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Concerning The Cost Of Rapid Control Of Certain Equations

YAMEOGO Pierre claver

Dr. En Mathématiques(Ph.D)

BV30246 Ouaga Pissy 10050 Ouagadougou BF

Abstract

Throughout this paper, we consider the cost of null controllability for a large class of linear equations of parabolic or dispersive type in one dimension in small time. We are able to give precise upper bounds on the time-dependence of the fast controls when the time of control T tends to 0. We also give a lower bound of the cost of fast controls for the same class of equations, which proves the optimality of the power of T involved in the cost of the control. These general results are then applied to treat notably the case of linear KdV equations and fractional heat or Schrodinger equations.

Keywords: self-adjoint; space operator; moment method; controllability, Schrödinger equations.

5.1 Introduction

5.1.1 Presentation of the problem

This paper is devoted to studying fast boundary controls for some evolution equations of parabolic or dispersive type, with the spatial derivative not necessarily of second order.

Let H be an Hilbert space (the state space) and U be another Hilbert space (the control space). Let $A: D(A) \longrightarrow H$ be a self-adjoint operator with compact resolvent, the eigenvalues (which can be assumed to be different from 0 without loss of generality) are called $(\lambda_k)_{k\geq 1}$, the eigenvector corresponding to the eigenvalue $\lambda_k$is called e_k. We assume that $-A$ generates on H a strongly continuous semigroup $S: t \mapsto S(t) = e^{-tA}$. The Hilbert space $D(A^*)'(= D(A)')$ is from now on equipped with the norm

$$\|x\|^2_{D(A)'} = \sum \frac{<x,e_k>^2_H}{\lambda_k^2}.$$

We call $B\epsilon L_c(U, D(A)')$ an admissible control operator for this semigroup, i. e. such that there exists some time $T_0 > 0$, there exists some constant $C > 0$ such that for every $z\epsilon D(A)$,

$$\int_0^{T_0} \|B^* S(t)^* z\|^2_U \leq C\|z\|^2_H.$$

We recall that if B is admissible, then necessarily the previous inequality holds at every time, that is to say for every time $T > 0$, there exists some constant $C(T) > 0$ such that for every $z \epsilon D(A)$, one has

$$\int_0^{T_0} \|B^* S(t)^* z\|_U^2 \leq C(T) \|z\|_H^2.$$

From now on, we consider control systems of the following form:

$$y_t + A_y = B_u \qquad (5.1.1)$$

Or

$$y_t + iA_y = B_u \, , \qquad (5.1.2)$$

Where A will always be supposed to be positive in the parabolic case (i.e. for Equation (5.1.1)).

Then, it is well-known (see for example [Cor07, chapter 2, Section 2.3], the operators -A or -iA generates a strongly continuous semigroup under the hypothesis given before thanks due to the Lummer-Phillips or Stone theorems) that if $u \epsilon L^2((0,T),U)$, System (5.1.1) or (5.1.2) with initial condition $y^0 \epsilon H$ has a unique solution satisfying $y \epsilon C^0([0,T],H)$. Moreover, control $u \epsilon L^2((0,T_0),U)$ such that $y(T_0,.) \equiv 0$), then there exists a unique optimal (for the optimal null control cost at time T_0 (or in a more concise form the cost of the control) and denoted C_{T_0}, which is also the smallest constant $C > 0$ such that for every $y^0 \epsilon H$, there exists some control u driving y^0 to 0 at time T_0 with

$$\|u\|_{L^2((0,T_0).U} \leq C \|y^0\|_H.$$

Our goal in work is to estimate precisely the cost of the control C_T when the time $T \longrightarrow 0$ for some families of operators A which are null controllable in arbitrary small time, and is of great interest in itself but it may also be applied to study the uniform controllability of transport-diffusion in the vanishing viscosity limit as explained in [Lis12]. (the strategy described in [Lis12] might probably be extended to the study of other problems of uniform controllability, for example in zero dispersion limit or in zero diffusion limit as in [GG08] or [GG09]) It is obvious that C_T must tend $to \infty \; when \, T \longrightarrow 0.$

5.2 <u>Proofs of Theorems</u>

5.2.1 <u>Proof of Theorem 5.1.1</u>

The following lemma is a refinement of the estimates proved in $[FR71, Lemma\ 3.1]$ and is strongly inspired by $[TT07, Lemma\ 4.1]$.

Lemma 5.2.1. Let $(\lambda_n)_{n\geq 1}$ be a regular increasing sequence of strictly positive numbers versifying moreover that there exists some $\alpha \geq 2$ and some constant $R > 0$ such that (5.1.3) holds.

Let Φ_n be defined as follows :

$$\Phi_n(z) := \prod_{k \neq n} \left(1 - \frac{z}{\lambda_k - \lambda_n}\right).$$

Then

1. If $z \in \mathbb{C}$,

$$\Phi_n(z) \lesssim e^{\frac{\pi}{\sqrt{R}\sin(\pi/\alpha)}|z|^{\frac{1}{\alpha}}} P(|z|), \qquad (5.2.1)$$

where P is a polynomial.

