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Neutrosophic Spherical Cubic Soft Set And Their Applications In Decision Making

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Abstract: We introduce the concept of Neutrosophic spherical cubic soft sets (NSCSSs) which can be considered as a generalization of Neutrosophic cubic soft sets. The notions of internal Neutrosophic spherical cubic soft sets (INSCSSs), external Neutrosophic spherical cubic soft sets (ENSCSSs), P-order, P-union, P-intersection, P-AND, P-OR and R-order, R-union, R-intersection, R-AND, R-OR have been defined for Neutrosophic spherical cubic soft sets (NSCSSs). We also investigate structural properties of these operations on Neutrosophic spherical cubic soft sets (NSCSSs). It has also been proved that Neutrosophic spherical cubic soft sets (NSCSESs) satisfy commutative, associative, De Morgan's, distributive, idempotent and absorption laws. In last section, we provide the application of a Neutrosophic spherical cubic soft sets (NSCSSs) in multi-criteria decision making problem. We present a numerical examples of Neutrosophic spherical cubic soft set.

Keywords: Neutrosophic spherical Cubic soft sets, Neutrosophic spherical internal cubic soft sets, Neutrosophic spherical external cubic soft sets, interval valued spherical fuzzy sets

1 Introduction

Fuzzy set is first introduced by Zadeh[31] to represent the degree of certainty of expert's in different statements. Zadeh also proposed the concept of a linguistic variable with application in [32]. Then Peng et al.[18, 17] presented an application in multi-criteria decision-making problems. After Zadeh, Turksen[26, 27, 28] extend fuzzy set to an interval valued fuzzy set. Interval value fuzzy sets have many applications in real life such as Sambuc[19], Kohout[9], Mukherjee and Sarkar[11, 12] also gave its applications in Medical, Turksen[26, 27] in interval valued logic. Molodstov pointed out that the important existing theories viz. probability theory, fuzzy set theory, intuitionistic fuzzy set theory, rough set theory etc, which can be considered as mathematical tools for dealing with uncertainties, have their own difficulties. He further pointed out that the reason for these difficulties is, possibly, the inadequacy of the parameterization tool of the theory. Molodtsov[14] has been given soft sets technique and its applications. In 2003, P. K .Maji, R. Biswas and A. R. Roy[13] studied the theory of soft sets initiated by Molodstov.

In 2009, M. Irfan Ali et al.,[1] gave some new notions such as the restricted intersection, the restricted union, the restricted difference and the extended intersection of two soft sets along with a new notion of complement of a soft set. Sezgin and Atagun[20] studied on soft set operations. Pei and Miao[16] and Chen et al. improved the work of Maji. Soft set theory has been applied to decision making problems. Jun et al.[6, 7, 8] introduced cubic set which is basically the combination of fuzzy sets with interval valued fuzzy sets. In [14], Muhiuddin and Al-roqi[15] have introduced the concept of cubic soft sets with applications in BCI/BCK-algebras.

Sezgin and Atagun[20] studied on soft set operations. Majumdar and Samanta, worked on soft mappings[28] were proposed and many related concepts were discussed too. Moreover, the theory of soft sets has gone through remarkably rapid strides with a wide-ranging applications especially in soft decision making and some other fields such as [11, 12, 13].

Motivating from the realisms of physical life phenomenon, i.e., different sports (win/ tie/ defeat), votes like yes/ NA/ NO and making a decision. In 1999, Smrandache[22, 23, 24, 25, 25] presented a fresh idea of NSs and neutrosophic logic, which is the generality of an FS and IFS, NS is defined by (truth-membership, indeterminacy membership and falsity-membership degrees). This idea of NS creates the NS theory by providing the illustration to indeterminate. This theory is well throughout as the whole demonstration of nearly each model of all actual difficulties. Thus, vagueness is complicated in problematical questions we use FS whereas, commerce indeterminacy, we must have a neutrosophic theory. This theory has numerous applications in countless fields such as control theory, records, medicinal judgment difficulties and decision-making questions. Such types of models have been studied by several authors[2, 3, 4, 30].

Spherical fuzzy sets (SFS) were introduced by Kahraman and Gndogdu[10] as an extension of Pythagorean, neutrosophic and picture fuzzy sets. The idea behind SFS is to let decision makers to generalize other extensions of fuzzy sets by defining a membership function on a spherical surface and independently assign the parameters of that membership function with a larger domain.

In this paper, we introduce neutrosophic spherical cubic soft set and define some new notions such as internal (external) neutrosophic spherical cubic soft sets.The notion of neutrosophic spherical cubic soft set generalizes the concept of neutrosophic cubic soft set. We also investigate some of the core properties of neutrosophic spherical cubic soft set. By using these new notions we then construct a decision making method called neutrosophic spherical cubic soft method. We finally present an application which shows that the methods can be successfully applied to many problems containing uncertainties. Finally we present an application of a neutrosophic spherical cubic soft set in decision making.

Definition 1.1 *Let U be a finite universe set containing n alternatives, E; a set of criteria and X; a set of experts (or decision makers).*

A pair (β_s, E_s, X_s) is called a Neutrosophic spherical cubic soft set over U if and only if $\beta_s: E_s \times X_s \to NSCP(U)$ is a mapping into the set of all Neutrosophic spherical cubic sets in U. Neutrosophic spherical cubic soft set is defined as

$$(\beta_s, E_s, X_s) = \{\beta_s(\breve{e}, x) = \{\langle u, A_s(\breve{e}, x)(u), \lambda_s(\breve{e}, x)(u)\rangle : u \in U,\\ \breve{e}, x \in E_s \times X_s\}\}$$

where $A_{s(\breve{e},x)}(u)$ is an interval valued Neutrosophic spherical set and $\lambda_s(\breve{e}, x)(u)$ is a Neutrosophic spherical set.

2 Interval valued Neutrosophic spherical set

Let U be a non-empty set. An interval valued Neutrosophic spherical set in U is of the form,

$$A_{s(\breve{e},x)} = \left\{X: \left[T^-_{A_{s(\breve{e},x)}}, T^+_{A_{s(\breve{e},x)}}\right] \\ \left[I^-_{A_{s(\breve{e},x)}}, I^+_{A_{s(\breve{e},x)}}\right] \\ \left[F^-_{A_{s(\breve{e},x)}}, F^+_{A_{s(\breve{e},x)}}\right] / x \in U\right\}$$

where $T^-_{A_{s(\breve{e},x)}}(x)$, $I^-_{A_{s(\breve{e},x)}}(x)$, $F^-_{A_{s(\breve{e},x)}}(x)/x \in U[0,1]$,

$$0 \le \left[T^-_{A_{s(\breve{e},x)}}(x)\right]^2 + \left[I^-_{A_{s(\breve{e},x)}}\right]^2 + \left[F^-_{A_{s(\breve{e},x)}}\right]^2 \le \sqrt{3}$$

and $T^+_{A_{s(\breve{e},x)}}(x)$, $I^+_{A_{s(\breve{e},x)}}(x)$, $F^+_{A_{s(\breve{e},x)}}(x)/x \in U[0,1]$,

$$0 \le \left[T^+_{A_{s(\breve{e},x)}}(x)\right]^2 + \left[I^+_{A_{s(\breve{e},x)}}\right]^2 + \left[F^+_{A_{s(\breve{e},x)}}\right]^2 \le \sqrt{3}$$

where

$$I^-_{A_{s(\breve{e},x)}}(x) = \min\left(\left\{1, 1 - \left[T^-_{A_{s(\breve{e},x)}}(x)\right] - \left[F^-_{A_{s(\breve{e},x)}}(x)\right]\right\}\right),$$

$$I^+_{A_{s(\breve{e},x)}}(x) = \max\left(\left\{1, 1 - \left[T^+_{A_{s(\breve{e},x)}}(x)\right] - \left[F^+_{A_{s(\breve{e},x)}}(x)\right]\right\}\right).$$

Example 2.1 Let $\mathrm{U} = \{\mathrm{u}_1, \mathrm{u}_2, \mathrm{u}_3\}$ be the set of countries, $\mathrm{E}_\mathrm{s} = \{\mathrm{e}_1 = \text{ physiological natality}, \ \mathrm{e}_2 = \text{ potential}$

mortality} be the set of factors affecting population, $X_s = \{x_1, x_2\}$ be the set of experts. Let

$$E_s \times X_s = \{(e_1,x_1),(e_1,x_2),(e_2,x_1),(e_2,x_2)\}.$$

Then the Neutrosophic spherical cubic soft set (β_s, E_s, X_s) in U is given by,

$$\beta_s(e_1,x_1) = \begin{Bmatrix} u_1, & ([0.3,0.4], & [0.5,1.0], & [0.2,0.4]), & (0.3,0.4,0.5) \\ u_2, & ([0.4,0.5], & [0.3,1.0], & [0.3,0.5]), & (0.5,0.4,0.5) \\ u_3, & ([0.1,0.3], & [0.7,1.0], & [0.2,0.4]), & (0.7,0.7,0.3) \end{Bmatrix}$$

$$\beta_s(e_2,x_1) = \begin{Bmatrix} u_1, & ([0.4,0.1], & [0.5,1.0], & [0.1,0.3]), & (0.7,0.7,0.5) \\ u_2, & ([0.3,0.4], & [0.6,1.0], & [0.1,0.3]), & (0.6,0.7,0.6) \\ u_3, & ([0.6,0.9], & [0.1,1.0], & [0.1,0.2]), & (0.8,0.8,0.4) \end{Bmatrix}$$

$$\beta_s(e_1,x_2) = \begin{Bmatrix} u_1, & ([0.3,0.5], & [0.7,1.0], & [0.0,0.2]), & (0.7,0.5,0.6) \\ u_2, & ([0.4,0.6], & [0.4,1.0], & [0.2,0.4]), & (0.6,0.5,0.7) \\ u_3, & ([0.4,0.5], & [0.6,1.0], & [0.0,0.5]), & (0.8,0.6,0.6) \end{Bmatrix}$$

$$\beta_s(e_2,x_2) = \begin{Bmatrix} u_1, & ([0.1,0.4], & [0.5,1.0], & [0.4,0.6]), & (0.5,0.4,0.5) \\ u_2, & ([0.3,0.6], & [0.5,1.0], & [0.2,0.1]), & (0.6,0.7,0.3) \\ u_3, & ([0.3,0.5], & [0.6,1.0], & [0.1,0.4]), & (0.5,0.4,0.6) \end{Bmatrix}$$

In this example, interval valued Neutrosophic spherical set indicates the experts opinion for future time period and Neutrosophic spherical set indicates the experts opinion for present time period under the different circumstances related to the given problem.

Definition 2.2 A Neutrosophic spherical cubic soft set is said to be an internal Neutrosophic spherical cubic soft(INSCS) set if

$$A^-_{s(\breve{e},x)}(u) \le \lambda_{s(\breve{e},x)}(u) \le A^+_{s(\breve{e},x)}(u)$$

for all $(\breve{e},x) \in E_s \times X_s$ and for all $u \in U$.

Definition 2.3 A Neutrosophic spherical cubic soft set is said to be external Neutrosophic spherical cubic soft(ENSCS) set if

$$\lambda_{s(\breve{e},x)}(u) \not\subseteq A^-_{s(\breve{e},x)}(u), A^+_{s(\breve{e},x)}(u)$$

for all $(\breve{e},x) \in E_s \times X_s$ and for all $u \in U$.

