda_i|^2}$$

$$E(G) \ge \sqrt{\sum_{i=1}^{n} |\lambda i|^2 + n(n-1)(\det A)^{2/n}}$$

3. Energy Cover Pebbling Graphs

3.1 Bull graph

The cover pebbling bull graph with five vertices v_1,v_2,v_3,v_4 and v_5 is shown in **Figure 3.1**

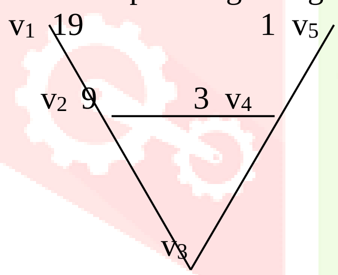

Figure 3.1 cover pebbling Bull graph

Pebbling moves are as follows:

$v_1(19,1) \xrightarrow{9} v_2(9)$ (a pebble is retained at v_1)

$v_2(9,3) \xrightarrow{3} v_4(3)$

$v_2(3,1) \xrightarrow{1} v_3(1)$ (a pebble is retained at each of v_2 and v_3)

$v_4(3,1) \xrightarrow{1} v_5(1)$ (a pebble is retained at each of v_4 and v_5)

Adjacency matrix of cover pebbling Bull graph G is

$$A_{cp}(G) = \begin{pmatrix} 0 & 9 & 0 & 0 & 0 \\ 9 & 0 & 1 & 3 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

The Eigen values are

$\lambda_1 = -0.94874, \lambda_2 = 0.94874, \lambda_3 = -9.5446,$

$\lambda_4 = 9.5446$ and $\lambda_5 = 0$

Its energy $E(G) = \sum_{i=1}^{5} |\lambda i| = 20.98668$

Determinant of $A_{cp}(G) = 0$

Also $\sum_{i=1}^{5} \lambda i^2 = 183.999$

Lower bound of energy =13.565

Upper bound of energy = 42.895

3.2 Banner Graph

The Cover pebbling Banner graph with five vertices v_1, v_2, v_3, v_4 and v_5 is shown in **Figure 3**.2

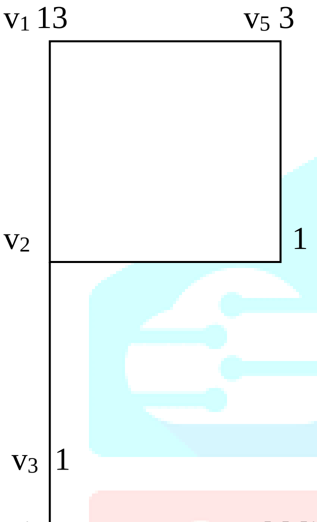

Figure 3.2 cover pebbling Banner Graph

Pebbling moves are as follows:

$v_1(13,7) \xrightarrow{3} v_5(3)$

$v_1(7,1) \xrightarrow{3} v_2(3)$ (a pebble is retained at v_1)

$v_5(3,1) \xrightarrow{1} v_4(1)$ (a pebble is retained at each of v_5 and v_4)

$v_2(3,1) \xrightarrow{1} v_3(1)$ (a pebble is retained at each of v_2 and v_3)

Its Adjacency matrix for Cover Pebbling Banner graph G is

$$A_{cp}(G) = \begin{pmatrix} 0 & 3 & 0 & 0 & 3 \\ 3 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 3 & 0 & 0 & 1 & 0 \end{pmatrix}$$

Determinant of $A_{cp}(G)=0$

The Eigen values are $\lambda_1 = 1, \lambda_2 = -1, \lambda_3 = -4.3588$

$\lambda_4 = 4.3588$ and $\lambda_5 = 0$

Energy E(G)=10.7177

Also $\sum_{i=1}^{5} \lambda_i^2 = 39.998$

Lower bound of energy =6.324

Upper bound of energy = 19.999

3.3 Caterpillar Graph:

The cover pebbling caterpillar graph with nine vertices v_1,v_2, v_3, v_4, v_5, v_6, v_7, v_8 and v_9 is shown in **Figure 3.3**