2. If If $z \in \mathbb{R}$,

$$\Phi_n(-ix - \lambda_n) \lesssim e^{\frac{\pi}{2\sqrt{R}\sin(\pi/\alpha)}|x|^{\frac{1}{\alpha}}} P(\lambda_n, |x|), \quad (5.2.2)$$

Where $\overline{P}$ is a polynomial.

(In the previous inequalities, the implicit constant may depend on α but not on z, x or n)

Remark 5.2.1. One can see numerically that inequalities (5.2.1) and (5.2.2) are optimal for $\alpha \geq 2$, but are false for $\alpha \in (1,2)$ (but one could find a less precise estimate).

Proof of lemma 5.2.1. Without loss of generality, we can assume that $R = 1$ (one can go back to the general case by an easy scaling argument). We have then the existence of some constant $C > 0$ such that $|\lambda_n - n^\alpha| \leq Cn^{\alpha-1}$. From now on we call $\gamma := \gamma\big((\lambda_n)n \geq 1\big)$. As in [TT07, Page 81], one has.

$$\ln|\Phi_n(z)| \leq \int_0^{|z|}\int_\gamma^\infty \frac{L_n(s)}{(t+s)^2} dsdt,$$

where

$$L_n(s) := \#\{k\| \lambda_k - \lambda \leq s\}.$$

Let us estimate precisely $L_n(s)$.

One has

$$|\lambda_k - \lambda_n| \leq s$$

if and only if

$$\lambda_k - \lambda_n \leq s$$

and

$$\lambda_k - \lambda_k \leq s.$$

1. Assume that (5.2.4) holds. Then

$$k^{\alpha-1}(k - C) \leq \lambda_n + s.$$

Let

$$\overline{R}(X) = X^{\alpha-1}(X - C).$$

We call $D = \lambda_n + s$. By studying function $\overline{R}(0) = 0$, $\overline{R}(+\infty) = +\infty$ and that $\overline{R}$ is strictly decreasing on $[0, C(1 - 1/\alpha)\,]$ and then strictly increasing on $[C(1\text{-}1/\alpha), \infty]$. Hence the equation $\overline{R}(X) \leq D$ is equivalent to $0 \leq X \leq \tilde{X}$. Moreover,

$$\overline{R}\left(D^{\frac{1}{\alpha}}\right) - D = -CD^{\frac{\alpha-1}{\alpha}} < 0$$

and

$$\overline{R}\left(D^{\frac{1}{\alpha}}\right) - D = \left(D^{\frac{1}{\alpha}} + C\right)^{\alpha-1} D^{\frac{1}{\alpha}} - D = D\left(\left(1 + CD^{-\frac{1}{\alpha}}\right)^{\alpha-1} - 1\right) > 0.$$

So $\tilde{X} \in \left[D^{1/\alpha}, D^{1/\alpha} + C\right]$ and

$$0 \leq k \leq \tilde{X}$$

Implies

$$k \leq (\lambda_n + s)^{\frac{1}{\alpha}} + C. \qquad (5.2.6)$$

2. Assume now that (5.2.5) holds.

$$(5.2.7)$$

$$\lambda_n - s \leq k^{\alpha-1}(k + C).$$

We call $E = \lambda_n - s$. If $\lambda_n - s < 0$ then inequality (5.2.7) is always true. If $\lambda_n + s \geq 0$, we introfuce

$$\tilde{R}(X) = X^{\alpha-1}(X + C).$$

By studying function $\tilde{R}$, we see that $\tilde{R}(0) = 0$, $\tilde{R}(+\infty) = +\infty$ and that $\tilde{R}$ is strictly increasing on $[0, \infty)$. Hence the equation $\tilde{R}(X) = E$ has a unique solution $\tilde{X} \in [0, \infty)$ and the inequality $\tilde{R}(X) \geq D$ is equivalent to $X \geq \tilde{X}$. Moreover,

$$\tilde{R}\left(E^{\frac{1}{\alpha}}\right) - E = CE^{\frac{\alpha-1}{\alpha}} > 0$$

and

$$\tilde{R}\left(\left(E^{\frac{1}{\alpha}} - C\right)^+\right) - E = \left(\left(E^{\frac{1}{\alpha}} - C\right)^+\right)^{\alpha-1} E^{\frac{1}{\alpha}} - E\left(\left(\left(1 - CE^{\frac{1}{\alpha}}\right)^+\right)^{\alpha-1} - 1\right) < 0.$$

So

$$\tilde{X} \in \left[E^{1/\alpha} - C, E^{1/\alpha}\right]$$

and $k \geq \tilde{X}$ implies

$$k \geq \left((\lambda_n - s)^{\frac{1}{\alpha}} - C\right)^+ \geq \left((\lambda_n - s)^{\frac{1}{\alpha}} - C\right). \qquad (5.2.8)$$