Definition 2.4 Let (β_s, E_s, X_s) be a Neutrosophic spherical cubic soft set over U. For any $e_1, e_2 \in E_s$, $x_1, x_2 \in X_s$ if

$$\beta_s(e_1,x_1) = \{\langle u, A_{1s(e_1,x_1)}(u), \lambda_{1s(e_1,x_1)}(u)\rangle : u \in U\}$$

and

$$\beta_s(e_2,x_2) = \{\langle u, A_{2s(e_2,x_2)}(u), \lambda_{2s(e_2,x_2)}(u)\rangle : u \in U\}$$

the P −order(R −order) denoted by

$$\beta_s(e_1,x_1) \subseteq_{P_s} \beta_s(e_2,x_2)$$

$$\beta_s(e_1,x_1) \subseteq_{R_s} \beta_s(e_2,x_2)$$

are defined as follows:

1. $A_{1s(e_1,x_1)}(u) \le A_{2s(e_2,x_2)}(u)$, $A_{1s(e_1,x_1)}(u) \le A_{2s(e_2,x_2)}(u)$ for all $u \in U$
2. $\lambda_{1s(e_1,x_1)}(u) \le \lambda_{2s(e_2,x_2)}(u)$, $\lambda_{1s(e_1,x_1)}(u) \ge \lambda_{2s(e_2,x_2)}(u)$ for all $u \in U$

Definition 2.5 A Neutrosophic spherical cubic soft set over U is said to be P −order(R −order) contained in another Neutrosophic spherical cubic soft set $(\beta_{2s}, E_{2s}, X_{2s})$ over U denoted by

$$(\beta_{1s}, E_{1s}, X_{1s}) \subseteq_{P_s} (\beta_{2s}, E_{2s}, X_{2s}),$$

$$\beta_s(e_1,x_1) \subseteq_{R_s} \beta_s(e_2,x_2)$$

are defined as below:

1. $E_{1s} \subseteq E_{2s}, (E_{1s} \subseteq E_{2s})$
2. $X_{1s} \subseteq X_{2s}, (X_{1s} \subseteq X_{2s})$
3. $\beta_{1s}(\breve{e},x) \subseteq_{P_s} \beta_{2s}(\breve{e},x),\ \beta_{1s}(\breve{e},x) \subseteq_{R_s} \beta_{2s}(\breve{e},x)$ for all $\breve{e} \in E_{1s},\ x \in X_{1s}$

Definition 2.6 Two Neutrosophic spherical cubic soft sets $(\beta_{1s}, E_{1s}, X_{1s})$ and $(\beta_{2s}, E_{2s}, X_{2s})$ over U are equal, denoted by

$$(\beta_{1s}, E_{1s}, X_{1s}) = (\beta_{2s}, E_{2s}, X_{2s})$$

if $E_{1s} = E_{2s},\ X_{1s} = X_{2s}$ and

$$\beta_{1s}(\breve{e},x) = \beta_{2s}(\breve{e},x)$$

for all $\breve{e} \in E_{1s} = E_{2s},\ x \in X_{1s} = X_{2s}$.

Corollary 2.7 For any two Neutrosophic spherical cubic soft sets $(\beta_{1s}, E_{1s}, X_{1s})$ and $(\beta_{2s}, E_{2s}, X_{2s})$ over U. If

$$(\beta_{1s}, E_{1s}, X_{1s}) \subseteq_{P_s} (\beta_{2s}, E_{2s}, X_{2s})$$

and

$$(\beta_{2s}, E_{2s}, X_{2s}) \subseteq_{P_s} (\beta_{1s}, E_{1s}, X_{1s})$$

then

$$(\beta_{1s}, E_{1s}, X_{1s}) = (\beta_{2s}, E_{2s}, X_{2s}).$$

Similar result holds for R −order.

Definition 2.8 The P −union of two Neutrosophic spherical cubic soft sets(NSCSS $_s$) $(\beta_{1s}, E_{1s}, X_{1s})$ and $(\beta_{2s}, E_{2s}, X_{2s})$ over U is denoted by

$$(\beta_{3s}, F_s, Y_s) = (\beta_{1s}, E_{1s}, X_{1s}) \cup_{P_s} (\beta_{2s}, E_{2s}, X_{2s})$$

where $F = E_{1s} \cup E_{2s},\ Y = X_{1s} \cup X_{2s}$ and for all $f \in F_s$ and $z \in Y_s$ it is defined as

$$\beta_{3s}(f,z) = \begin{cases} \{\langle u, A_{1s(f,z)}(u), \lambda_{1s(f,z)}(u)\rangle\} & if \quad (f,z) \in (E_{1s} \times X_{1s}) \backslash (E_{2s} \times X_{2s}) \\ \{\langle u, A_{2s(f,z)}(u), \lambda_{2s(f,z)}(u)\rangle\} & if \quad (f,z) \in (E_{2s} \times X_{2s}) \backslash (E_{1s} \times X_{1s}) \\ \{\langle u, \sup\{A_{1s(f,z)}(u), A_{2s(f,z)}(u)\}, & \\ \sup\{\lambda_{1s(f,z)}(u), \lambda_{2s(f,z)}(u)\}\rangle\} & if \quad (f,z) \in (E_{1s} \cap E_{1s}) \times (X_{1s} \cap X_{2s}) \end{cases}$$

whenever $\beta_{1s(f,z)} = \{\langle u, A_{1s(f,z)}(u), \lambda_{1s(f,z)}(u)\rangle; u \in U\}$
and $\beta_{2s(f,z)} = \{\langle u, A_{2s(f,z)}(u), \lambda_{2s(f,z)}(u)\rangle; u \in U\}$

Definition 2.9 The P −intersection of two Neutrosophic spherical cubic soft sets $(\beta_{1s}, E_{1s}, X_{1s})$ and $(\beta_{2s}, E_{2s}, X_{2s})$ over U is denoted by

$$(\beta_{3s}, F_s, Y_s) = (\beta_{1s}, E_{1s}, X_{1s}) \cap_{P_s} (\beta_{2s}, E_{2s}, X_{2s})$$

where $F_s = E_{1s} \cap E_{2s},\ Y_s = X_{1s} \cap X_{2s}$ and for all $f \in F_s$ and $z \in Y_s$. It is defined as

$$\beta_{3s}(f,z) = \{\langle u, \inf\{A_{1s(f,z)}(u), A_{2s(f,z)}(u)\}, \inf\{\lambda_{1s(f,z)}(u), \lambda_{2s(f,z)}(u)\rangle\}.$$

Definition 2.10 The R −union of two Neutrosophic spherical cubic soft sets $(\beta_{1s}, E_{1s}, X_{1s})$ and $(\beta_{2s}, E_{2s}, X_{2s})$ over U is denoted by

$$(\beta_{3s}, F_s, Y_s) = (\beta_{1s}, E_{1s}, X_{1s}) \cup_{R_s} (\beta_{2s}, E_{2s}, X_{2s})$$

where $F_s = E_{1s} \cup E_{2s},\ Y_s = X_{1s} \cup X_{2s}$ and for all $f \in F_s$ and $z \in Y_s$ it is defined as

$$\beta_{3s}(f,z) = \begin{cases} \{\langle u, A_{1s(f,z)}(u), \lambda_{1s(f,z)}(u)\rangle\} & if \quad (f,z) \in (E_{1s} \times X_{1s}) \backslash (E_{2s} \times X_{2s}) \\ \{\langle u, A_{2s(f,z)}(u), \lambda_{2s(f,z)}(u)\rangle\} & if \quad (f,z) \in (E_{2s} \times X_{2s}) \backslash (E_{1s} \times X_{2s}) \\ \{\langle u, \sup\{A_{1s(f,z)}(u), A_{2s(f,z)}(u)\}, & \\ \inf\{\lambda_{1s(f,z)}(u), \lambda_{2s(f,z)}(u)\}\rangle\} & if \quad (f,z) \in (E_{1s} \cap E_{2s}) \times (X_{2s} \cap X_{2s}) \end{cases}$$

Definition 2.11 The R −intersection of two Neutrosophic spherical cubic soft sets $(\beta_{1s}, E_{1s}, X_{1s})$ and $(\beta_{2s}, E_{2s}, X_{2s})$ over U is denoted by

$$(\beta_{3s}, F_s, Y_s) = (\beta_{1s}, E_{1s}, X_{1s}) \cap_{P_s} (\beta_{2s}, E_{2s}, X_{2s})$$

where $F_s = E_{1s} \cap E_{2s}$, $Y_s = X_{1s} \cap X_{2s}$ and for all $f \in F_s$ and $z \in Y_s$ it is defined as

$$\beta_{3s}(f,z) = \{\langle u, \inf\{A_{1s(f,z)}(u), A_{2s(f,z)}(u)\}, \sup\{\lambda_{1s(f,z)}(u), \lambda_{2s(f,z)}(u)\rangle\}.$$

Definition 2.12 The complement of a NSCSS (β_s, E_s, X_s) is denoted and defined as

$$(\beta_s, E_s, X_s)^c = (\beta_s^c, E_s^c, X_s^c)$$

where $\beta_s^c: E_s^c \times X_s \to NSCP(U)$ is a mapping given as,

$$\beta_s^c(e^c, x) = \{\langle u, A_{s(\check{e},x)}^c(u), \lambda_{s(\check{e},x)}^c(u)\rangle : u \in U, (e^c, x) \in E^c \times X\}$$

where

$$A_{s(\check{e},x)}^c(u) = [1 - A_{s(\check{e},x)}^+(u), 1 - A_{s(\check{e},x)}^-(u)]$$

and

$$\lambda_{s(\check{e},x)}^c(u) = 1 - \lambda_{(\check{e},x)}(u)$$

whenever

$$\beta(\check{e}, x) = \{\langle u, A_{(\check{e},x)}(u), \lambda_{(\check{e},x)}(u)\rangle : u \in U\}.$$

Theorem 2.13 For any NSCSSs, $(\beta_{1s}, E_{1s}, X_{1s})$, $(\beta_{2s}, E_{2s}, X_{2s})$, $(\beta_{3s}, E_{3s}, X_{3s})$ and $(\beta_{4s}, E_{4s}, X_{4s})$ over U the following properties hold:

1. **Idempotent** $(\beta_{1s}, E_{1s}, X_{1s}) \cup_{P_s} (\beta_{1s}, E_{1s}, X_{1s}) = (\beta_{1s}, E_{1s}, X_{1s}) \cap_{P_s} (\beta_{1s}, E_{1s}, X_{1s})$
2. **Commutative** $(\beta_{1s}, E_{1s}, X_{1s}) \cup_{P_s} (\beta_{2s}, E_{2s}, X_{2s}) = (\beta_{2s}, E_{2s}, X_{2s}) \cup_{P_s} (\beta_{1s}, E_{1s}, X_{1s})$
3. **Associative** $(\beta_{1s}, E_{1s}, X_{1s}) \cup_{P_s} (\beta_{2s}, E_{2s}, X_{2s}) \cup_{P_s} (\beta_{3s}, E_{3s}, X_{3s}) = (\beta_{1s}, E_{1s}, X_{1s}) \cup_{P_s} (\beta_{2s}, E_{2s}, X_{2s}) \cup_{P_s} (\beta_{3s}, E_{3s}, X_{3s})$
4. **Distributive** $(\beta_{1s}, E_{1s}, X_{1s}) \cup_{P_s} (\beta_{2s}, E_{2s}, X_{2s}) \cap_{P_s} (\beta_{3s}, E_{3s}, X_{3s}) = (\beta_{1s}, E_{1s}, X_{1s}) \cup_{P_s} (\beta_{2s}, E_{2s}, X_{2s}) \cap_{P_s} (\beta_{1s}, E_{1s}, X_{1s}) \cup_{P_s} (\beta_{3s}, E_{3s}, X_{3s})$
   $(\beta_{1s}, E_{1s}, X_{1s}) \cap_{P_s} (\beta_{2s}, E_{2s}, X_{2s}) \cup_{P_s} (\beta_{3s}, E_{3s}, X_{3s}) = (\beta_{1s}, E_{1s}, X_{1s}) \cap_{P_s} (\beta_{2s}, E_{2s}, X_{2s}) \cup_{P_s} (\beta_{1s}, E_{1s}, X_{1s}) \cap_{P_s} (\beta_{3s}, E_{3s}, X_{3s})$
5. **De Morgan's laws** $(\beta_{1s}, E_{1s}, X_{1s}) \cup_{P_s} (\beta_{2s}, E_{2s}, X_{2s})^c = (\beta_{1s}, E_{1s}, X_{1s})^c \cap_{P_s} (\beta_{2s}, E_{2s}, X_{2s})^c$
   $(\beta_{1s}, E_{1s}, X_{1s}) \cap_{P_s} (\beta_{2s}, E_{2s}, X_{2s})^c = (\beta_{1s}, E_{1s}, X_{1s})^c \cup_{P_s} (\beta_{2s}, E_{2s}, X_{2s})^c$
6. **Involution law** $((\beta_{1s}, E_{1s}, X_{1s})^c)^c = (\beta_{1s}, E_{1s}, X_{1s})$

Similar results hold for R −order, R −union and R −intersection.

Proof. These properties can be verified by using definitions 2.8,2.9,2.10,2.11 and 2.12.

Proposition 2.14 For any two NSCSS $(\beta_{1s}, E_{1s}, X_{1s})$ and $(\beta_{2s}, E_{2s}, X_{2s})$ over U the following are equivalent

1. $(\beta_{1s}, E_{1s}, X_{1s}) \subseteq_{P_s} (\beta_{2s}, E_{2s}, X_{2s})$
2. $(\beta_{1s}, E_{1s}, X_{1s}) \cap_{P_s} (\beta_{2s}, E_{2s}, X_{2s}) = (\beta_{1s}, E_{1s}, X_{1s})$
3. $(\beta_{1s}, E_{1s}, X_{1s}) \cup_{P_s} (\beta_{2s}, E_{2s}, X_{2s}) = (\beta_{2s}, E_{2s}, X_{2s})$

Proof. (2.14)⇒(2.14)

By definition 2.9, we have

$$(\beta_{1s}, E_{1s}, X_{1s}) \cap_{P_s} (\beta_{2s}, E_{2s}, X_{2s}) = (\beta_{1s} \cap_{P_s} \beta_{2s}, E_{1s} \cap_{P_s} E_{2s}, X_{1s} \cap_{P_s} X_{2s})$$
$$= (\beta_{1s} \cap_{P_s} \beta_{2s}, E_{1s}, X_{1s}) \quad as \quad E_1 \subseteq E_2 \quad and \quad X_1 \subseteq X_2$$

by hypothesis.

Now for any $(\check{e}, x) \in (E_{1s} \times X_{1s})$, since

$$\beta_{1s}(\check{e}, x) \subseteq_{P_s} \beta_{2s}(\check{e}, x)$$

definition 2.4 implies that

$$A_{1s(\check{e},x)}(u) \leq A_{2s(\check{e},x)}(u)$$

and

$$\lambda_{1s(\breve{e},x)}(u) \leq \lambda_{2s(\breve{e},x)}(u)$$

for any $u \in U$ where

$$\beta_{1s}(\breve{e},x) = \{\langle u, A_{1s(\breve{e},x)}(u), ë_{1s(\breve{e},x)}(u)\rangle : u \in U\}.$$

By definition 2.4, we have

$$A^{-}_{1s(\breve{e},x)}(u) \leq A^{-}_{2s(\breve{e},x)}(u)$$
$$and \;\; A^{+}_{1s(\breve{e},x)}(u) \leq A^{+}_{2s(\breve{e},x)}(u)$$

Thus,

$$\inf\{A_{1s(\breve{e},x)}(u), A_{2s(\breve{e},x)}(u)\} = \inf\{A^{-}_{1s(\breve{e},x)}(u) \leq A^{-}_{2s(\breve{e},x)}(u)\}, \inf\{A^{+}_{1s(\breve{e},x)}(u) \leq A^{+}_{2s(\breve{e},x)}(u)\}$$
$$= [A^{-}_{1s(\breve{e},x)}(u) \leq A^{+}_{1s(\breve{e},x)}(u)]$$

and

$$\inf\{\lambda_{1s(\breve{e},x)}(u), \lambda_{2s(\breve{e},x)}(u)\} = \lambda_{1s(\breve{e},x)}(u).$$

By using definition 2.9,

$$\beta_{1s(\breve{e},x)} \cap_{P_s} \beta_{2s(\breve{e},x)} = \{\langle u, \inf\{A_{1s(\breve{e},x)}(u), A_{2s(\breve{e},x)}(u)\}, \inf\{\lambda_{1s(\breve{e},x)}(u), \lambda_{2s(\breve{e},x)}(u)\}\rangle : u \in U\}$$
$$= \{\langle u, A_{1s(\breve{e},x)}(u), \lambda_{1s(\breve{e},x)}(u)\}\rangle : u \in U\}$$
$$= \beta_{1s(\breve{e},x)}.$$

Hence, $\beta_{1s(\breve{e},x)} \cap_{P_s} \beta_{2s(\breve{e},x)} = \beta_{1s(\breve{e},x)}$.

(2.14)⇒(2.14)

By definition 2.8, we have

$$(\beta_{1s}, E_{1s}, X_{1s}) \cup_{P_s} (\beta_{2s}, E_{2s}, X_{2s}) = (\beta_{1s} \cup_{P_s} \beta_{2s}, E_{1s} \cup_{P_s} E_{2s}, X_{1s} \cup_{P_s} X_{2s})$$
$$= (\beta_{1s} \cup_{P_s} \beta_{2s}, E_{1s}, X_{1s})$$

as $E_{1s} \cap E_{2s} = E_{1s}$ and $X_{1s} \cap X_{2s} = X_1$ by hypothesis.

Now for any $(\breve{e},x) \in (E_{1s} \times X_{1s})$. Since

$$\beta_{1s}(\breve{e},x) \cap_{P_s} \beta_{2s}(\breve{e},x) = \beta_{1s}(\breve{e},x).$$

By definition 2.9 we have

$$\inf\{A_{1s(\breve{e},x)}(u), A_{2s(\breve{e},x)}(u)\} = A_{1s(\breve{e},x)}(u)$$

and

$$\inf\{\lambda_{1s(\breve{e},x)}(u), \lambda_{2s(\breve{e},x)}(u)\} = \lambda_{1s(\breve{e},x)}(u).$$

This implies that

$$\sup\{A_{1s(\breve{e},x)}(u), A_{2s(\breve{e},x)}(u)\} = A_{2s(\breve{e},x)}(u)$$

and

$$\sup\{\lambda_{1s(\breve{e},x)}(u), \lambda_{2s(\breve{e},x)}(u)\} = \lambda_{2s(\breve{e},x)}(u).$$

Thus we have,

$$\beta_{1s}(\breve{e},x) \cup_{P_s} \beta_{2s}(\breve{e},x) = \{\langle u, \sup\{A_{1s(\breve{e},x)}(u), A_{2s(\breve{e},x)}(u)\}, \sup\{\lambda_{1s(\breve{e},x)}(u), \lambda_{2s(\breve{e},x)}(u)\}\rangle : u \in U\}$$
$$= \{\langle u, A_{2s(\breve{e},x)}(u), \lambda_{2s(\breve{e},x)}(u)\}\rangle : u \in U\}$$
$$= \beta_{2s(\breve{e},x)}.$$

(2.14)⇒(2.14)

By hypothesis, we have

$$(\beta_{1s}, E_{1s}, X_{1s}) \cup_{P_s} (\beta_{2s}, E_{2s}, X_{2s}) = (\beta_{1s} \cup_{P_s} \beta_{2s}, E_{1s} \cup_{P_s} E_{2s}, X_{1s} \cup_{P_s} X_{2s})$$
$$= (\beta_{1s} \cup_{P_s} \beta_{2s}, E_{1s}, X_{1s})$$

as $E_{1s} \cup E_{2s} = E_{2s}$ and $X_{1s} \cup X_{2s} = X_{2s} \Rightarrow E_{1s} \subseteq E_{2s}$ and $X_{1s} \subseteq X_{2s}$.

Also,

$$\beta_{1s}(\breve{e},x) \cup_{P_s} \beta_{2s}(\breve{e},x) = \{\langle u, \sup\{A_{1s(\breve{e},x)}(u), A_{2s(\breve{e},x)}(u)\}, \sup\{\lambda_{1s(\breve{e},x)}(u), \lambda_{2s(\breve{e},x)}(u)\}\rangle : u \in U\}$$
$$= \{\langle u, A_{2s(\breve{e},x)}(u), \lambda_{2s(\breve{e},x)}(u)\}\rangle : u \in U\}$$
$$= \beta_{2s(\breve{e},x)} \Rightarrow A_{1s(\breve{e},x)}(u) \leq A_{2s(\breve{e},x)}(u)$$

and $\lambda_{1s(\breve{e},x)}(u) \leq \lambda_{2s(\breve{e},x)}(u)$ for any $u \in U$.

Hence, $(\beta_{1s}, E_{1s}, X_{1s}) \subseteq_{P_s} (\beta_{2s}, E_{2s}, X_{2s})$.

Corollary 2.15 If we take $X_{1s} = X_{2s} = X_s$ in above proposition then the following are equivalent.