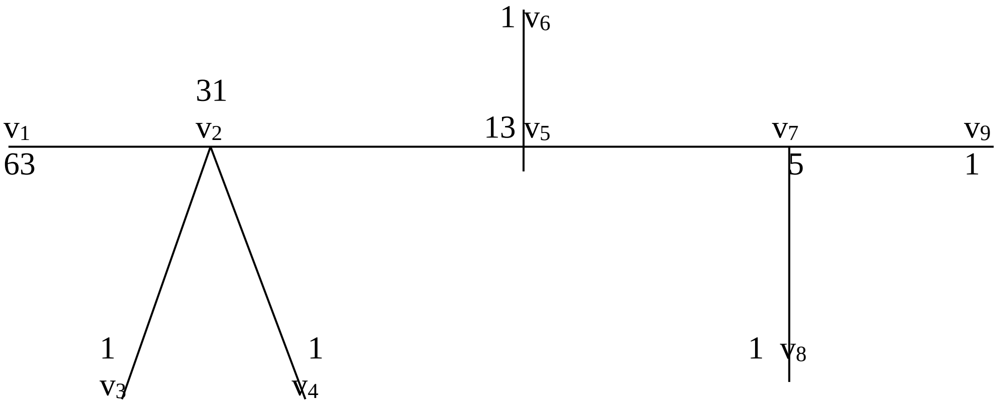

Figure 3.3 cover pebbling caterpillar graph

Pebbling moves are as follows:

$v_1(63,1)\xrightarrow{31} v_2(31)$ (a pebble is at v_1)

$v_2(31,29)\xrightarrow{1} v_3(1)$ (a pebble is at v_3)

$v_2(29,27)\xrightarrow{1} v_4(1)$ (a pebble is at v_4)

$v_2(27,1)\xrightarrow{13} v_5(13)$ (a pebble is at v_2)

$v_5(13,11)\xrightarrow{1} v_6(1)$ (a pebble is at v_6)

$v_5(11,1)\xrightarrow{5} v_7(5)$ (a pebble is at v_5)

$v_7(5,3)\xrightarrow{1} v_8(1)$ (a pebble is at v_8)

$v_7(3,1)\xrightarrow{1} v_9(1)$ (a pebble is at each of v_7 and v_9)

The Adjacency matrix of cover pebbling caterpillar graph G is

$$A_{cp}(G)=\begin{pmatrix} 0 & 31 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 31 & 0 & 1 & 1 & 13 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 13 & 0 & 0 & 0 & 1 & 5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 5 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

The Eigen values are $\lambda_1=0, \lambda_2=0,$

$\lambda_3=-33.704, \lambda_4=-4.895, \lambda_5=-0.266$

$\lambda_6=0.266, \lambda_7=4.895, \lambda_8=33.704$

Energy E(G)=77.73

Determinant of A_{cp}(G)=0

$$\sum_{i=1}^{9}\lambda_i^2 = 2319.983$$

Lower bound = 48.166

Upper bound = 204.3519

3.4 Firecracker graph

The cover pebbling Firecracker graph with four vertices v_1,v_2,v_3 and v_4 is shown in **Figure 3.4**

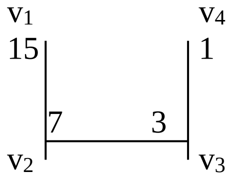

Figure 3.4 cover pebbling firecracker graph

Pebbling moves are as follows:

$v_1(15,1) \xrightarrow{7} v_2(7)$ (a pebble is at v_1)

$v_2(7,1) \xrightarrow{3} v_3(3)$ (a pebble is at v_2)

$v_3(3,1) \xrightarrow{1} v_4(1)$ (a pebble is at each of v_3 and v_4)

Adjacency matrix $A_{cp}(G)$ of cover pebbling firecracker graph G is given by

$$A_{cp}(G) = \begin{pmatrix} 0 & 7 & 0 & 0 \\ 7 & 0 & 3 & 0 \\ 0 & 3 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

The Eigen values are $\lambda_1 = -0.91789, \lambda_2 = 0.91789, \lambda_3 = -7.626104, \lambda_4 = 7.626104$

Energy E(G)=17.088

Determinant of $A_{cp}(G)$=49

$$\sum_{i=1}^{4} \lambda_i^2 = 118$$

Lower bound = 14.213, Upper bound =30.7246

3.5 Sunlet graph

The Sunlet graph with six vertices v_1,v_2, v_3, v_5, and v_6 is shown in **Figure 3.5**