Finally, if we have simultaneously the conditions (5.2.4) and (5.2.5) and (5.2.5), then combining inequalities (5.2.6) and (5.2.8) necessarily.

$$k \in \left[((\lambda_n - s)^+)^{\frac{1}{\alpha}} - C, (\lambda_n + s)^{\frac{1}{\alpha}} + C\right]$$

and

$$L_n(s) \leq (\lambda_n + s)^{\frac{1}{\alpha}} - ((\lambda_n - s)^+)^{\frac{1}{\alpha}} + 2C. \qquad (5.2.9)$$

Finally, from (5.2.3) and (5.2.9).

$$|\Phi_n(z)| \lesssim (1 + |z|/\gamma)^{2C} e^{\int_\gamma^{|z|} \frac{(\lambda_n+s)^{\frac{1}{\alpha}} - ((\lambda_n - s)^+)^{\frac{1}{\alpha}}}{(i+s)^2} ds dt} \qquad (5.2.10)$$

One has (using the change of variables $v = s / \lambda_n$ for the last inequality)

$$\int_0^{|z|} \int_\gamma^\infty \frac{(\lambda_n + s)^{\frac{1}{\alpha}} - ((\lambda_n - s)^+)^{\frac{1}{\alpha}}}{(y + s)^2} \leq |z| \int_\gamma^\infty \frac{(\lambda_n + s)^{\frac{1}{\alpha}} - ((\lambda_n - s)^+)^{\frac{1}{\alpha}}}{S(s + |z|)^2} ds \qquad (5.2.11)$$

$$\leq \lambda_n^{1-\frac{1}{\alpha}} \left(U\left(\frac{|z|}{\lambda_n}\right) + V\left(\frac{|z|}{\lambda_n}\right)\right),$$

where

$$U(x) := \int_0^1 \frac{(1 + s)^{\frac{1}{\alpha}} - ((1 - s)^+)^{\frac{1}{\alpha}}}{v(v + x)^2} dv \qquad (5.2.12)$$

and

$$(5.2.13)$$

$$V(x) := \int_1^\infty \frac{(v+1)^{\frac{1}{\alpha}}}{v(v+x)} dv$$

To prove inequality (5.2.1),, in view of (5.2.10) and (5.1.11) it is now enough to prove

$$x^{1-\frac{1}{\alpha}}\big(U(x)+V(x)\big) \leq \frac{\pi}{\sin\left(\frac{\pi}{\alpha}\right)} \qquad (5.2.14)$$

For every $x \geq 0$.

Let us now prove inequality (5.2.14). Let us first study $x^{1-1/\alpha}V(x)$. We remark that

$$x^{1-1/\alpha}V(x) = x^{1-1/\alpha}\int_1^\infty \frac{(v+1)^{\frac{1}{\alpha}}}{v(v+x)} dv = \int_1^\infty \frac{(v/x+1/x)^{\frac{1}{\alpha}}}{v(v/x+1)} dv.$$

By considering the change of variables $t = x/v$, we obtain

$$x^{1-1/\alpha}V(x) = \int_0^x \frac{(1/t+1/x)^{\frac{1}{\alpha}}}{1+t} dt. \qquad (5.2.15)$$

Similarly one has

$$x^{1-1/\alpha}U(x) = \int_x^\infty \frac{(1/t+1/x)^{\frac{1}{\alpha}} - (1/x+1/\mathrm{t})^{\frac{1}{\alpha}}}{1+t} dt. \qquad (5.2.16)$$

Using the dominated convergence Theorem, one proves easily that

$$x^{1-1/\alpha}V(x)\overrightarrow{x\to\infty}\int_0^\infty \frac{dt}{t^{\frac{1}{\alpha}}(1+t)}$$

and

$$x^{1-1/\alpha}U(x)\overrightarrow{x\to\infty}\ 0.$$

Let us call

$$I(\alpha) := \int_0^\infty \frac{dt}{t^{\frac{1}{\alpha}}(1+t)}.$$

One can compute explicitly this integral.

Lemma 5.2.2 $Let x > 1. Then$

$$I(x) = \frac{\pi}{\sin(\pi/x)}.$$

Proof of Lemma 5.2.2. We remind the following Definition of the Euler Beta function:

$$B(x,y) := \int_0^\infty \frac{t^{x-1}}{(1+t)^{x+y}} dt.$$

We the have

$$I(x) = B(1-1/x, 1/x). \qquad (5.2.17)$$

Using the link between the B function and the Γ function, we obtain

$$B(1\text{-}1/x, 1/x) = \frac{\Gamma(1-1/x)\Gamma(1/x)}{\Gamma(1-1/x+1/x)} = \Gamma(1-1/x)\Gamma(1/x). \quad (5.2.18)$$

Using the Euler reflection formula (which can be applied here because $1/x \epsilon(0,1)$), we obtain the desired result.