1. $(\beta_{1s}, E_{1s}, X_s) \subseteq_{P_s} (\beta_{2s}, E_{2s}, X_s)$
2. $(\beta_{1s}, E_{1s}, X_s) \cap_{P_s} (\beta_{2s}, E_{2s}, X_s) = (\beta_{1s}, E_{1s}, X_s)$
3. $(\beta_{1s}, E_{1s}, X_s) \cup_{P_s} (\beta_{2s}, E_{2s}, X_s) = (\beta_{2s}, E_{2s}, X_s)$
4. $(\beta_{2s}, E_{2s}, X_s)^c \subseteq_{P_s} (\beta_{1s}, E_{1s}, X_s)^c$

Definition 2.16 Let $\{\mathcal{L}_{is}\}_{i\in J}$ be a family of Neutrosophic spherical cubic soft sets over U, where

$$\beta_{is}(\breve{e}, x) = \{\langle u, A_{is(\breve{e},x)}(u), \lambda_{is(\breve{e},x)}(u)\rangle : u \in U\}$$

for any $\breve{e} \in E_i$, $x \in X_i$. Then P −union, P −intersection, R −union and R −intersection are defined as below:

1. $\bigcup_{P_s\, i\in J} \{\mathcal{L}_i\} = \{\langle u, (\sup_{i\in J} A_{is(\breve{e},x)})(u), (\vee_{i\in J}\, \lambda_{is(\breve{e},x)})(u)\rangle; u \in U\}$
2. $\bigcap_{P_s\, i\in J} \{\mathcal{L}_i\} = \{\langle u, (\inf_{i\in J} A_{is(\breve{e},x)})(u), (\wedge_{i\in J}\, \lambda_{is(\breve{e},x)})(u)\rangle; u \in U\}$
3. $\bigcup_{R_s\, i\in J} \{\mathcal{L}_i\} = \{\langle u, (\sup_{i\in J} A_{is(\breve{e},x)})(u), (\wedge_{i\in J}\, \lambda_{is(\breve{e},x)})(u)\rangle; u \in U\}$
4. $\bigcap_{R_s\, i\in J} \{\mathcal{L}_i\} = \{\langle u, (\inf_{i\in J} A_{is(\breve{e},x)})(u), (\vee_{i\in J}\, \lambda_{is(\breve{e},x)})(u)\rangle; u \in U\}$

Theorem 2.17 Let $\{\mathcal{L}_i\}_{i\in J} = \{\beta_{is}, E_{is}, X_{is}\}_{i\in J}$ be a family of Internal Neutrosophic spherical cubic soft sets (INSCSSs) over U, where

$$\beta_{is}(\breve{e}, x) = \{\langle u, A_{is(\breve{e},x)}(u), \lambda_{is(\breve{e},x)}(u)\rangle : u \in U$$

for any $\breve{e} \in E_{is}$, $x \in X_{is}$. Then the $\bigcup_{P_s\, i\in J} \{\mathcal{L}_i\}$ and $\bigcap_{P_s\, i\in J} \{\mathcal{L}_i\}$ are INSCSSsover U.

Proof. By using definitions 2.16 and 2.2, we can easily prove this theorem.

Theorem 2.18 *Let* $(\beta_{1s}, E_{1s}, X_{1s})$ *and* $(\beta_{2s}, E_{2s}, X_{2s})$ *are two INSCSSs over* U, *where*

$$\beta_{1s}(\breve{e}, x) = \{\langle u, A_{1s(\breve{e},x)}(u), \lambda_{1s(\breve{e},x)}(u)\rangle : u \in U\}$$

for any $(\breve{e}, x) \in E_1 \times X_1$ and

$$\beta_{2s}(f, z) = \{\langle u, A_{2s(f,z)}(u), \lambda_{2s(f,z)}(u)\rangle : u \in U\}$$

for any $(f, z) \in (\beta_{2s}, E_{2s}, X_{2s})$ is also an INSCSSs over U.

Proof. Since $(\beta_{1s}, E_{1s}, X_{1s})$ and $(\beta_{2s}, E_{2s}, X_{2s})$ are INSCSSs over U,

$$A^-_{1s(f,z)}(u) \leq \lambda_{1s(f,z)}(u) \leq A^+_{1s(f,z)}(u)$$

for all $u \in U$ and

$$A^-_{2s(f,z)}(u) \leq \lambda_{2s(f,z)}(u) \leq A^+_{2s(f,z)}(u)$$

for all $u \in U$. Then we have

$$\sup\{A^-_{1s(f,z)}(u), A^-_{2s(f,z)}(u)\} \leq \lambda_{1s(f,z)}(u) \vee \lambda_{2s(f,z)}(u) \leq \sup\{A^+_{1s(f,z)}(u), A^+_{2s(f,z)}(u)\}$$

for all $u \in U$ and $(f, z) \in (E_{1s} \cup E_{2s} \times X_{1s} \cup X_{2s})$.

By definition of 2.8, we have

$$(\beta_{1s}, F_s, Y_s) = (\beta_{1s}, E_{1s}, X_{1s}) \cup_{P_s} (\beta_{2s}, E_{2s}, X_{2s})$$

where $F_s = E_{1s} \cup E_{2s}$ and $Y_s = X_{1s} \cup X_{2s}$ and for any $f \in F$ and $z \in Y$.

$$\beta_{3s}(f, z) = \begin{cases} \{\langle u, A_{1s(f,z)}(u), \lambda_{1s(f,z)}(u)\rangle\} & if \quad (f, z) \in (E_{1s} \times X_{1s}) \backslash (E_{2s} \times X_{2s}) \\ \{\langle u, A_{2s(f,z)}(u), \lambda_{2s(f,z)}(u)\rangle\} & if \quad (f, z) \in (E_{2s} \times X_{2s}) \backslash (E_{1s} \times X_{1s}) \\ \{\langle u, \sup\{A_{1s(f,z)}(u), A_{2s(f,z)}(u)\}, & \\ \sup\{\lambda_{1s(f,z)}(u), \lambda_{2s(f,z)}(u)\}\}\rangle\} & if \quad (f, z) \in (E_{1s} \cap E_{2s}) \times (X_{1s} \cap X_{2s}) \end{cases}$$

If $(f, z) \in (E_{1s} \cap E_{2s} \times X_{1s} \cap X_{2s})$ then,

$$\beta_{3s}(f, z) = \{\langle u, \sup\{A_{1s(f,z)}(u), A_{2s(f,z)}(u)\}, \lambda_{1s(f,z)}(u) \vee \lambda_{2s(f,z)}(u)\rangle : u \in U\}.$$

Thus

$$(\beta_{1s}, E_{1s}, X_{1s}) \cup_{P_s} (\beta_{2s}, E_{2s}, X_{2s})$$

is INSCSS.

If $(f,z) \in (E_{1s} \times X_{1s}) \backslash (E_{2s} \times X_{2s})$ or if $(f,z) \in (E_{2s} \times X_{2s}) \backslash (E_{1s} \times X_{1s})$, then the result is trivial. Hence $(\beta_{1s}, E_{1s}, X_{1s}) \cup_{P_s} (\beta_{2s}, E_{2s}, X_{2s})$ is an INSCSS over U.

Theorem 2.19 Let $(\beta_{1s}, E_{1s}, X_{1s})$ and $(\beta_{2s}, E_{2s}, X_{2s})$ are two INSCSs over U, where
$$\beta_{1s}(\breve{e},x) = \{\langle u, A_{1s(\breve{e},x)}(u), \lambda_{1s(\breve{e},x)}(u)\rangle : u \in U\}$$
for any $(\breve{e},x) \in E_{1s} \times X_{1s}$ and
$$\beta_{2s}(f,z) = \{\langle u, A_{2s(f,z)}(u), \lambda_{2s(f,z)}(u)\rangle : u \in U\}$$
for any $(f,z) \in E_{2s} \times X_{2s}$. Then the P −intersection of $(\beta_{1s}, E_{1s}, X_{1s})$ and $(\beta_{2s}, E_{2s}, X_{2s})$ is also an INSCSS.

Proof. By similar way as theorem 2.18, we can prove this theorem.

The following theorem gives the condition that R −union of INSCSSs is also an INSCSS.

Theorem 2.20 Let $(\beta_{1s}, E_{1s}, X_{1s})$ and $(\beta_{2s}, E_{2s}, X_{2s})$ are two INSCSSs over U, where
$$\beta_{1s}(\breve{e},x) = \{\langle u, A_{1s(\breve{e},x)}(u), \lambda_{1s(\breve{e},x)}(u)\rangle : u \in U\}$$
for any $(\breve{e},x) \in E_{1s} \times X_{1s}$ and
$$\beta_{2s}(f,z) = \{\langle u, A_{2s(f,z)}(u), \lambda_{2s(f,z)}(u)\rangle : u \in U\}$$
for any $(f,z) \in E_{2s} \times X_{2s}$ such that
$$\sup\{A^-_{1s(f,z)}(u), A^-_{2s(f,z)}(u)\} \le (\lambda_{1s(f,z)}(u) \wedge \lambda_{2s(f,z)}(u)$$
for all $u \in U$ and $(f,z) \in (E_{1s} \cap E_{2s} \times X_{2s} \cap X_{2s})$.

Then the R −union of $(\beta_{1s}, E_{1s}, X_{1s})$ and $(\beta_{2s}, E_{2s}, X_{2s})$ is also an INSCSS.

Proof. By definition 2.10 and 2.2 we can easily prove this theorem.

The following theorem gives the conclusion that R −intersection of two $INSCSSs$ is also the $\pm$NSCSS.

Theorem 2.21 Let $(\beta_{1s}, E_{1s}, X_{1s})$ and $(\beta_{2s}, E_{2s}, X_{2s})$ are two INSCSSs over U, where
$$\beta_{1s}(\breve{e},x) = \{\langle u, A_{1s(\breve{e},x)}(u), \lambda_{1s(\breve{e},x)}(u)\rangle : u \in U\}$$
for any $(\breve{e},x) \in E_{1s} \times X_{1s}$ and
$$\beta_{2s}(f,z) = \{\langle u, A_{2s(f,z)}(u), \lambda_{2s(f,z)}(u)\rangle : u \in U\}$$
for any $(f,z) \in E_{2s} \times X_{2s}$ such that
$$\inf\{A^+_{1s(f,z)}(u), A^-_{2s(f,z)}(u)\} \ge (\lambda_{1s(f,z)}(u) \vee \lambda_{2s(f,z)}(u))$$
for all $u \in U$ and $(f,z) \in (E_{1s} \cap E_{2s}) \times (X_{2s} \cap X_{2s})$.

Then the R −intersection of $(\beta_{1s}, E_{1s}, X_{1s})$ and $(\beta_{2s}, E_{2s}, X_{2s})$ is also an INSCSS over U.

Proof. By definition 2.11, we have
$$(\beta_{3s}, E_{3s}, X_{3s}) = (\beta_{1s}, E_{1s}, X_{1s}) \cap_R (\beta_{2s}, E_{2s}, X_{2s})$$
where, $E_{3s} = E_{1s} \cap E_{2s}$ and $X_{3s} = X_{1s} \cap X_{2s}$, $f \in E_{3s}$ and $z \in X_{3s}$.
$$\hat{a}_{3s}(f,z) = \{\langle u, \inf\{A_{1s}(f,z)(u), A_{2s}(f,z)(u)\}, \sup\{\lambda_{1s}(f,z)(u), \lambda_{2s}(f,z)(u)\}\rangle\}$$
since $(\beta_{1s}, E_{1s}, X_{1s})$ and $(\beta_{2s}, E_{2s}, X_{2s})$ are INSCSSs over U. So we have,
$$A^-_{1s(f,z)}(u) \le \lambda_{1s(f,z)}(u) \le A^+_{1s(f,z)}(u)$$
for all $u \in U$ and
$$A^-_{2s(f,z)}(u) \le \lambda_{2s(f,z)}(u) \le A^+_{2s(f,z)}(u)$$
for all $u \in U$. Also
$$\inf\{A^-_{1s(f,z)}(u), A^-_{2s(f,z)}(u)\} \le \lambda_{1s(f,z)}(u) \vee \lambda_{2s(f,z)}(u) \le \inf\{A^+_{1s(f,z)}(u), A^+_{2s(f,z)}(u)\}$$
for all $u \in U$ and $(f,z) \in (E_{1s} \cap E_{2s} \times X_{1s} \cap X_{2s})$.

