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GRAPH THEORY: VERTEX COLORINGS AND UPPER BOUNDS

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Abstract: Graph theory is a branch of Mathematics that deals with the study of graphs, which are mathematical structures used to model relationships between objects. It consists of set of vertices and a set of edges that connect pairs of vertices. Vertex coloring is a fundamental concept that assigns colors to the vertices of a graph such that no two adjacent receive the same color. The minimum number of colors required to color the vertices of a graph is known as the chromatic number of the graph. An upper bound on the chromatic number of a Graph is a value that is guaranteed to be greater than or equal to the chromatic number. In this paper some common upper bounds for the chromatic number is studied.

Keywords: Graph coloring, Vertex coloring, Chromatic number, Color-class, Greedy coloring, Gallai-Roy-Vitaver Theorem

Introduction: Graph theory has come a long way from the days when it was considered as an off-shoot of topology. It is well known that it has been independently discovered many a time, although the earliest record mention of the subject is found in the works of Euler. Since then graph theory has been expanding its branches quite enormously, obviously due to its well-defined and interesting applications in various fields. A Graph can be used to represent almost any physical situation involving discrete objects and a relationship among them.

Graph: A graph **G** is an ordered pair (V, E) where V is a finite non-empty set of vertices and E is a set of unordered pairs of distinct vertices of V. V is called the vertex set of G i.e.,V(G) and E is called the Edge set of G i.e., E(G). The Graph with p vertices and q edges is called a (p,q) graph. p is called the order of G and q the size of G.

Graph Coloring: In its simplest form, it is a way of coloring the vertices of a graph such that no two adjacent vertices share the same color, called a vertex coloring. Similarly, an edge coloring assigns a color to each edge so that no two incident edges share the same color, and a face coloring of a planar graph assigns a color to each face or region so that no two faces that share a boundary share the same color.

Vertex coloring: When used without any qualification, a coloring of a graph is always assumed to be a *vertex coloring*, namely an assignment of colors to the vertices of the graph. Again, when used without any qualification, a coloring is nearly always assumed to be proper, meaning no two vertices are assigned the same color. Here, "adjacent" means sharing the same. A coloring using at most k colors is called a (proper) k-coloring and is equivalent to the problem of partitioning the vertex set into k or fewer.

Vertex coloring is the starting point of the subject, and other coloring problems can be transformed into a vertex version. The convention of using colors comes from graph drawings of graph colorings, where each node or edge is literally colored to indicate its mapping. In computer representations it's more typical to use nonnegative integers, and in general any mapping from the graph objects into a finite set can be used.

Chromatic number:

The least number of colors needed to color the graph is called its chromatic number. It is denoted by the symbol χ(G), where G is a graph. For example the chromatic number of a K_n of n vertices (a graph with an edge between every two vertices i.,e a Complete graph with n vertices), is $\chi(K_n) = n$. A graph that can be assigned a (proper) k-coloring is k-colorable, and it is k-chromatic if its chromatic number is exactly k.

Chromatic polynomial:

The chromatic polynomial counts the number of ways a graph can be colored using no more than a given number of colors.

Color Class:

A color class is a set of vertices of a graph which are having the same color when a coloring is done to a graph.

Definition: A k – coloring of G is a labeling f: V (G) $\to$ $\{1\ldots\ldots k\}$. The labels are colors ; the vertices with color i are a color class. A k coloring f is proper if x $\leftrightarrow$ y implies f(x) $\neq$ f(y). A graph G is k – colorable if it has a proper k – coloring. The chromatic number χ(G) is the minimum k such that, G is k – colorable if $\chi(G)= k$, then G is k – chromatic if $\chi(G) = k$ but $\chi(H) < k$ for every proper sub graph H of G, then G is color critical or k – critical.

UPPER BOUNDS:

An upper bound on the chromatic number of a Graph is a value that is guaranteed to be greater than or equal to the chromatic number.