We will prove that for every $x > 0$ one has

$$x^{1-\frac{1}{\alpha}}\big(U(x) + V(x)\big) \leq I(\alpha). \quad (5.2.19)$$

Let us remark that one can compute explicitly V in terms of linear combining of hypergeometric functions: one can use for example Mathematica to check that

$$x^{1-1/\alpha}V(x) = -\alpha x^{-1/\alpha}\, 2F_1(-1/\alpha, -1/\alpha, 1-1/\alpha, -1)$$

$$+\alpha(1+1/x)^{\frac{1}{\alpha}} 2F1(-1/\alpha,, -1/\alpha, 1-1/\alpha, (x-1)/(x+1)), \quad (5.2.20)$$

Where 2F1is the ordinary hypergeometric function. It is the easy to prove that for every $\alpha \geq 2, x \mapsto x^{1-1/\alpha}V$ is increasing by differentiating (5.2.20) with respect to x. Let us consider two different cases:

1. Assume $x < 1$. In this case,

$$x^{1-\frac{1}{\alpha}}V(x) \leq -\alpha_2 F_1(-1/\alpha, 1-1/\alpha, \text{-}1) + \alpha 2^{1/\alpha}. \quad (5.2.21)$$

We remark (by differentiating $x^{1-1/\alpha}U(x)$ with respect to α in expression (5.2.16) that $\alpha \mapsto x^{1-1/\alpha}U(x)$ is increasing, so

$$x^{1-1/\alpha}U(x) \leq \sqrt{x}\int_0^1 \frac{(1+v)^{\frac{1}{2}} - (1-v)^{\frac{1}{2}}}{v(v+x)}\,dv \leq 1. \quad (5.2.22)$$

(the last inequality in (5.2.22) can be checked numerically for $x\epsilon[0,1]$)

We also have (the function $\alpha \mapsto 2F_1(-1/\alpha, -1/\alpha, 1-1/\alpha, -1)$ is increasing)

$$-\alpha_2 F_1(-1/\alpha, -1/\alpha, 1-1/\alpha, -1) \leq -\alpha_2 F_1(-1/2, -1/2, 1-1/2, -1) \leq -0.52\alpha. \quad (5.2.23)$$

Combining (5.2.21), (5.2.22) and (5.2.23), we deduce

$$x^{1-1/\alpha}\big(U(x) + V(x)\big) \leq 1 + \alpha 2^{1/\alpha} - 0.52\alpha$$

We just have to prove that

$$1 - 0.52\alpha + \alpha 2^{1/\alpha} \leq \frac{\pi}{\sin(\pi/\alpha)}. \quad (5.2.24)$$

One verifies numerically that (5.2.24) it is true for $\alpha\epsilon[2,3]$, and one verifies easily by differentiating the expression with respect to α that $\alpha \mapsto 1 - 0.52\alpha + \alpha 2^{1/\alpha} - \frac{\pi}{\sin(\pi/\alpha)}$ is decreasing at least on $(3,\infty)$. Inequality (5.2.19) is proved at least for $x < 1$.

2. Assume $x \geq 1$. We have (the equality can be easily obtained thanks to Mathematica for example)

$$x^{1-1/\alpha}U(x) \leq x^{-1/\alpha}\int_0^1 \frac{(1+v)^{\frac{1}{\alpha}} - (1-v)^{\frac{1}{\alpha}}}{v}\,dv$$

$$= x^{-1/\alpha}(H_{1/\alpha} + {}_2F_1(-1/\alpha, -1/\alpha, 1-1/\alpha, -1)),$$

(5.2.25)

Where we call $H_{1/\alpha}$ the (generalized) harmonic number of order $1/\alpha$. We have

$$H_{1/\alpha} \leq H_{1/\alpha} \leq 0.62. \qquad (5.2.26)$$

Using (5.2.20), (5.2.25) and (5.2.26), we deduce

$$x^{1-1/\alpha}\big(U(x)+V(x)\big) \leq x^{-1/\alpha}A(\alpha)+B(x) \qquad (5.2.27)$$

with

$$A(\alpha) = (0.62-\alpha)_2F_1(-1/\alpha,-1/\alpha,1-1/\alpha,-1)$$

and

$$B(x) = \alpha(1+1/x)^{1/\alpha}\ \ 2F_1(-1/\alpha,-1/\alpha,1-1/\alpha,(x-1)/(x+1)).$$

One has $A(\alpha)<0$, moreover, one easily proves that B is increasing with respect to x and tends to $\alpha_2F_1(-1/\alpha,-1/\alpha,1-1/\alpha,1)=I(\alpha)$. Hence inequality (5.2.27) implies that inequality (5.2.19) is also proved for $x\geq 1$ and finally (5.2.14) is proved.