Hence $(\beta_{1s}, E_{1s}, X_{1s}) \cap_{R_s} (\beta_{2s}, E_{2s}, X_{2s})$ is an INSCSS over U.

Theorem 2.22 Let $(\beta_{1s}, E_{1s}, X_{1s})$ and $(\beta_{2s}, E_{2s}, X_{2s})$ are two ENSCSSs over U, where

$$\beta_{1s}(\breve{e},x)=\{\langle u, A_{1s(\breve{e},x)}(u), \lambda_{1s(\breve{e},x)}(u)\rangle : u\in U\}$$

for any $(\breve{e},x)\in E_{1s}\times X_{1s}$ and

$$\beta_{2s}(f,z)=\{\langle u, A_{2s(f,z)}(u), \lambda_{2s(f,z)}(u)\rangle : u\in U\}$$

for any $(f,z)\in E_{2s}\times X_{2s}$ such that

$$\lambda_{1s(f,z)}(u)\wedge\lambda_{2s(f,z)}(u)\in\{\inf\{\sup\{A^{+}_{1s(f,z)}(u), A^{-}_{2s(f,z)}(u)\}, \sup\{A^{-}_{1s(f,z)}(u), A^{+}_{2s(f,z)}(u)\}, \inf\{A^{+}_{1s(f,z)}(u), A^{-}_{2s(f,z)}(u)\}, \inf\{A^{-}_{1s(f,z)}(u), A^{+}_{2s(f,z)}(u)\}\}\}$$

for all $u\in U$ and $(f,z)\in(E_{1s}\cap E_{2s})\times(X_{1s}\cap X_{2s})$.

Then

$$(\beta_{1s}, E_{1s}, X_{1s})\cup_{R_s}(\beta_{2s}, E_{2s}, X_{2s})$$

is also an ENSCSS over U.

Proof. By definition 2.10, we have

$$(\beta_{3s}, E_{3s}, X_{3s})=(\beta_{1s}, E_{1s}, X_{1s})\cup_{R_s}(\beta_{2s}, E_{2s}, X_{2s})$$

where,

$$\beta_{3s}(f,z)=\left\{\begin{array}{ll}\{\langle u, A_{1s(f,z)}(u), \lambda_{1s(f,z)}(u)\rangle\} & if \quad (f,z)\in(E_{1s}\times X_{1s})\backslash(E_{2s}\times X_{2s})\\ \{\langle u, A_{2s(f,z)}(u), \lambda_{2s(f,z)}(u)\rangle\} & if \quad (f,z)\in(E_{2s}\times X_{2s})\backslash(E_{1s}\times X_{1s})\\ \{\langle u, \sup\{A_{1s(f,z)}(u), A_{2s(f,z)}(u)\}, & \\ \inf\{\lambda_{1s(f,z)}(u), \lambda_{2s(f,z)}(u)\}\rangle\} & if \quad (f,z)\in(E_{1s}\cap E_{2s})\times(X_{1s}\cap X_{2s})\end{array}\right\}$$

if $(f,z)\in(E_{1s}\cap E_{2s}\times X_{1s}\cap X_{2s})$.

Take

$$h=\inf\{\sup\{A^{+}_{1s(f,z)}(u), A^{-}_{2s(f,z)}(u)\}, \sup\{A^{-}_{1s(f,z)}(u), A^{+}_{2s(f,z)}(u)\}\},$$

and

$$k=\sup\{\inf\{A^{+}_{1s(f,z)}(u), A^{-}_{2s(f,z)}(u)\}, \inf\{A^{-}_{1s(f,z)}(u), A^{+}_{2s(f,z)}(u)\}\}.$$

Then h on of $A^{+}_{1s(f,z)}(u)$, $A^{-}_{2s(f,z)}(u)\}$, $A^{-}_{1s(f,z)}(u)$, $A^{+}_{2s(f,z)}(u)$, we only consider $h=A^{-}_{2s(f,z)}(u)$ Or $A^{+}_{2s(f,z)}(u)$ because remaining cases are similar to this one. If $h=A^{-}_{2s(f,z)}(u)$ then

$$A^{-}_{1s(f,z)}(u)\le A^{+}_{1s(f,z)}(u)\le A^{-}_{2s(f,z)}(u)\le A^{+}_{2s(f,z)}(u)$$

and $k=A^{+}_{1s(f,z)}(u)$.

Thus

$$(\sup\{A_{1s(f,z)}, A_{2s(f,z)}\}^{-})(u)=A_{2s(f,z)}\}^{-}(u)=h>(\lambda_{1s(f,z)}\wedge\lambda_{2s(f,z)})(u).$$

Hence

$$(\lambda_{1s(f,z)}\wedge\lambda_{2s(f,z)})(u)\not\in(\sup\{A_{1s(f,z)}, A_{2s(f,z)}\}^{-}(u), \sup\{A_{1s(f,z)}, A_{2s(f,z)}\}^{+}(u)).$$

If $h=A^{-}_{2s(f,z)}(u)$ then

$$A^{-}_{1s(f,z)}(u)\le A^{+}_{2s(f,z)}(u)\le A^{+}_{1s(f,z)}(u)$$

so

$$k=\sup\{A^{-}_{1s(f,z)}(u), A^{-}_{2s(f,z)}(u)\}.$$

Assume $k=A^{-}_{2s(f,z)}(u)$, then we have

$$A^{-}_{2s(f,z)}(u)\le A^{-}_{1s(f,z)}(u)\le(\lambda_{1s(f,z)}\wedge\lambda_{2s(f,z)})(u)<A^{+}_{2s(f,z)}(u)\le A^{+}_{1s(f,z)}(u).$$

So we can write

$$A^{-}_{2s(f,z)}(u)\le A^{-}_{1s(f,z)}(u)<(\lambda_{1s(f,z)}\wedge\lambda_{2s(f,z)})(u)<A^{+}_{2s(f,z)}(u)\le A^{+}_{1s(f,z)}(u)$$

or

$$A^{-}_{2s(f,z)}(u)\le A^{-}_{1s(f,z)}(u)=(\lambda_{1s(f,z)}\wedge\lambda_{2s(f,z)})(u)\le A^{+}_{2s(f,z)}(u)\le A^{+}_{1s(f,z)}(u).$$

For the case

$$A^{-}_{2s(f,z)}(u)\le A^{-}_{1s(f,z)}(u)<(\lambda_{1s(f,z)}\wedge\lambda_{2s(f,z)})(u)<A^{+}_{2s(f,z)}(u)\le A^{+}_{1s(f,z)}(u)$$

which contradicts the fact that $(\beta_{1s}, E_{1s}, X_{1s})$ and $(\beta_{2s}, E_{2s}, X_{2s})$ are ENSCSSs.

For the case

$$A^-_{2s(f,z)}(u) < A^-_{1s(f,z)}(u) = (\lambda_{1s(f,z)} \wedge \lambda_{2s(f,z)})(u) \leq A^+_{2s(f,z)}(u) \leq A^+_{1s(f,z)}(u),$$

we have

$$(\lambda_{1s(f,z)} \wedge \lambda_{2s(f,z)})(u) \neq \sup\{A_{1s(f,z)}, A_{2s(f,z)}\}^-(u), \sup\{A_{1s(f,z)}, A_{2s(f,z)}\}^+(u)$$

because

$$\sup\{A_{1s(f,z)}, A_{2s(f,z)}\}^-(u) = A^-_{2s(f,z)}(u) = (\lambda_{1s(f,z)} \wedge \lambda_{2s(f,z)})(u).$$

If $(f,z) \in (E_{1s} \times X_{1s}) \backslash (E_{2s} \times X_{2s})$ or $(f,z) \in (E_{2s} \times X_{2s}) \backslash (E_{1s} \times X_{1s})$ then the result holds trivially. Hence $(\beta_{1s}, E_{1s}, X_{1s}) \cup_{R_s} (\beta_{2s}, E_{2s}, X_{2s})$ is an ENSCSS over U.

Theorem 2.23 Let $(\beta_{1s}, E_{1s}, X_{1s})$ and $(\beta_{2s}, E_{2s}, X_{2s})$ are two ENSCSSs over U, where

$$\beta_{1s}(\breve{e}, x) = \{\langle u, A_{1s(\breve{e},x)}(u), \lambda_{1s(\breve{e},x)}(u)\rangle : u \in U\}$$

for any $(\breve{e}, x) \in E_{1s} \times X_{1s}$ and

$$\beta_{2s}(f, z) = \{\langle u, A_{2s(f,z)}(u), \lambda_{2s(f,z)}(u)\rangle : u \in U\}$$

for any $(f, z) \in E_{2s} \times X_{2s}$ such that

$$\begin{aligned} &\lambda_{1s(f,z)}(u) \vee \lambda_{2s(f,z)}(u) \in \{\inf\{\sup\{A^+_{1s(f,z)}(u), A^-_{2s(f,z)}(u)\},\\ &\sup\{A^-_{1s(f,z)}(u), A^+_{2s(f,z)}(u)\},\\ &\inf\{A^+_{1s(f,z)}(u), A^-_{2s(f,z)}(u)\},\\ &\inf\{A^-_{1s(f,z)}(u), A^+_{2s(f,z)}(u)\}\}\} \end{aligned}$$

for all $u \in U$ and $(f,z) \in (E_{1s} \cap E_{2s}) \times (X_{1s} \cap X_{2s})$.

Then

$$(\beta_{1s}, E_{1s}, X_{1s}) \cap_{R_s} (\beta_{2s}, E_{2s}, X_{2s})$$

is also an ENSCSS over U.

Proof. We can prove this theorem similar to theorem 2.22

In the next theorem we derive condition that P −union of two ENSCSSs are ENSCSS over U.

Theorem 2.24 Let $(\beta_{1s}, E_{1s}, X_{1s})$ and $(\beta_{2s}, E_{2s}, X_{2s})$ are two ENSCSSs over U, where

$$\beta_{1s}(\breve{e}, x) = \{\langle u, A_{1s(\breve{e},x)}(u), \lambda_{1s(\breve{e},x)}(u)\rangle : u \in U\}$$

for any $(\breve{e}, x) \in E_{1s} \times X_{1s}$ and

$$\beta_{2s}(f, z) = \{\langle u, A_{2s(f,z)}(u), \lambda_{2s(f,z)}(u)\rangle : u \in U\}$$

for any $(f, z) \in E_{2s} \times X_{2s}$ such that

$$\begin{aligned} &\lambda_{1s(f,z)}(u) \vee \lambda_{2s(f,z)}(u) \in \{\inf\{\sup\{A^+_{1s(f,z)}(u), A^-_{2s(f,z)}(u)\},\\ &\sup\{A^-_{1s(f,z)}(u), A^+_{2s(f,z)}(u)\},\\ &\inf\{A^+_{1s(f,z)}(u), A^-_{2s(f,z)}(u)\},\\ &\inf\{A^-_{1s(f,z)}(u), A^+_{2s(f,z)}(u)\}\}\} \end{aligned}$$

for all $(f,z) \in (E_{1s} \cap E_{2s}) \times (X_{1s} \cap X_{2s})$ and $u \in U$.

Then

$$(\beta_{1s}, E_{1s}, X_{1s}) \cup_{P_s} (\beta_{2s}, E_{2s}, X_{2s})$$

is also an ENSCSS over U.

Proof. By definition 2.8, we can prove this theorem.

In the next theorem we derive condition that P −intersection of two NSCSSs are ENSCSS(INSCSS).