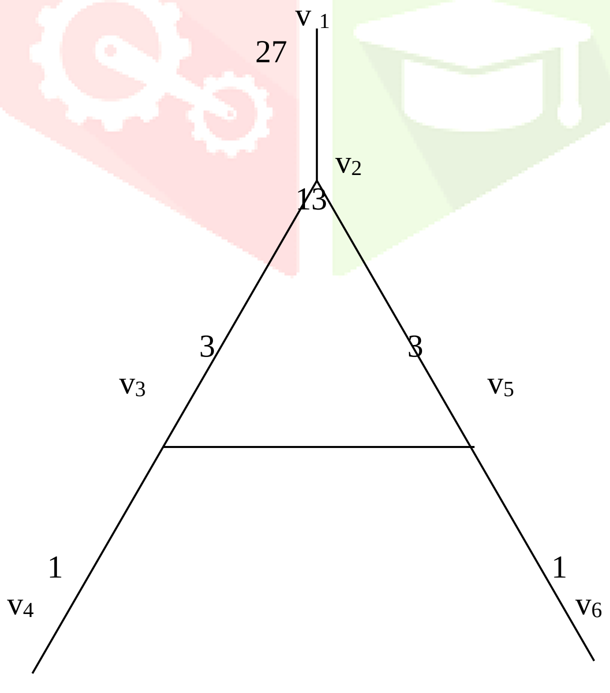

Figure 3.5 Cover Pebbling Sunlet Graph

Pebbling moves are as follows:

$v_1(27,1) \xrightarrow{13} v_2(13)$ (a pebble is at v_1)

$v_2(13,7) \xrightarrow{3} v_3(3)$

$v_2(7,1) \xrightarrow{3} v_5(3)$ (a pebble is at v_2)

$v_3(3,1) \xrightarrow{1} v_4(1)$ (a pebble is at each of v_3 and v_4)

$v_5(3,1) \xrightarrow{1} v_6(1)$ (a pebble is at each of v_5 and v_6)

Its Adjacency matrix for cover pebbling sunlet graph G is

$$A_{cp}(G) = \begin{pmatrix} 0 & 13 & 0 & 0 & 0 & 0 \\ 13 & 0 & 3 & 0 & 3 & 0 \\ 0 & 3 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

The Eigen values are $\lambda_1 = 1, \lambda_2 = -1,$

$\lambda_3 = 0.95135, \lambda_4 = -0.95135, \lambda_5 = -13.677,$ and $\lambda_6 = 13.677$

Energy E(G) =31.25670

Determinant of $A_{cp}(G)$ = -169

$$\sum_{i=1}^{6} \lambda_i^2 = 376.1206$$

Lower bound =14.465

Upper bound = 67.182

3.6 Gear graph

The cover pebbling Gear graph with nine vertices $v_1,v_2,v_3,v_4,v_5,v_6,v_7,v_8$ and v_9 is shown in **Figure 3.6**

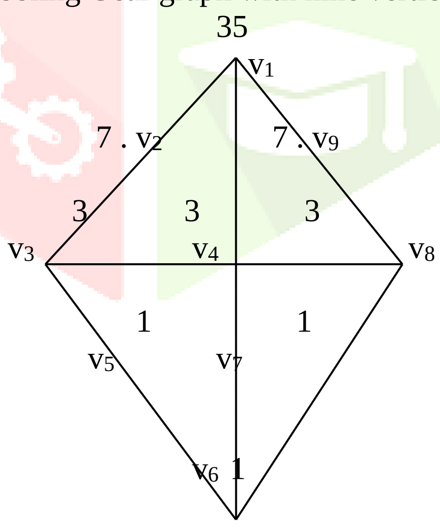

Figure 3.6 Cover Pebbling Gear Graph

Pebbling moves are as follows:

$v_1(35,21)\xrightarrow{7}v_9(7)$

$v_1(21,7)\xrightarrow{7}v_2(7)$

$v_1(7,1)\xrightarrow{3}v_4(3)$ (a pebble is retained v_1)

$v_9(7,1)\xrightarrow{3}v_8(3)$ (a pebble is retained at v_9)

$v_2(7,1)\xrightarrow{3}v_3(3)$ (a pebble is at v_2)

$v_4(3,1)\xrightarrow{1}v_6(1)$ (a pebble is at each of v_4 and v_6)

$v_8(3,1)\xrightarrow{1}v_7(1)$ (a pebble is at each of v_8 and v_7)

$v_3(3,1)\xrightarrow{1}v_5(1)$ (a pebble is at each of v_3 and v_5)

Adjacency matrix for cover pebbling Gear graph G is

$$A_{cp}(G)=\begin{pmatrix} 0&7&0&3&0&0&0&0&7\\ 7&0&3&0&0&0&0&0&0\\ 0&3&0&0&1&0&0&0&0\\ 3&0&0&0&0&1&0&0&0\\ 0&0&1&0&0&0&0&0&0\\ 0&0&0&1&0&0&0&0&0\\ 0&0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&1&0&3\\ 7&0&0&0&0&0&0&3&0 \end{pmatrix}$$