Inequality (5.2.2) is easier to prove. Doing as in $[TT07, Page 83]$, we have

$$|\Phi_n(-ix-\lambda_n)|^2 = \prod_{k\neq n}\frac{|1+ix/\lambda_k|^2}{(1-\lambda_n/\lambda_k)^2} = B_n^2\prod_{k\neq n}|1+x^2/\lambda_k^2| \qquad (5.2.28)$$

where

$$B_n := \prod_{k\neq n}(1-\lambda_n/\lambda_k)^{-1}.$$

Let us remark that

$$\sum_{k\geq 1}\ln(1+x^2/\lambda_k^2 = \int_0^{|x|^2/\lambda_1^2}\frac{M(t)}{1+t}dt, \qquad (5.2.29)$$

where

$$M(t) := \sum_{\lambda_k\leq |x|/\sqrt{t}} 1.$$

One easily observe using same computations as before that

$$M(t) \leq |x|^{\frac{1}{\alpha}}t^{-1/(2\alpha)} + \ C. \qquad (5.2.30)$$

$$\sum_{k\geq 1}\ln(1+x^2/\lambda_k^2 \leq Cln\big(1+|x|^2/\lambda_k^2\big)+|x|^{\frac{1}{\alpha}}\int_0^{\pi}\frac{1}{t^{1/(2\alpha)}(1+t)}dt \leq Cln(1+|x|^2/\lambda_1^2+|x|^{\frac{1}{\alpha}}I(2\alpha).$$

We deduce by Lemma 5.2.2 and (5.2.28) that

$$\Phi_n(-ix-\lambda_n) \lesssim B_n(1+|x|^2/\lambda_1^2)^{C/2}e^{\pi|x|^{\frac{1}{\alpha}}/(2\sin(\pi/(2\alpha)))}$$

and it can be proved that B_n is at most polynomial in λ_n (the computations are the same as in $[TT07, Pages\ 83-84]$) as wished. This proves inequality (5.2.2).

Now, we study the multiplier, which is very similar to the one studied in [TT07]. Let $v>0$ and $\beta>0$ be linked by the following relation:

(5.2.31)

$$\beta v^{\alpha-1} = \left(4(\alpha-1)\right)^{\alpha-1}\left(\frac{\pi+\delta}{\alpha\sin(\pi/\alpha)}\right)^{\alpha},$$

where $\delta > 0$ is a small parameter.

We call

$$\sigma_v(t) := \exp\left(-\frac{v}{(1-t^2)}\right)$$

Prolonged by 0 outside $(-1;1)$. σ_v is analytic on $B(0,1)$. We call

$$H_\beta(z) := C_v\int_{-1}^{1}\sigma_v(t)e^{-i\beta tz}dt,$$

Where

$$C_v := 1/\|\sigma_v\|_1.$$

Thanks to [TT07, Lemma 4.3], we have

$$H\beta(0) = 1, \tag{5.2.32}$$

$$H\beta(ix) \gtrsim \frac{e^{\beta|x|/(2\sqrt{v+1})}}{\sqrt{v+1}}, \tag{5.2.33}$$

$$\frac{1}{2}e^{v} \leq C_v \leq \frac{3}{2}\sqrt{v+1}e^{v}, \tag{5.2.34}$$

$$\left|H_\beta(x)\right| \leq e^{\beta|Im(z)|}. \tag{5.2.35}$$

The main estimate is the following:

Lemma 5.2.3. For $x\epsilon\mathbb{R}$, we have

$$H_\beta(x) \lesssim \sqrt{v+1}e^{3v/4-\left((\pi+\delta/2)|x|^{\frac{1}{\alpha}}\right)/(\sin(\pi/\alpha))}$$

(The implicit constant may depend on α)

Remark 5.2.2 Lemma 5.2.3 is false for $\alpha\epsilon(1,2)$. This explain why we were not able to extend Theorem 5.1.1 to the case where $\alpha\epsilon(1,2)$. However, we know that systems like (5.1.1) and (5.1.2) are null controllable as soon as $\alpha > 1$, so one can conjecture that there is a way to extend the estimates for $\alpha\epsilon(1,2)$.