Theorem 2.25 Let $(\beta_{1s}, E_{1s}, X_{1s})$ and $(\beta_{2s}, E_{2s}, X_{2s})$ are two NSCSSs over U, where
$$\beta_{1s}(\breve{e}, x) = \{\langle u, A_{1s(\breve{e},x)}(u), \lambda_{1s(\breve{e},x)}(u)\rangle : u \in U\}$$
for any $(\breve{e}, x) \in E_{1s} \times X_{1s}$ and
$$\beta_{2s}(f, z) = \{\langle u, A_{2s(f,z)}(u), \lambda_{2s(f,z)}(u)\rangle : u \in U\}$$
for any $(f, z) \in E_{2s} \times X_{2s}$ such that
$$\lambda_{1s(f,z)}(u) \wedge \lambda_{2s(f,z)}(u) \in \{\inf\{\sup\{A^+_{1s(f,z)}(u), A^-_{2s(f,z)}(u)\},$$
$$\sup\{A^-_{1s(f,z)}(u), A^+_{2s(f,z)}(u)\},$$
$$\inf\{A^+_{1s(f,z)}(u), A^-_{2s(f,z)}(u)\},$$
$$\inf\{A^-_{1s(f,z)}(u), A^+_{2s(f,z)}(u)\}\}\}$$
for all $(f, z) \in (E_{1s} \cap E_{2s}) \times (X_{1s} \cap X_{2s})$ and $u \in U$.
Then
$$(\beta_{1s}, E_{1s}, X_{1s}) \cap_{P_s} (\beta_{2s}, E_{2s}, X_{2s})$$
is both an ENSCSS and INSCSS over U.

Proof. We can prove this theorem by similar way to theorem 2.22.

Theorem 2.26 Let $(\beta_{1s}, E_{1s}, X_{1s})$ and $(\beta_{2s}, E_{2s}, X_{2s})$ are two NSCSSs over U, where
$$\beta_{1s}(\breve{e}, x) = \{\langle u, A_{1s(\breve{e},x)}(u), \lambda_{1s(\breve{e},x)}(u)\rangle : u \in U\}$$
for any $(\breve{e}, x) \in E_{1s} \times X_{1s}$ and
$$\beta_{2s}(f, z) = \{\langle u, A_{2s(f,z)}(u), \lambda_{2s(f,z)}(u)\rangle : u \in U\}$$
for any $(f, z) \in E_{2s} \times X_{2s}$ such that
$$\lambda_{1s(f,z)}(u) \vee \lambda_{2s(f,z)}(u) \in \{\inf\{\sup\{A^+_{1s(f,z)}(u), A^-_{2s(f,z)}(u)\},$$
$$\sup\{A^-_{1s(f,z)}(u), A^+_{2s(f,z)}(u)\},$$
$$\inf\{A^+_{1s(f,z)}(u), A^-_{2s(f,z)}(u)\},$$
$$\inf\{A^-_{1s(f,z)}(u), A^+_{2s(f,z)}(u)\}\}\}$$
for all $(f, z) \in (E_{1s} \cap E_{2s}) \times (X_{1s} \cap X_{2s})$ and $u \in U$.
Then
$$(\beta_{1s}, E_{1s}, X_{1s}) \cap_{R_s} (\beta_{2s}, E_{2s}, X_{2s})$$
is both an ENSCSS and INSCSS over U.

Proof. We can prove this theorem by similar way to theorem 2.22.

Theorem 2.27 For any two NSCSS $(\beta_{1s}, E_{1s}, X_{1s})$ and $(\beta_{2s}, E_{2s}, X_{2s})$ the following absorption law hold [1)]

1. $(\beta_{1s}, E_{1s}, X_{1s}) \cup_{R_s} ((\beta_{1s}, E_{1s}, X_{1s}) \cap_{P_s} (\beta_{2s}, E_{2s}, X_{2s})) = (\beta_{1s}, E_{1s}, X_{1s})$
2. $(\beta_{1s}, E_{1s}, X_{1s}) \cap_{P_s} ((\beta_{1s}, E_{1s}, X_{1s}) \cup_{P_s} (\beta_{2s}, E_{2s}, X_{2s})) = (\beta_{1s}, E_{1s}, X_{1s})$
3. $(\beta_{1s}, E_{1s}, X_{1s}) \cup_{R_s} ((\beta_{1s}, E_{1s}, X_{1s}) \cap_{P_s} (\beta_{2s}, E_{2s}, X_{2s})) = (\beta_{1s}, E_{1s}, X_{1s})$
4. $(\beta_{1s}, E_{1s}, X_{1s}) \cap_{R_s} ((\beta_{1s}, E_{1s}, X_{1s}) \cup_{R_s} (\beta_{2s}, E_{2s}, X_{2s})) = (\beta_{1s}, E_{1s}, X_{1s})$

Proof. [1)]

1. By definitions 2.8 and 2.9, we have

$$(\beta_{1s}, E_{1s}, X_{1s}) \cup_{P_s} ((\beta_{1s}, E_{1s}, X_{1s}) \cap_{P_s} (\beta_{2s}, E_{2s}, X_{2s}))$$
$$= (\beta_{3s}, (E_{1s} \cup_{P_s} (E_{1s} \cap_{P_s} E_{2s})), (X_{1s} \cup_{P_s} (X_{1s} \cap_{P_s} X_{2s})))$$
$$= (\beta_{3s}, E_{1s}, X_{1s})$$
such that for any $f \in E_{1s}$ and $z \in X_{1s}$ we have
$$\beta_{3s}(f, z) = \beta_{1s}(f, z) \cup_{P_s} (\beta_{1s}(f, z) \cap_{P_s} \beta_{2s}(f, z)).$$
If $(f, z) \in E_{1s} \times X_{1s}$,

$$\beta_{1s}(f,z) \cup_{P_s} (\beta_{1s}(f,z) \cap_{P_s} \beta_{2s}(f,z))$$
$$= \{\langle u, A_{1s}(\breve{e},x)(u), \lambda_{1s}(\breve{e},x)(u)\rangle, u \in U, (\breve{e},x) \in E_{1s} \times X_{1s}\}$$
$$\cup_{P_s} \{\{\langle u, A_{1s}(\breve{e},x)(u), \lambda_{1s}(\breve{e},x)(u)\rangle, u \in U, (\breve{e},x) \in E_{1s} \times X_{1s}\}$$
$$\cap_{P_s} \{\langle u, A_{2s}(f,z)(u), \lambda_{2s}(f,z)(u)\rangle, (f,z) \in E_{2s} \times X_{2s}\}\}$$
$$= \{\langle u, A_{1s}(\breve{e},x)(u), \lambda_{1s}(\breve{e},x)(u)\rangle, u \in U, (\breve{e},x) \in E_{1s} \times X_{1s}\}$$
$$\cup_{P_s} \{\langle u, r\inf\{A_{1s}(\breve{e},x)(u), A_{2s}(f,z)(u)\}, \inf\{\lambda_{1s}(\breve{e},x)(u), \lambda_{2s}(f,z)(u)\}\rangle\}$$
$$= \{\langle u, r\sup\{A_{1s}(\breve{e},x)(u), r\inf\{A_{1s}(\breve{e},x)(u), A_{2s}(f,z)(u)\},$$
$$\sup\{\lambda_{1s}(\breve{e},x)(u), \inf\{\lambda_{1s}(\breve{e},x)(u), \lambda_{2s}(f,z)(u)\}\rangle\}$$
$$\subseteq \{\langle u, A_{1s}(\breve{e},x)(u), \lambda_{1s}(\breve{e},x)(u)\rangle, u \in U, (\breve{e},x) \in E_{1s} \times X_{1s}\}$$
$$= \beta_{1s}(\breve{e},x)$$
$$\subseteq \{\langle u, r\inf\{A_{1s}(\breve{e},x)(u), r\sup\{A_{1s}(\breve{e},x)(u), A_{2s}(f,z)(u)\}\},$$
$$\inf\{\lambda_{1s}(\breve{e},x)(u), \sup\{\lambda_{1s}(\breve{e},x)(u), \lambda_{2s}(f,z)(u)\}\}\rangle\}$$
$$= \{\langle u, r\sup\{A_{1s}(\breve{e},x)(u), r\inf\{A_{1s}(\breve{e},x)(u), A_{2s}(f,z)(u)\}\},$$
$$\sup\{\lambda_{1s}(\breve{e},x)(u), \inf\{\lambda_{1s}(\breve{e},x)(u), \lambda_{2s}(f,z)(u)\}\}\rangle\}$$
$$= \beta_{1s}(\breve{e},x) \cup_{P_s} ((\beta_{1s}(\breve{e},x) \cap_{P_s} \beta_{2s}(f,z))).$$

In second case, when $(f,z) \in (E_{1s} \times X_{1s})\backslash(E_{2s} \times X_{2s})$, using definitions 2.8 and 2.9, we have

$$\beta_{1s}(\breve{e},x) \cup_{P_s} ((\beta_{1s}(\breve{e},x) \cap_{P_s} \beta_{2s}(f,z))) = \beta_{1s}(\breve{e},x) \cup_{P_s} \beta_{1s}(\breve{e},x) = \beta_{1s}(\breve{e},x)$$

which is one required result for both the cases.

Similarly we can prove 2.27),2.27) and 2.27).

Definition 2.28 *For two NSCSSs* $(\beta_{1s}, E_{1s}, X_{1s})$ *and* $(\beta_{2s}, E_{2s}, X_{2s})$ *over* U, $P-AND$ *is denoted and defined as,*

$$(\beta_{1s}, E_{1s}, X_{1s}) \wedge_{P_s} (\beta_{2s}, E_{2s}, X_{2s}) = \beta_{3s}(E_{1s} \times E_{2s}), (X_{1s} \times X_{2s})$$

where $\beta_{3s}((\breve{e},f),(x,y)) = \beta_{1s}(\breve{e},x) \cap_{P_s} \beta_{2s}(f,z)$ for all $((\breve{e},f),(x,y)) \in ((E_{1s} \times E_{2s}) \times (X_{1s} \times X_{2s}))$.

Definition 2.29 For two NSCSSs $(\beta_{1s}, E_{1s}, X_{1s})$ and $(\beta_{2s}, E_{2s}, X_{2s})$ over U, $R-AND$ is denoted and defined as,

$$(\beta_{1s}, E_{1s}, X_{1s}) \wedge_{R_s} (\beta_{2s}, E_{2s}, X_{2s}) = \beta_{3s}(E_{1s} \times E_{2s}), (X_{1s} \times X_{2s})$$

where $\beta_{3s}((\breve{e},f),(x,y)) = \beta_{1s}(\breve{e},x) \cap_{R_s} \beta_{2s}(f,z)$ for all $((\breve{e},f),(x,y)) \in ((E_{1s} \times E_{2s}) \times (X_{1s} \times X_{2s}))$.

Definition 2.30 For two NSCSSs $(\beta_{1s}, E_{1s}, X_{1s})$ and $(\beta_{2s}, E_{2s}, X_{2s})$ over U, $P-OR$ is denoted and defined as,

$$(\beta_{1s}, E_{1s}, X_{1s}) \wedge_{P_s} (\beta_{2s}, E_{2s}, X_{2s}) = \beta_{3s}(E_{1s} \times E_{2s}), (X_{1s} \times X_{2s})$$

where $\beta_{3s}((\breve{e},f),(x,y)) = \beta_{1s}(\breve{e},x) \cup_{P_s} \beta_{2s}(f,z)$ for all $((\breve{e},f),(x,y)) \in ((E_{1s} \times E_{2s}) \times (X_{1s} \times X_{2s}))$.