The Eigen values are $\lambda_1=0.;\lambda_2=3.162,\lambda_3=-3.162,\lambda_4=-10.744,\lambda_5=-1.471,$ $\lambda_6=-0.626,\lambda_7=0.626,\lambda_8=1.471$ and $\lambda_9=10.744,$

Energy $E(G)=\sum_{i=1}^{9}|\lambda_i|=32.0066$

Determinant of $A_{cp}(G)$=0

$\sum_{i=1}^{9}\lambda_i^2=255.976$

Lower bound = 15.999

Upper bound = 67.879

3.7 Paw Graph

The cover pebbling paw graph with four vertices v_1,v_2,v_3 and v_4 is shown in **Figure 3.7**

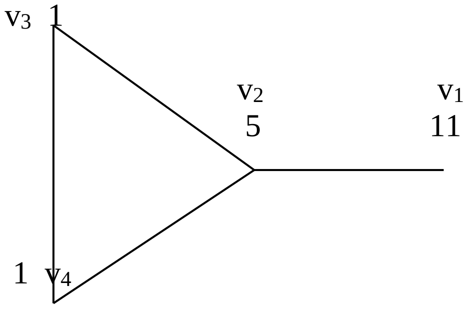

Figure 3.7 Cover Pebbling Paw Graph

Pebbling moves are as follows:

$v_1(11,1)\xrightarrow{5}v_2(5)$ (a pebble is at v_1)
$v_2(5,3)\xrightarrow{1}v_3(1)$ (a pebble is at v_3)
$v_2(3,1)\xrightarrow{1}v_4(1)$ (a pebble is at each of v_2 and v_4)

Adjacency matrix for cover pebbling paw graph G is

$$A_{cp}(G)=\begin{pmatrix}0&5&0&0\\5&0&1&1\\0&1&0&0\\0&1&0&0\end{pmatrix}$$

The Eigen value are $\lambda_1=0,\lambda_2=0,\lambda_3=3\sqrt{3},\lambda_4=-3\sqrt{3}$

Energy = $\sum_{i=1}^{4}|\lambda_i|=6\sqrt{3}=10.392$

Determinant of $A_{cp}(G)=0$

$\sum_{i=1}^{4}\lambda_i^2=54$

Lower bound = 7.348, Upper bound = 20.78

3.8 Pan graph

The cover pebbling Pan graph with five vertices v_1,v_2,v_3, v_4 and v_5 is shown in **Figure 3.8**

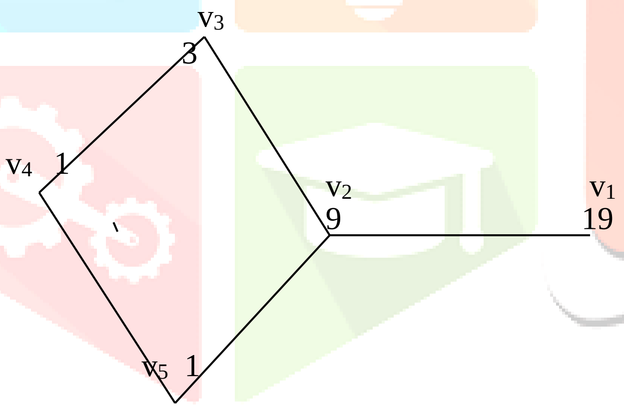

Figure 3.8 Cover Pebbling Pan Graph

Pebbling moves are as follows:
$v_1(19,1)\xrightarrow{9}v_2(9)$ (a pebble is retained at v_1)
$v_2(9,7)\xrightarrow{1}v_5(1)$ (a pebble is at v_5)
$v_2(7,1)\xrightarrow{3}v_3(3)$ (a pebble is at v_2)
$v_3(3,1)\xrightarrow{1}v_4(1)$ (a pebble is at each of v_3 and v_4)

Adjacency matrix of Cover Pebbling Pan graph G is

$$A_{cp}(G)=\begin{pmatrix} 0 & 9 & 0 & 0 & 0 \\ 9 & 0 & 3 & 0 & 1 \\ 0 & 3 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

The Eigen values are $\lambda_1 = 0, \lambda_2 = 0.9492, \lambda_3 = -0.9492, \lambda_4 = 9.545$ and $\lambda_5 = -9.545$