Most upper bounds on χ (G) come from coloring algorithms. The bound

χ (G) $\leq n$ (G) where n (G) is the order of G, uses nothing about the structure of G. We can improve the bound by coloring vertices successively using the "least available" color.

Algorithm: (Greedy coloring). The *greedy coloring* with respect to a vertex ordering $v_1, \ldots\ldots v_n$ of V (G) is obtained by coloring vertices in the order $v_1, \ldots\ldots v_n$, assigning to V_i the smallest-indexed color not already used its lower-indexed neighbors.

Proposition: (Welsh – Powell) [1967] If a graph G has degree sequence $d_1 \geq \ldots\ldots \geq d_n$, then $\chi(G) \leq 1 + \max_i \min \{ d_i, i-1\}$.

Proof: we apply greedy coloring with the vertices in nonincreasing order of degree. When we color the *i* th vertex, at most min { d_i …… i-1} of its neighbors already have colors, so its color is at most 1+ min { d_i, i-1}. Maximizing this over *i* yields the upper bound.

The greedy coloring algorithm runs rapidly. It is "on – line" in the sense that it produces a proper coloring even if it sees only one new vertex at each step and must color it with no option to change earlier colors. For a random vertex ordering in a random graph, greedy coloring almost always uses only about twice as many colors as the minimum, although with a bad ordering it may use many colors on a tree.

We began with greedy coloring to underscore the constructive aspect of upper bounds on chromatic number. Other bounds follow from the properties of *k* – critical graphs but don't produce proper colorings: every *k* – chromatic graph has a *k* – critical sub graph, but we have no good algorithm for finding one. We derive the next bound using critical sub graphs; it can also be proved using greedy coloring.

Proposition: $\chi(G) \leq \Delta(G) + 1$

Proof: In a vertex ordering, each vertex has at most Δ (G) earlier neighbors, so the greedy coloring cannot be forced to use more than Δ(G) + 1 colors. This proves constructively that $\chi(G) \leq \Delta(G) + 1$. The bound $\chi(G) \leq \Delta(G) + 1$ results from every vertex ordering. By choosing the ordering carefully, we may obtain a better bound indeed, every graph G has a vertex ordering on which the greedy algorithm uses only $\chi(G)$ colors.

Lemma: If H is a *k* – critical graph, then $\delta(H) \geq k - 1$

Proof: suppose *x* is a vertex of H. Because H is *k* – critical, H – *x* is *k – 1* colorable. If $d_H(x) < k-1$, then the k-1 colors used on H – *x* do not all appear on N(*x*), and we can assign a missing one to *x* to extend the coloring to H. This contradicts our hypothesis that H has no proper *k -1*- coloring. Hence every vertex of H has degree at least *k -1*.

Proposition: If G is an interval graph, then $\chi(G) = \omega(G)$.

Proof: Order the vertices according to the left endpoints of the intervals in an interval representation. Apply greedy coloring, and suppose that *x* receives k, the maximum color assigned. Since *x* does not receive a smaller color, the left endpoint a of its interval belongs also to intervals that already have colors 1 through k-1. These intervals all share the point a, so we have a k-clique consisting of *x* and neighbors of *x* with colors 1 through k-1. Hence $\omega(G) \geq k \geq \chi(G)$. Since $\chi(G) \geq \omega(G)$ always, this coloring is optimal.

Theorem : Gallai-Roy-Vitaver Theorem

If D is an orientation of G with longest path length $l(D)$, then $\chi(G) \leq 1 + l(D)$. Furthermore, equality holds for some orientation of G.