Definition 2.31 For two NSCSSs $(\beta_{1s}, E_{1s}, X_{1s})$ and $(\beta_{2s}, E_{2s}, X_{2s})$ over U, $R-OR$ is denoted and defined as,

$$(\beta_{1s}, E_{1s}, X_{1s}) \vee_{R_s} (\beta_{2s}, E_{2s}, X_{2s}) = \beta_{3s}(E_{1s} \times E_{2s}), (X_{1s} \times X_{2s})$$

where $\beta_{3s}((\breve{e},f),(x,y)) = \beta_{1s}(\breve{e},x) \cup_{P_s} \beta_{2s}(f,z)$ for all $((\breve{e},f),(x,y)) \in ((E_{1s} \times E_{2s}) \times (X_{1s} \times X_{2s}))$.

Example 2.32 Let $U = \{u_1, u_2, u_3\}$ be the initial universe. $E_s = \{e_1, e_2\}$ be the set of attributes, $X_s = \{x_1, x_2\}$ be the set of experts. Then the cubic set $(\beta_{1s}, E_{1s}, X_{1s})$ over U is given below.

$$\beta_{1s}(e_1,x_1) = \begin{Bmatrix} u_1, & ([0.1,0.4],[0.6,1.0],[0.3,0.4]), & (0.2,0.5,0.3) \\ u_2, & ([0.4,0.5],[0.5,1.0],[0.1,0.2]), & (0.4,0.4,0.2) \\ u_3, & ([0.3,0.5],[0.6,1.0],[0.1,0.4]), & (0.5,0.4,0.5) \end{Bmatrix}$$

$$\beta_{1s}(e_2,x_2) = \begin{Bmatrix} u_1, & ([0.1,0.3],[0.8,1.0],[0.1,0.4]), & (0.2,0.7,0.3) \\ u_2, & ([0.3,0.4],[0.5,1.0],[0.2,0.3]), & (0.4,0.5,0.3) \\ u_3, & ([0.1,0.2],[0.6,1.0],[0.3,0.4]), & (0.1,0.7,0.4) \end{Bmatrix}$$

Let $U = \{u_1, u_2, u_3\}$ be the initial universe, $F_s = \{f_1, f_2\}$ be the set of attributes and $Y_s = \{y_1, y_2\}$ be the set of experts. Then the cubic set (β_{2s}, F_s, Y_s) over U is given below.

$$\beta_{2s}(f_1,y_1)=\begin{Bmatrix} u_1, & ([0.2,0.4],[0.5,1.0],[0.3,0.5]), & (0.2,0.6,0.4)\\ u_2, & ([0.3,0.5],[0.5,1.0],[0.2,0.3]), & (0.3,0.6,0.2)\\ u_3, & ([0.1,0.2],[0.7,1.0],[0.2,0.4]), & (0.1,0.8,0.3)\end{Bmatrix}$$

$$\beta_{2s}(f_2,y_2)=\begin{Bmatrix} u_1, & ([0.2,0.4],[0.2,1.0],[0.3,0.5]), & (0.3,0.5,0.4)\\ u_2, & ([0.1,0.4],[0.8,1.0],[0.1,0.2]), & (0.2,0.8,0.1)\\ u_3, & ([0.3,0.4],[0.3,1.0],[0.4,0.5]), & (0.4,0.6,0.5)\end{Bmatrix}$$

By using definitions 2.28, 2.29, 2.30 and 2.31, we have

$$(\beta_{1s},E_{1s},X_s)\wedge_{P_s}(\beta_{2s},F_s,Y_s)$$
$$=\beta_{3s},(E_{1s}\times F_s),(X_s\times Y_s)$$
$$=\beta_{3s}((\breve{e},f),(x,y))$$
$$=\begin{Bmatrix} u_1, & ([0.1,0.4],[0.5,1.0],[0.3,0.4]), & (0.2,0.5,0.3)\\ u_2, & ([0.3,0.5],[0.5,1.0],[0.1,0.2]), & (0.3,0.4,0.2)\\ u_3, & ([0.1,0.2],[0.6,1.0],[0.1,0.4]), & (0.1,0.4,0.3)\end{Bmatrix}$$

$$\beta_{3s}((e_2,f_2),(x_2,y_2))$$
$$=\begin{Bmatrix} u_1, & ([0.1,0.3],[0.2,1.0],[0.1,0.4]), & (0.2,0.5,0.3)\\ u_2, & ([0.1,0.4],[0.5,1.0],[0.1,0.2]), & (0.2,0.5,0.1)\\ u_3, & ([0.1,0.2],[0.3,1.0],[0.3,0.4]), & (0.1,0.6,0.4)\end{Bmatrix}$$

$$(\beta_{1s},E_s,X_s)\wedge_{R_s}(\beta_{2s},F_s,Y_s)$$
$$=\beta_{3s},(E_s\times F_s),(X_s\times Y_s)$$
$$=\beta_{3s}((e_1,f_1),(x_1,y_1))$$
$$=\begin{Bmatrix} u_1, & ([0.1,0.4],[0.5,1.0],[0.3,0.4]), & (0.2,0.6,0.4)\\ u_2, & ([0.3,0.5],[0.5,1.0],[0.1,0.2]), & (0.4,0.6,0.2)\\ u_3, & ([0.1,0.2],[0.6,1.0],[0.1,0.4]), & (0.5,0.8,0.5)\end{Bmatrix}$$

$$\beta_{3s}((e_2,f_2),(x_2,y_2))$$
$$=\begin{Bmatrix} u_1, & ([0.1,0.3],[0.2,1.0],[0.1,0.4]), & (0.3,0.7,0.4)\\ u_2, & ([0.1,0.4],[0.5,1.0],[0.1,0.2]), & (0.4,0.8,0.3)\\ u_3, & ([0.1,0.2],[0.3,1.0],[0.3,0.4]), & (0.4,0.7,0.5)\end{Bmatrix}$$

$$(\beta_{1s},E_s,X_s)\vee_{P_s}(\beta_{2s},F_s,Y_s)$$
$$=\beta_{3s},(E_s\times F_s)(X_s\times Y_s)$$
$$=\beta_{3s}((\breve{e},f),(x,y))$$
$$=\begin{Bmatrix} u_1, & ([0.2,0.4],[0.6,1.0],[0.3,0.5]), & (0.2,0.6,0.4)\\ u_2, & ([0.4,0.5],[0.5,1.0],[0.2,0.3]), & (0.4,0.6,0.2)\\ u_3, & ([0.3,0.5],[0.7,1.0],[0.2,0.4]), & (0.5,0.8,0.5)\end{Bmatrix}$$

$$\beta_{3s}((e_2,f_2),(x_2,y_2))$$
$$=\begin{Bmatrix} u_1, & ([0.2,0.4],[0.8,1.0],[0.3,0.5]), & (0.3,0.5,0.4)\\ u_2, & ([0.3,0.4],[0.8,1.0],[0.2,0.3]), & (0.4,0.8,0.3)\\ u_3, & ([0.3,0.4],[0.6,1.0],[0.4,0.5]), & (0.4,0.7,0.5)\end{Bmatrix}$$

$$(\beta_{1s}, E_s, X_s) \vee_{R_s} (\beta_{2s}, F_s, Y_s)$$
$$= \beta_{3s}, (E_s \times F_s), (X_s \times Y_s)$$
$$= \beta_{3s}((e_1, f_1), (x_1, y_1))$$
$$= \begin{Bmatrix} u_1, & ([0.2,0.4],[0.6,1.0],[0.3,0.5]), & (0.2,0.5,0.3) \\ u_2, & ([0.4,0.5],[0.5,1.0],[0.2,0.3]), & (0.3,0.6,0.2) \\ u_3, & ([0.3,0.5],[0.7,1.0],[0.2,0.4]), & (0.1,0.4,0.3) \end{Bmatrix}$$

$$\beta_{3s}((e_2, f_2), (x_2, y_2))$$
$$= \begin{Bmatrix} u_1, & ([0.2,0.4],[0.8,1.0],[0.3,0.5]), & (0.2,0.5,0.3) \\ u_2, & ([0.3,0.4],[0.8,1.0],[0.2,0.3]), & (0.2,0.5,0.1) \\ u_3, & ([0.3,0.4],[0.6,1.0],[0.4,0.5]), & (0.1,0.6,0.4) \end{Bmatrix}$$

respectively.

Theorem 2.33 Let $(\beta_{1s}, E_{1s}, X_{1s})$ be a NSCSS over U. If $(\beta_{1s}, E_{1s}, X_{1s})$ is an INSCSS(ENSCSS) then $(\beta_{1s}, E_{1s}, X_{1s})^C$ INSCSS(ENSCSS) respectively.

Proof. By using definition 2.2 and 2.3 we can proof this theorem.

Definition 2.34 Let $A_{s(\breve{e},x_i)}, \lambda_{s(\breve{e},x_i)} \in$ NSCSES over U, $1 \leq i \leq n$. The Neutrosophic Spherical Cubic soft Weighted Average Quotient Operator(NSCSWAQO) is denoted and defined as,

$$P_{w_{s_i}}(A_{s(\breve{e},x_i)}, \lambda_{s(\breve{e},x_i)})$$
$$= \left(\frac{\prod_{i=1}^{n}(1+A^-_{s(\breve{e},x_i)}(u))^{w_{s_i}} - \prod_{i=1}^{n}(1-A^-_{s(\breve{e},x_i)}(u))^{w_{s_i}}}{\prod_{i=1}^{n}(1+A^-_{s(\breve{e},x_i)}(u))^{w_{s_i}} + \prod_{i=1}^{n}(1-A^-_{s(\breve{e},x_i)}(u))^{w_{s_i}}} \right.$$
$$\frac{\prod_{i=1}^{n}(1+A^+_{s(\breve{e},x_i)}(u))^{w_{s_i}} - \prod_{i=1}^{n}(1-A^+_{s(\breve{e},x_i)}(u))^{w_{s_i}}}{\prod_{i=1}^{n}(1+A^+_{s(\breve{e},x_i)}(u))^{w_{s_i}} + \prod_{i=1}^{n}(1-A^+_{s(\breve{e},x_i)}(u))^{w_{s_i}}}$$
$$\left. \prod_{i=1}^{n} (\lambda_{s(\breve{e},x_i)}(u))^{w_{s_i}} \right)$$

where w_{s_i} is the weight of experts opinion $w_{s_i} \in [0,1]$ and $\sum_{i=1}^{n} w_s^i = 1$.

Definition 2.35 Let $\beta_s = \langle [A^-_{s(\breve{e},x)}, A^+_{s(\breve{e},x)}], \lambda_{s(\breve{e},x)} \rangle$ be a NSCS value. A score function $\breve{S}$ of NSCSS value defined as

$$\breve{S}(\beta_s) = \frac{A^-_{s(\breve{e},x)} + A^+_{s(\breve{e},x)} - \lambda_{s(\breve{e},x)}}{3}$$

where $\breve{S}(\beta_s) \in [-1,1]$.