Energy $E(G) = \sum_{i=1}^{5} |\lambda_i| = 20.9884$

Determinant of $A_{cp}(G)=0$

$\sum_{i=1}^{5} \lambda_i^2 = 184.015$

Lower bound = 13.56

Upper bound = 42.896

3.9 Comb graph

The cover pebbling comb graph with ten vertices $v_1,v_2,v_3,v_4,v_5,v_6,v_7,v_8$,v_9 and v_{10} is shown in **Figure 3.9**

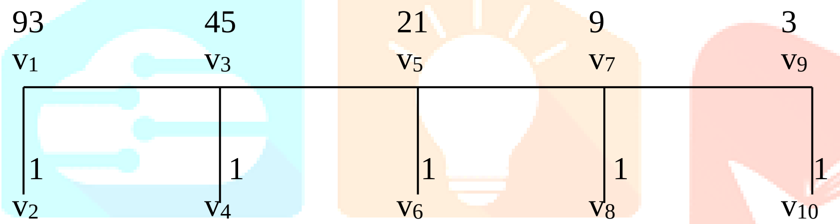

Figure 3.9 Cover Pebbling Comb Graph

Pebbling moves are as follows:

$v_1(93,91) \xrightarrow{1} v_2(1)$ (a pebble remains at v_2)

$v_1(91,1) \xrightarrow{45} v_3(45)$ (a pebble remains at v_1)

$v_3(45,43) \xrightarrow{1} v_4(1)$ (a pebble remains at v_4)

$v_3(43,1) \xrightarrow{21} v_5(21)$ (a pebble remains at v_3)

$v_5(21,19) \xrightarrow{1} v_6(1)$ (a pebble remains at v_6)

$v_5(19,1) \xrightarrow{9} v_7(9)$ (a pebble remains at v_5)

$v_7(9,7) \xrightarrow{1} v_8(1)$ (a pebble remains at v_8)

$v_7(7,1) \xrightarrow{3} v_9(3)$ (a pebble at v_7)

$v_9(3,1) \xrightarrow{1} v_{10}(1)$ (a pebble at each of v_9 and v_{10})

Adjacency matrix of cover pebbling comb graph G is

$$A_{cp}(G)=\begin{pmatrix} 0 & 1 & 45 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 45 & 0 & 0 & 1 & 21 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 21 & 0 & 0 & 1 & 9 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 9 & 0 & 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 3 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

The Eigen values are $\lambda_1=1, \lambda_2=-1, \ \lambda_3=0.114 \ \lambda_4=-0.114, \lambda_5=8.778, \lambda_6=-8.778$
$\lambda_7=0.020, \lambda_8=-0.020, \lambda_9=49.829, \lambda_{10}=-49.829$

Energy $E(G)=\sum_{i=1}^{10}|\lambda_i|=119.482$

Determinant of $A_{cp}(G)=0$

$\sum_{i=1}^{10}\lambda_i^2=5121.9926$

Lower bound = 71.568, Upper bound =320.062

3.10 Centipede graph

The cover pebbling centipede graph with six vertices v_1, v_2, v_3, v_4, v_5 and v_6 is shown in **Figure 3.10.**

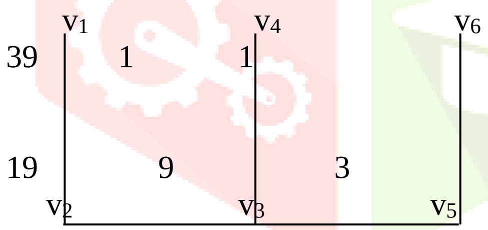

Figure 3.10 Cover Pebbling Centipede Graph

Pebbling moves are as follows:

$v_1(39,1)\xrightarrow{19}v_2(19)$ (a pebble remains at v_1)

$v_2(19,1)\xrightarrow{9}v_3(9)$ (a pebble remains at v_2)

$v_3(9,7)\xrightarrow{1}v_4(1)$ (a pebble remains at v_4)

$v_3(7,1)\xrightarrow{3}v_5(3)$ (a pebble remains at v_3)

$v_5(3,1)\xrightarrow{1}v_6(1)$ (a pebble at each of v_5 and v_6)

Adjacency matrix of cover pebbling centipede graph G is

$$A_{cp}(G)=\begin{pmatrix} 0 & 19 & 0 & 0 & 0 & 0 \\ 19 & 0 & 9 & 0 & 0 & 0 \\ 0 & 9 & 0 & 1 & 3 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