Proof of Lemma 5.2.3. First of all, consider some $t\epsilon[0,1)$ and $\theta\epsilon(-\pi,\pi)$. We call $\rho := 1-t \; and \; z := t+\rho e^{i\theta}$. One has (see [TT07, Page 85])

$$Re\frac{1}{1-z^2} \geq \frac{1}{4\rho} + \frac{1}{4} \geq \frac{1}{4\rho 1/(\alpha-1)} + \frac{1}{4},$$

because $\rho \leq 1 \; and \; \alpha \geq 2$. So, doing as in [TT07], we obtain by applying the Cauchy formula for holomorphic functions

$$\left|\sigma_v^{(j)}(t)\right| \leq j!\,e^{\frac{v}{4}}\sup_{\rho>0}\frac{e^{-\frac{1}{4\rho^{1/(\alpha-1)}}}}{\rho j}$$

Computing the supremum on $\rho\epsilon\mathbb{R}^{+*}$, we obtain

(5.2.36)

$$\left|\sigma_v^{(j)}(t)\right| \leq j!\, e^{-\frac{v}{4}} e^{-(\alpha-1)j}\left(\frac{4(\alpha-1)j}{v}\right)^{(\alpha-1)j}, t\epsilon[0,1).$$

Using the fact that σ_v is even, inequality (5.2.36) is true for every $t\epsilon(-1,1)$. Using inequality $j! > j^j e^{-j}$ in (5.2.36), we obtain

$$\left|\sigma_v^{(j)}(t)\right| \leq (j!)^{\alpha} e^{-\frac{v}{4}}\left(\frac{4(\alpha-1)}{v}\right)^{(\alpha-1)j} \quad (5.2.37)$$

Since all derivatives of σ_v vanish at $t=-1$ *and* $t=1$, we have

$$H_\beta(x) \leq \frac{2C_v \left\|\sigma_v^{(j)}\right\| \infty}{(\beta x)^j}, \quad (5.2.38)$$

For every $x > 0$ *and* $j\epsilon\mathbb{N}$. Combining (5.2.37), (5.2.38) and (5.2.34), we deduce that

$$H_\beta(x) \lesssim \sqrt{v+1}(j!)^{\alpha} e^{\frac{3v}{1}} \frac{\left(4(\alpha-1)\right)^{(\alpha-1)j}}{(\beta x)^j}, j \in \mathbb{N}. \quad (5.2.39)$$

We set

$$j := \left\lfloor (1/\alpha)(\beta x)^{1/\gamma} \right\rfloor \quad (5.2.40)$$

With some constants α and γ which will be chosen correctly soon. Then we have

$$\beta x \geq (\alpha j)^{\gamma}. \quad (5.2.41)$$

Using (5.2.41) and (5.2.39) we obtain

$$H_\beta(x) \lesssim \sqrt{v+1}(j!)^{\alpha} e^{\frac{3v}{4}} \frac{4(\alpha-1)j}{(\alpha j)^{\gamma j}}. \quad (5.2.42)$$

We choose $\gamma = \alpha$ *and* $\alpha = (4(\alpha-1)^{1-1/\alpha}$. Combining (5.2.42), (5.2.40), (5.2.31) and inequality

$$(j!)^{\alpha} \lesssim j^{\alpha/2} j^{\alpha j} e^{-\alpha j},$$

We deduce (5.2.43)

$$\left|H_\beta(x)\right| \lesssim \sqrt{v+1} e^{\frac{3v}{4}} e^{-\alpha j} j^{\alpha/2} \leq \sqrt{v+1} e^{\frac{3v}{4}} e^{-(\pi+\delta/2)/(\sin(\pi/\alpha))|x|^{\frac{1}{\alpha}}}.$$

Proof of Theorem 5.1.1.

The proof follows the proof of [TT07, Theorem 3.1 and 3.4]. We still assume without loss of generality that $R=1$. Let us first consider the dispersive case (Equation (5.1.2)). We call

$$g_n(z) := \Phi_n(-z - \lambda_n H\beta(z+\lambda_n). \quad (5.2.44)$$

We want to apply at the end the Paley-Wiener Theorem (see estimate (5.2.35)) in an optimal way, so we want β to be close to $T/2$. Assume that $\beta < T/2$ and close to $T/2$, for example

$$\beta = \frac{T(1-\delta)}{2}. \quad (5.2.45)$$

One has $g_n(-\lambda_k) = \delta_{kn}$. Moreover, thanks to (5.2.44), (5.2.1), Lemma 5.2.3, (5.2.31) and (5.2.45)

$$|g_n(x)| \lesssim e^{\frac{3v}{4}+\pi/\sin(\pi/\alpha)|x+\lambda_n|^{\frac{1}{\alpha}}-(\pi+\delta/2)/\sin(\pi/\alpha)|x+\lambda_n|^{\frac{1}{\alpha}}} P(|x+\lambda_n|) \quad (5.2.46)$$

$$\lesssim e^{\frac{3v}{4}-\delta/(2\sin(\pi/\alpha))|x+\lambda_n|^{\frac{1}{\alpha}}}P(|x+\lambda_n|) \tag{5.2.47}$$

$$\lesssim \frac{e^{3(\alpha-1)(\pi+\delta)^{\alpha-1)}/((a\sin(\pi/\alpha))^{\alpha/(\alpha-1)}\beta^{1/(\alpha-1)}}}{1+(x+\lambda_n)^2} \tag{5.2.48}$$

$$\lesssim \frac{e^{2^{\frac{1}{\alpha-1}}3(\alpha-1)(\pi+\delta)^{\alpha(\alpha-1)/((\alpha\sin(\pi/\alpha))^{\alpha/(\alpha-1)}}(T(1-\delta))^{1/(\alpha-1)})}}{1+(x+\lambda_n)^2} \tag{5.2.49}$$