Example 2.36 Let $U = \{u_1 =$Kenya, $u_2 =$Uganda, $u_3 =$Algeria Sudan, $u_4 =$Morocco$\}$ be the set of countries, $E = \{e_1 =$Dry Cough, $e_2 =$Diarrhea, $e_3 =$Nausea and Vomitting, $e_4 =$Severe Headache$\}$ be the set of attributes, $X_s = \{x_1, x_2\}$ be the set of symptoms of COVID patients $X = (x_1, x_2, x_3)$ be the set of Physicians.

Step 1:

$$\beta_1(e_1, x_1) = \begin{Bmatrix} u_1, & ([0.2,0.4],[0.6,1.0],[0.2,0.4]), & (0.2,0.7,0.3) \\ u_2, & ([0.3,0.4],[0.5,1.0],[0.2,0.4]), & (0.3,0.5,0.2) \\ u_3, & ([0.4,0.5],[0.4,1.0],[0.2,0.5]), & (0.5,0.5,0.3) \\ u_4, & ([0.1,0.3],[0.3,1.0],[0.6,0.7]), & (0.2,0.4,0.6) \end{Bmatrix}$$

$$\beta_1(e_2, x_1) = \begin{Bmatrix} u_1, & ([0.3,0.5],[0.5,1.0],[0.2,0.3]), & (0.3,0.6,0.2) \\ u_2, & ([0.2,0.4],[0.3,1.0],[0.5,0.6]), & (0.2,0.3,0.5) \\ u_3, & ([0.1,0.3],[0.3,1.0],[0.6,0.7]), & (0.1,0.4,0.6) \\ u_4, & ([0.3,0.4],[0.2,1.0],[0.5,0.6]), & (0.3,0.2,0.5) \end{Bmatrix}$$

$$\beta_1(e_3,x_1)=\begin{Bmatrix} u_1, & ([0.1,0.4],[0.6,1.0],[0.3,0.5]), & (0.2,0.6,0.4) \\ u_2, & ([0.3,0.5],[0.6,1.0],[0.1,0.4]), & (0.5,0.4,0.5) \\ u_3, & ([0.1,0.3],[0.6,1.0],[0.3,0.4]), & (0.3,0.6,0.3) \\ u_4, & ([0.5,0.6],[0.2,1.0],[0.2,0.4]), & (0.5,0.6,0.3) \end{Bmatrix}$$

$$\beta_1(e_4,x_1)=\begin{Bmatrix} u_1, & ([0.3,0.6],[0.7,1.0],[0.0,0.2]), & (0.2,0.8,0.1) \\ u_2, & ([0.2,0.6],[0.7,1.0],[0.1,0.4]), & (0.1,0.6,0.2) \\ u_3, & ([0.3,0.5],[0.6,1.0],[0.1,0.4]), & (0.2,0.5,0.2) \\ u_4, & ([0.2,0.3],[0.7,1.0],[0.1,0.2]), & (0.2,0.7,0.1) \end{Bmatrix}$$

Step 2:

$$\beta_1(e_1,x_2)=\begin{Bmatrix} u_1, & ([0.2,0.3],[0.5,1.0],[0.3,0.5]), & (0.2,0.6,0.3) \\ u_2, & ([0.1,0.4],[0.7,1.0],[0.2,0.4]), & (0.2,0.8,0.3) \\ u_3, & ([0.5,0.6],[0.3,1.0],[0.2,0.3]), & (0.5,0.7,0.2) \\ u_4, & ([0.1,0.2],[0.7,1.0],[0.2,0.3]), & (0.1,0.7,0.2) \end{Bmatrix}$$

$$\beta_1(e_2,x_2)=\begin{Bmatrix} u_1, & ([0.4,0.6],[0.4,1.0],[0.2,0.3]), & (0.4,1.0,0.2) \\ u_2, & ([0.5,0.6],[0.3,1.0],[0.2,0.3]), & (0.5,1.0,0.3) \\ u_3, & ([0.2,0.4],[0.7,1.0],[0.1,0.2]), & (0.3,0.5,0.4) \\ u_4, & ([0.0,0.1],[0.7,1.0],[0.3,0.4]), & (0.1,0.7,0.4) \end{Bmatrix}$$

$$\beta_1(e_3,x_2)=\begin{Bmatrix} u_1, & ([0.1,0.2],[0.5,1.0],[0.4,0.4]), & (0.1,0.6,0.4) \\ u_2, & ([0.2,0.3],[0.2,1.0],[0.6,0.6]), & (0.1,0.2,0.6) \\ u_3, & ([0.2,0.3],[0.5,1.0],[0.3,0.4]), & (0.1,0.5,0.3) \\ u_4, & ([0.0,0.1],[0.8,1.0],[0.2,0.4]), & (0.2,0.7,0.2) \end{Bmatrix}$$

$$\beta_1(e_4,x_2)=\begin{Bmatrix} u_1, & ([0.5,0.5],[0.4,1.0],[0.1,0.2]), & (0.6,0.4,1.0) \\ u_2, & ([0.2,0.3],[0.4,1.0],[0.4,0.5]), & (0.1,0.0,1.0) \\ u_3, & ([0.0,0.1],[0.6,1.0],[0.4,0.5]), & (0.2,0.5,0.4) \\ u_4, & ([0.2,0.3],[0.6,1.0],[0.2,0.3]), & (0.2,0.6,0.2) \end{Bmatrix}$$

Step 3:

$$\beta_1(e_1,x_3)=\begin{Bmatrix} u_1, & ([0.2,0.3],[0.6,1.0],[0.2,0.3]), & (0.2,0.6,0.3) \\ u_2, & ([0.4,0.6],[0.5,1.0],[0.1,0.2]), & (0.4,0.5,0.2) \\ u_3, & ([0.3,0.6],[0.2,1.0],[0.6,0.5]), & (0.3,0.2,0.5) \\ u_4, & ([0.2,0.3],[0.5,1.0],[0.3,0.4]), & (0.1,0.5,0.3) \end{Bmatrix}$$

$$\beta_1(e_2,x_3)=\begin{Bmatrix} u_1, & ([0.0,0.1],[0.6,1.0],[0.4,0.5]), & (0.0,1.0,0.5) \\ u_2, & ([0.2,0.3],[0.5,1.0],[0.3,0.4]), & (0.2,0.5,0.3) \\ u_3, & ([0.5,0.6],[0.4,1.0],[0.1,0.2]), & (0.5,0.4,1.0) \\ u_4, & ([0.3,0.4],[0.3,1.0],[0.2,0.5]), & (0.4,1.0,0.2) \end{Bmatrix}$$

$$\beta_1(e_3,x_3)=\begin{Bmatrix} u_1, & ([0.2,0.4],[0.3,1.0],[0.5,0.6]), & (0.2,1.0,0.3) \\ u_2, & ([0.1,0.3],[0.8,1.0],[0.1,0.2]), & (0.0,0.1,0.2) \\ u_3, & ([0.4,0.5],[0.6,1.0],[0.0,0.2]), & (0.4,0.1,0.2) \\ u_4, & ([0.5,0.6],[0.2,1.0],[0.3,0.4]), & (0.5,0.2,0.3) \end{Bmatrix}$$

$$\beta_1(e_4,x_3)=\begin{Bmatrix} u_1, & ([0.2,0.3],[0.5,1.0],[0.3,0.4]), & (0.2,0.5,0.2) \\ u_2, & ([0.4,0.6],[0.2,1.0],[0.4,0.5]), & (0.5,0.2,0.5) \\ u_3, & ([0.5,0.6],[0.4,1.0],[0.1,0.2]), & (0.5,0.4,0.2) \\ u_4, & ([0.1,0.2],[0.7,1.0],[0.2,0.3]), & (0.1,0.8,0.2) \end{Bmatrix}$$

Step 4: The cubic soft expert weighted average of each attribute.

$$\left\{\begin{array}{ll} e_1, & ([0.20,0.34],[0.58,1.0],[0.22,0.39]),(0.59,0.64,0.30) \\ & ([0.29,0.47],[0.54,1.0],[0.17,0.33]),(0.31,0.57,0.22) \\ & ([0.39,0.56],[0.31,1.0],[0.31,0.46]),(0.43,0.44,0.35) \\ & ([0.16,0.28],[0.46,1.0],[0.41,0.50]),(0.14,0.50,0.41) \\ e_2, & ([0.22,0.38],[0.47,1.0],[0.27,0.37]),(0.22,0.83,0.31) \\ & ([0.27,0.41],[0.37,1.0],[0.37,0.47]),(0.27,0.52,0.38) \\ & ([0.26,0.36],[0.42,1.0],[0.32,0.42]),(0.28,0.42,0.38) \\ & ([0.23,0.33],[0.33,1.0],[0.42,0.52]),(0.29,0.59,0.37) \\ e_3, & ([0.12,0.36],[0.47,1.0],[0.39,0.51]),(0.18,0.43,0.37) \\ & ([0.21,0.37],[0.58,1.0],[0.21,0.37]),(0.24,0.57,0.42) \\ & ([0.26,0.37],[0.58,1.0],[0.19,0.33]),(0.49,0.40,0.27) \\ & ([0.39,0.49],[0.33,1.0],[0.24,0.40]),(0.43,0.48,0.30) \\ e_4, & ([0.31,0.47],[0.56,1.0],[0.13,0.27]),(0.29,0.61,0.33) \\ & ([0.21,0.53],[0.20,1.0],[0.21,0.39]),(0.24,0.33,0.28) \\ & ([0.34,0.41],[0.53,1.0],[0.17,0.37]),(0.31,0.47,0.24) \\ & ([0.17,0.27],[0.68,1.0],[0.16,0.26]),(0.17,0.71,0.23) \end{array}\right\}$$

Step 5: Calculate the $\vee_{Ps}$ of 1st, 2nd, 3rd and 4th column of above table by using definition 3.30, so we have

$$\widehat{U_{1s}} = \bigvee_{j=1}^{4} \{e_j, u_1\} = \{([0.39,0.56],[0.58,1.0],[0.41,0.46]),(0.59,0.69,0.14)\}$$

$$\widehat{U_{2s}} = \bigvee_{j=1}^{4} \{e_j, u_2\} = \{([0.27,0.41],[0.47,1.0],[0.42,0.52]),(0.29,0.83,0.38)\}$$

$$\widehat{U_{3s}} = \bigvee_{j=1}^{4} \{e_j, u_3\} = \{([0.26,0.37],[0.58,1.0],[0.42,0.51]),(0.49,0.57,0.42)\}$$

$$\widehat{U_{4s}} = \bigvee_{j=1}^{4} \{e_j, u_4\} = \{([0.54,0.53],[0.68,0.1],[0.27,0.37]),(0.31,0.71,0.33)\}$$

Step 6: Now calculate the score of above NSCSS elements by using definition 3.34

$$\breve{S}(\widehat{U_{1s}}) = (0.19)$$
$$\breve{S}(\widehat{U_{2s}}) = (0.01)$$
$$\breve{S}(\widehat{U_{3s}}) = (0.18)$$
$$\breve{S}(\widehat{U_{4s}}) = (0.61)$$

Step 7: Generate the non-decreasing order of the score of NCSES set values corresponding to $\vee_P$ we have the following order

$$u_4 > u_1 > u_3 > u_2.$$

In above example, we want to check which country is much affected by COVID-19. Here Morocco is more affected by COVID-19.

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