The Eigen values are

$\lambda_1 = -21.068 \; \lambda_2 = -3.007, \lambda_3 = 0.300, \lambda_4 = -0.300, \lambda_5 = 3.007, \lambda_6 = 21.068$

Energy $E(G) = \sum_{i=1}^{6} |\lambda_i| = 48.75$

Determinant of $A_{cp}(G)$=-361

$\sum_{i=1}^{6} \lambda_i^2 = 905.9852$

Lower bound =26.31, Upper bound =104.268

3.11 Ladder Graph

The cover pebbling ladder Graph with eight vertices $v_1,v_2,v_3,v_4,v_5,v_6,v_7$ and v_8 is Shown in **Figure 3.11**

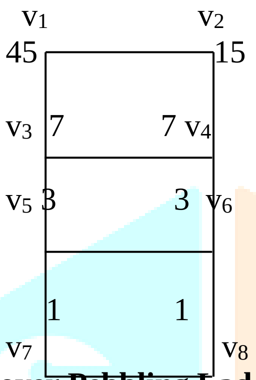

Figure 3.11 Cover Pebbling Ladder Graph

Pebbling moves are as follows:

$v_1(45,15) \xrightarrow{15} v_2(15)$

$v_1(15,1) \xrightarrow{7} v_3(7)$ (a pebble remains at v_1)

$v_2(15,1) \xrightarrow{7} v_4(7)$ (a pebble at v_2)

$v_3(7,1) \xrightarrow{3} v_5(3)$ (a pebble at v_3)

$v_4(7,1) \xrightarrow{3} v_6(3)$ (a pebble at v_4)

$v_5(3,1) \xrightarrow{1} v_7(1)$ (a pebble at each of v_5 and v_7)

$v_6(3,1) \xrightarrow{1} v_8(1)$ (a pebble at each of v_6 and v_8)

Adjacency matrix of cover pebbling ladder graph G is

$$A_{cp}(G) = \begin{pmatrix} 0 & 15 & 7 & 0 & 0 & 0 & 0 & 0 \\ 15 & 0 & 0 & 7 & 0 & 0 & 0 & 0 \\ 7 & 0 & 0 & 0 & 3 & 0 & 0 & 0 \\ 0 & 7 & 0 & 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

The Eigen values are $\lambda_1 = -17.829, \lambda_2 = -2.038, \lambda_3 = 0.295, \lambda_4 = 4.571, \lambda_5 = -4.571,$

$\lambda_6 = -0.295, \lambda_7 = 2.038, \lambda_8 = 17.829$

Energy E(G)= $\sum_{i=1}^{8} |\lambda_i| = 49.466$

Determinant of $A_{cp}(G)$=2401

$\sum_{i=1}^{8} \lambda_i^2 = 686.014$

Lower bound = 32.83, Upper bound = 104.268

3.12 Banana Tree Graph

The cover pebbling Banana tree Graph with nine vertices $v_1,v_2,v_3,v_4,v_5,v_6,v_7,v_8$, and v_9 is shown in **Figure 3.12.**

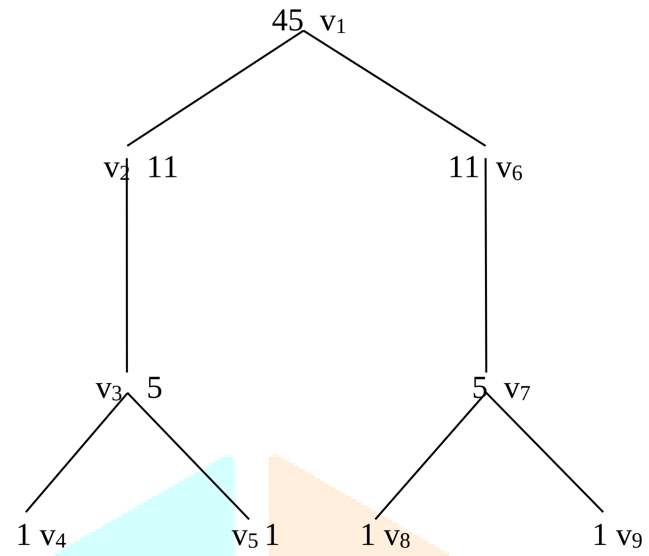

Figure 3.12 Cover Pebbling Banana Tree Graph

Pebbling moves are as follows:

$v_1(45,23)\xrightarrow{11} v_2(11)$

$v_1(23,1)\xrightarrow{11} v_6(11)$) (a pebble retains at v_1)