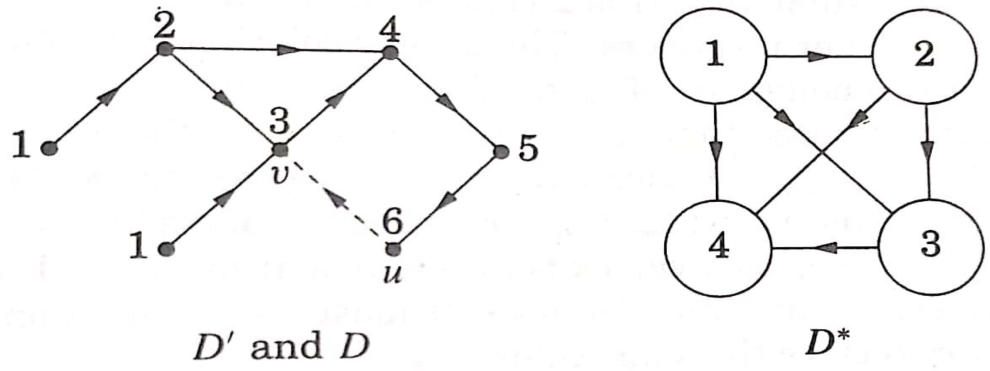
Proof: Let D be an orientation of G. Let D' be a maximal subdigraph of D that contains no cycle (in the example below, uv is the only edge of D not in D'), here D' includes all vertices of G. Color V(G) by letting $f(v)$ be 1 plus the length of the longest path in D' that ends at v.

Let P be a path in D', and let u be the first vertex of P. Every path in D' ending at u has no other vertex on P, since D' is acyclic. Therefore, each path ending at u (including the longest such path) can be lengthened along P. This implies that f strictly increases along each path in D'.

The coloring f uses colors 1 through $1+l(D')$ on $V(D')$ (which is also $V(G)$). We claim that f is a proper coloring of G. For each $uv \in E(D)$, there is a path in D' between its endpoints (since uv is an edge of D' or its addition to D' creates a cycle). This implies that $f(u) \neq f(v)$, since f increases along paths of D'.

To prove the second statement, we construct an orientation D^* such that

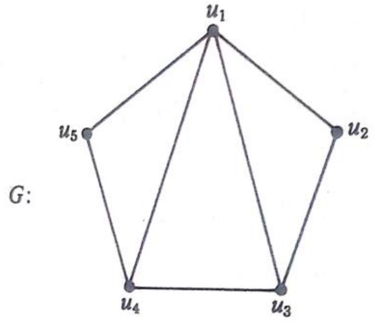
$l(D^*) \leq \chi(G) - 1$. Let f be an optimal coloring of G. For each edge uv in G, orient it from u to v in D* if and only if $f(u) < f(v)$. Since f is a proper coloring, this defines an orientation. Since the labels used by f increase along each path in D*, and there are only $\chi(G)$ labels $in\ f$, we have $l(D^*) \leq \chi(G) - 1$.



Fig(1)

Uniquely colorable graphs:

Let G be a labelled graph. Any $\chi(G)$-coloring of G induces a partition of the point set of G into $\chi(G)$ colour classes. If $\chi(G) = n$ and every n-coloring of G induces the same partition of V, then G is called uniquely *n-colorable* or simply *uniquely colorable* . The graph G of Fig(2) shown below is uniquely 3-colorable since every 3-coloring of G has the partition $\{u_1\}$, $\{u_2, u_4\}$, $\{u_3, u_5\}$ while the pentagon is not uniquely 3-colorable; indeed, five different partitions of its point set are possible.



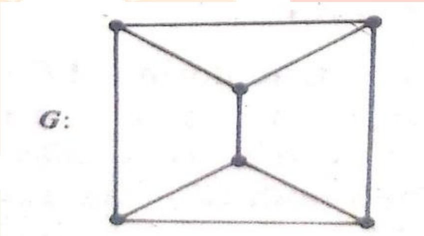
Fig(2): A uniquely colorabe graph

We begin with a few elementary observations concerning uniquely colorable graphs. First, in any n-coloring of a uniquely n-colorable graph G, every point v of G is adjacent with at least one point of every color different from that assigned to v ; for otherwise a different n-coloring of G could be obtained by recoloring v. This further implies that $\delta(G) \geq n-1$. A necessary condition for a graph to be uniquely colorable was found by Cartwright and Harary[CH2].