Let us fix some

$$K > 3(\alpha-1)2^{1/(\alpha-1)}\pi^{\alpha/(\alpha-1)}/(\alpha\sin(\pi/\alpha))^{\alpha/(\alpha-1)}.$$

Considering δ as close as 0 as needed, we deduce that

$$|g_n(x)| \lesssim \frac{e^{\frac{K}{T^{1/(\alpha-1)}}}}{1+(x+\lambda_n)^2} \tag{5.2.50}$$

This notably proves that $g_n \epsilon L^2(\mathbb{R})$. Moreover, using (5.2.1), (5.2.44), (5.2.45) and (5.2.35), we obtain

$$|g_n(z)| \lesssim e^{T|z|/2}.$$

Hence, using the Paley-Wiener Theorem, g_n is the Fourier transform of a function $f_n \epsilon L^2(\mathbb{R})$ with compact support $[-T/2, T/2]$. Moreover, by construction $\{f_n\}$ is biorthogonal to the family $\{e^{i\lambda_n t}\}$. Then, one can create the control thanks to the family $\{f_n\}$. Let us consider $y^0 = \sum a_k e_k$ the initial condition, we call

$$u(t) := -\sum_{k\epsilon\mathbb{N}}(a_k/b_k)e^{-iT\lambda_k/2}f_k(t-T/2). \tag{5.2.51}$$

This expression is meaningful since $b_k \asymp 1$, moreover the corresponding solution y of (5.1.2) verifies $y(T,.) \equiv 0$. Using the Minkovski inequality, Parseval equality, (5.2.51), $b_k \asymp 1$ and (5.2.50), we obtain

$$\|u(t)\|_{L^2(0,T)} \lesssim e^{\frac{K}{T^{1/(\alpha-1)}}}(\Sigma|a_k|^2(\textstyle\int_{\mathbb{R}}\frac{dx}{(1+(x+\lambda_n)^2)^2}))1/2 \tag{5.2.52}$$

$$\lesssim e^{\frac{K}{T^{1/(\alpha-1)}}}(\pi/2\sum|a_k|^2)^{1/2} \tag{5.2.53}$$

$$\lesssim e^{\frac{K}{T^{1/(\alpha-1)}}}\|y^0\|H. \tag{5.2.54}$$

We now consider the parabolic case (Equation (5.1.1)). We call

$$h_n(z) := \frac{\Phi_n(-iz-\lambda_n)H\beta(z\sin(\pi/a)^{\alpha}/(2\alpha))^{\alpha}))}{H\beta(i\lambda_n\sin(\pi/\alpha)^{\alpha}/(2\sin(\pi/(2\alpha))^{\alpha}))}. \tag{5.2.55}$$

Assume that

$$\beta < \frac{T(2\sin(\pi/2\alpha))^{\alpha}}{2\sin(\pi/(\alpha))^{\alpha}}$$

and close to

$$\frac{T(2\sin(\pi/2\alpha))^{\alpha}}{2\sin(\pi/(\alpha))^{\alpha}},$$

For example

(5.2.56)

$$\beta < \frac{(1-\delta)T(2\sin(\pi/2\alpha))^{\alpha}}{2\sin(\pi/(\alpha))^{\alpha}}.$$

One has $h_n(i\lambda_k) = \delta_{kn}$. Moreover, thanks to (5.2.55), (5.2.2), (5.2.33), Lemma 5.2.3. (5.2.31) and (5.2.56), one has

$$|h_n(x)| \lesssim (v+1)e^{\frac{3}{4}v+\pi/(2\sin(\pi/2\alpha))|x|^{\frac{1}{\alpha}}-((\pi+\delta/2)/(2\sin(\pi/2\alpha)))|x|^{\frac{1}{\alpha}}-\frac{\beta|\lambda_n|}{2\sqrt{v+1}}}\overline{P}(|x|,|\lambda_n|) \qquad (5.2.57)$$

$$\lesssim (v+1)e^{\frac{3}{4}v-\delta/(2\sin(\pi/2\alpha))|x|^{\frac{1}{\alpha}}-\frac{\beta\lambda_n}{2\sqrt{v+1}}}P(|x|,\lambda_n|) \qquad (5.2.58)$$

$$\lesssim (v+1)\frac{e^{3(\alpha-1)(\pi+\delta)^{\alpha/(\alpha-1)}/((2\alpha\sin(\pi/\alpha))^{\alpha/(\alpha-1)}\beta^{1/(\alpha-1)})}}{(1+(x+\lambda_n)^2)} \qquad (5.2.59)$$

$$\lesssim (v+1)\frac{e^{3(\alpha-1)2^{\frac{1}{\alpha-1}}(\pi+\delta)^{\alpha/(\alpha-1)}/((2\alpha\sin(\pi/(2\alpha)))^{\alpha/(\alpha-1)}(T^{(1-\delta)})^{1/(\alpha-1)})}}{(1+(x+\lambda_n)^2)} \qquad (5.2.60)$$