$v_2(11,1)\xrightarrow{5} v_3(5)$ (a pebble retains at v_2)

$v_6(11,1)\xrightarrow{5} v_7(5)$ (a pebble retains at v_6)

$v_3(5,3)\xrightarrow{1} v_4(1)$ (a pebble retains at v_4)

$v_3(3,1)\xrightarrow{1} v_5(1)$ (a pebble retains at each of v_3 and v_5)

$v_7(5,3)\xrightarrow{1} v_8(1)$ (a pebble retains at v_8)

$v_7(3,1)\xrightarrow{1} v_9(1)$ (a pebble retains at v_8 and v_9)

Adjacency matrix of cover pebbling banana tree graph G is

$$A_{cp}(G)=\begin{pmatrix} 0 & 11 & 0 & 0 & 0 & 11 & 0 & 0 & 0 \\ 11 & 0 & 5 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 5 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 11 & 0 & 0 & 0 & 0 & 0 & 5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 5 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

The Eigen values are

$\lambda_1=0,\ \lambda_2=0\ \ \lambda_3=0,\ \lambda_4=5.196,\ \lambda_5=-5.196,\ \lambda_6=16.345,\ \lambda_7=1.345,\ \lambda_8=-16.345,\ \lambda_9=-1.345$ Energy

$$E(G)=\sum_{i=1}^{9}|\lambda_i|=45.772$$

Determinant of $A_{cp}(G)$ =0

$$\sum_{i=1}^{9}\lambda_i^2=591.932$$

Lower bound =24.32, Upper bound =103.221

3.13 Butterfly graph

The cover pebbling butterfly graph with five vertices v_1,v_2,v_3,v_4 and v_5 is shown in **Figure 3.13**

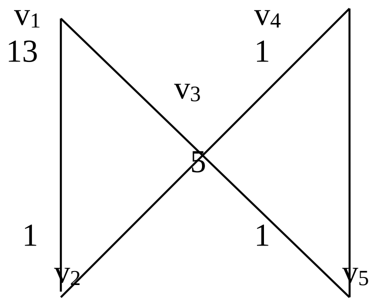

Figure 3.13 Cover Pebbling Butterfly Graph

Pebbling moves are as follows:

$v_1(13,11) \xrightarrow{1} v_2(1)$ (one pebble is at v_2)

$v_1(11,1) \xrightarrow{5} v_3(5)$ (one pebble is at v_1)

$v_3(5,3) \xrightarrow{1} v_4(1)$ (one pebble is at v_4)

$v_3(3,1) \xrightarrow{1} v_5(1)$ (one pebble is at each of v_3 and v_5)

Adjacency matrix of cover pebbling butterfly graph G is

$$A_{cp}(G) = \begin{pmatrix} 0 & 1 & 5 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 5 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

The Eigen values are

$\lambda_1 = 0.\ \lambda_2 = 0.2683, \lambda_3 = -0.2683, \lambda_4 = 5.285,$ and $\lambda_5 = -5.285,$

Energy E(G) =11.1066

Determinant of A_{cp}(G)=0

$$\sum_{i=1}^{5} \lambda_i^2 = 56.006$$

Lower bound =7.48, Upper bound =23.666

3.14 Sun graph

The cover pebbling sun graph with six vertices v_1,v_2,v_3,v_4,v_5 and v_6 is shown in **Figure 3.14**

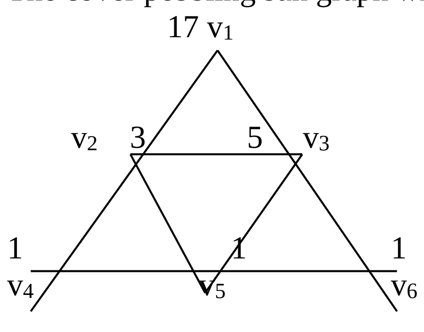

Figure 3.14 Cover Pebbling Sun Graph

Pebbling moves are as follows:

$v_1(17,7) \xrightarrow{5} v_3(5)$

$v_1(7,1) \xrightarrow{3} v_2(3)$ (a pebble is at v_1)

$v_3(5,3) \xrightarrow{1} v_5(1)$ (a pebble is at v_5)

$v_3(3,1) \xrightarrow{1} v_6(1)$ (a pebble is at each of v_3 and v_6)

$v_2(3,1) \rightarrow v_4(1)$ (a pebble is at each of v_2 and v_4)