Theorem (a): In the n-coloring of a uniquely n-colorable graph G, the subgraph induced by the union of any two color classes is connected.

Proof : Consider an n-coloring of a uniquely n-colorable graph G, and suppose there exist two color classes of G, say C_1 and C_2, such that the subgraph S of G induced by $C_1 \cup C_2$ is disconnected. Let S_1 and S_2 be two components of S. From our earlier remarks, each of S_1 and S_2 must contain points of both C_1 and C_2. An n-coloring different from the given one can now be obtained if the color of the points in $C_1 \cap S_1$ is interchanged with the color of the points in $C_2 \cap S_1$. This implies that G is not uniquely n-colorable , which is contradiction.

The converse of Theorem(a) is not true, however. This can be seen with the aid of the 3-chromatic graph G of Fig(3) shown below. It has the property that in any 3-coloring , the subgraph induced by the union of any 2 color classes is connected, but G is not uniquely 3-colorable.



Fig(3): A counterexample to the converse of Theorem(a)

From Theorem(a), it now follows that every uniquely n-colorable graph, $n \geq 2$, is connected. However, a stronger result can be given, due to Chartrand and Geller[CG1] , that G is neither complete nor $(n-1)$-connected so that there exists a set U of $n-2$ points whose removal disconnects G. Thus, there are at least two distinct colors, say c_1 and c_2, not assigned to any point of U. By Theorem(a), a point colored c_1 is connected to any point colored c_2 by a path all of whole points are colored c_1 or c_2 . Hence, the set of points of G colored c_1 or c_2 lies within the same component of $G - U$, say G_1. Another n-coloring of G can therefore be obtained by taking any point of $G - U$ which is not G_1 and recoloring it either c_1 or c_2. This contradicts the hypothesis that G is uniquely n-colorable ; thus G is $(n-1)$-connected.

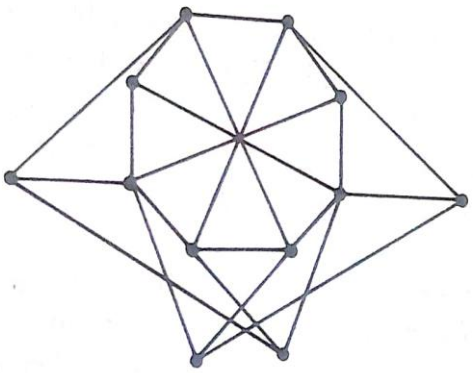
Since the union of any k color classes of a uniquely n-colorable graph, $2 \leq k \leq n$, induces a uniquely k-colorable graph, we arrive at the following consequence.

Corollary : In any n-colouring of a uniquely n-colourable graph, the subgraph induced by the union of any k color classes, $2 \leq k \leq n$, is $(k-1)$-connected.

It is easy to give examples of 3-chromatic graphs containing no triangles; indeed, we have seen in our study that for any n, there exists n-chromatic graphs with no triangles and hence no subgraphs isomorphic to K_n. In this connection, a stronger result was obtained by Harary, Hedetniemi, and Robinson [HHR1].

Theorem(b): For all n ≥ 3, there is a uniquely *n*-colorable graph which contains no subgraph isomorphic to K_n.

For *n=3*, the graph G of Fig(4) given below illustrates the theorem.



Fig(4): A uniquely 3-colorable graph having no triangles.

Naturally, a graph is uniquely 1-colorable if and only if it is 1-colorable, that is, totally disconnected. It is also well known that a graph G is uniquely **2**-colorable if and only if G is **2**-chromatic and connected. As might be expected, the information concerning uniquely *n*-colorable graphs, n ≥ 3, is very sparse. In the case where the graphs are planar, however, more can be said, although in view of the Five Color Theorem, we need to consider only the values $3 \le n \le 5$. The results in this area are due to Chartrand and Geller [CG1].

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