Adjacency matrix of cover pebbling sun graph G is

$$A_{cp}(G) = \begin{pmatrix} 0 & 3 & 5 & 0 & 0 & 0 \\ 3 & 0 & 0 & 1 & 0 & 0 \\ 5 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

The Eigen values are $\lambda_1 = 0, \lambda_2 = 0, \lambda_3 = 3.1432, \lambda_4 = -3.1432, \lambda_5 = 5.208, \lambda_6 = -5.208$

Energy $E(G) = \sum_{i=1}^{6} |\lambda_i| = 16.7024$

Determinant of $A_{cp}(G)=0$

$\sum_{i=1}^{6} \lambda_i^2 = 74.004$

Lower bound = 8.60

Upper bound = 29.800

3.15 Helm graph

The cover pebbling Helm graph G with nine vertices $v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8$ and v_9 is shown in **Figure 3.15**

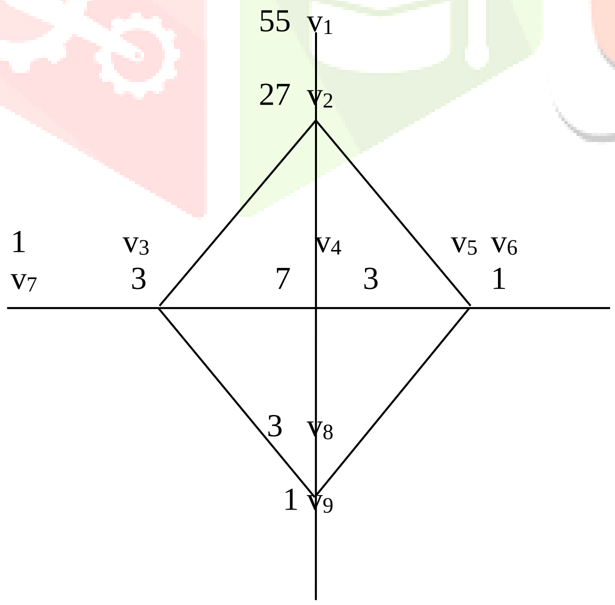

Figure 3.15 cover pebbling Helm graph

Pebbling moves are as follows:

$v_1(55,1) \xrightarrow{27} v_2(27)$ (a pebble is retained at v_1)

$v_2(27,13) \xrightarrow{7} v_4(7)$

$v_2(13,7) \xrightarrow{3} v_3(3)$

$v_2(7,1)\xrightarrow{3} v_5(3)$ (a pebble is retained at v_2)

$v_4(7,1)\xrightarrow{3} v_8(3)$ (a pebble is retained at v_4)

$v_3(3,1)\xrightarrow{1} v_7(1)$ (a pebble is retained at each of v_3 and v_7)

$v_5(3,1)\xrightarrow{1} v_6(1)$ (a pebble is retained at each of v_5 and v_6)

$v_8(3,1)\xrightarrow{1} v_9(1)$ (a pebble is retained at each of v_8 and v_9)

Adjacency matrix of cover pebbling helm graph G is

$$A_{cp}(G)=\begin{pmatrix} 0 & 27 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 27 & 0 & 3 & 7 & 3 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 7 & 0 & 0 & 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 3 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

The Eigen values of G are

$\lambda_1 = 0.\ \lambda_2 = 1,\ \lambda_3 = -1,\ \lambda_4 = -28.023,\ \lambda_5 = -2.950,\ \lambda_6 = -0.989,\ \lambda_7 = 0.999,\ \lambda_8 = -2.859,\ \lambda_9 = 28.104$

Energy E(G)=65.924

Determinant of $A_{cp}(G)$=0

$$\sum_{i=1}^{9} \lambda_i^2 = 1595.974$$

Lower Bound = $\sqrt{\sum_{i=1}^{n} |\lambda i|^2 + n(n-1)(\det A)^{2/n}}$ = 39.949

Upper Bound= $\sqrt{2n\sum_{i=1}^{n} |\lambda_i|^2}$ =169.492

4. Conclusion.

We have found the energy of cover pebbling graphs and their bounds. Also we have concluded that

$$\sqrt{\sum_{i=1}^{n} \lambda_i^2 + n(n-1)|A|^{\frac{2}{n}}} \le E(G) \le \sqrt{2n\sum_{i=1}^{n} |\lambda_i|^2}$$

That is lower bound $\le$ Energy $\le$ Upper bound

Open Problem. Find different types of energy in covering cover pebbling graph and also to find the relation between them.

5 References

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