Let us fix some

$$K > 3(\alpha-1)2^{1/(\alpha-1)}\pi^{\alpha/(\alpha-1)}/\left(\left(2\alpha\sin\left(\pi/(2\alpha)\right)\right)^{\alpha/(\alpha-1)}\right).$$

Considering δ as close as 0 as needed, we deduce that

$$|h_n(x)| \lesssim \frac{e^{\frac{k}{T^{1/(\alpha-1)}}}}{(1+(x+\lambda_n)^2)}, \qquad (5.2.61)$$

This notably implies that $h_n(x)\epsilon L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ and

$$\|h_n\|_{L^1(\mathbb{R})} \lesssim e^{\frac{K}{T^{\frac{1}{\alpha-1}}}}. \qquad (5.2.62)$$

Moreover, using (5.2.2), (5.2.55), (5.2.35) and (5.2.56)

$$|h_n(z)| \lesssim e^{T|z|/,2}$$

So using the Paley-Wiener Theorem, h_n is the Fourier transform of a function $w_n \epsilon L^2(\mathbb{R})$ with compact support [-T/2,T/2]. Moreover, by construction $\{w_n\}$ is biorthogonal to the family $\{e^{-\lambda_n t}\}$. Then, one can create the control thanks to the family $\{h_n\}$. Let us consider $y^0 = \sum \alpha_k e_k$ the initial condition, we call

$$u(t) := -\sum (a_k/b_k)e^{-T\lambda_k/2}w_k(t-T/2), \qquad (5.2.63)$$

This expression is meaning since $b_k \lesssim 1$, moreover the corresponding solution y of (5.1.1) verifies $y(T,.) \equiv 0$. One easily verifies that $u\epsilon C^0([0,T],\mathbb{R})$. Using (5.2.63), $|b_k| \simeq 1$ and inequality (5.2.62), we obtain

$$\|u(t)\|_L\infty(0,T) \lesssim e^{\frac{k}{T^{1/(\alpha-1)}}}\sum |a_k|e^{-T\lambda_k/2}.$$

Using the Cauchy-Schwarz inequality, one deduces

$$\|u(t)\|_L\infty(0,T) \lesssim e^{\frac{K}{T^{1/(\alpha-1)}}}\|y^0\|H.$$

5.3 Applications

5.3.1 Linear KdV equations controlled on the boundary: the case of periodic boundary conditions with a boundary control on the derivative of the state

In this section, we consider the following controlled linearized KdV equation posed on (0, T) x (0, L) (this is the first example studied in [Ros97]. Let us first introduce the following family of periodic Sobolev spaces (endowed with the usual Sobolev norm)

$$H_p^k := \{\mathcal{Y} \in H^k(0,L) \big| u^{(j)}(0) = u^{(j)}(L), j = 0 \dots k-1 \big|\}.$$

We consider the following equation :

$$\begin{cases} \mathcal{Y}_t + \mathcal{Y}_{xxx} = 0 & in\ (0,T) \times (0,L), \\ \mathcal{Y}(t,0) = \mathcal{Y}(t,L) & in\ (0,T), \\ \mathcal{Y}_x(t,0) = \mathcal{Y}_x(t,L) + u(t) & in\ (0,L), \\ \mathcal{Y}_{xx}(t,0) = \mathcal{Y}_{xx}(t,L) & in\ (0,L), \end{cases} \quad \textbf{(5.3.1)}$$

with initial condition $\mathcal{Y}^0 \in H := (H_p^1)'$ and control $u \in L^2(0,T)$. This system was first studied in [$RZ93$] where the authors proved a result of exact controllability under the technical condition that the integral in space of the initial state had to be equal to the one of the final state. This case there exists a unique solution $\mathcal{Y} \in C^0\left([0,T], (H_p^1)'\right)$ to (5.3.1). Moreover, it is explained in [Ros97, Remark 2.3] that this equation is exactly controllable (and then null controllable) at all time $T > 0$ for every length $L > 0$ (in fact the case which is treated in [Ros97] is $L = 2\pi$ but it can be easily extended to all L). We call A the operator ∂_{xxx}^3 with domain $\mathcal{D}(A) := H_p^2(0,L)$. This operator is skew-adjoint, the eigenvalues are $i\lambda_k := 8i\pi^3 k^3/L^3$ for $k \in \mathbb{Z}$, the corresponding eigenfunction is (normed in $(H_p^1)'$)

$$ek : x \mapsto \frac{(1 + 4\pi^2 k^2/L^2)^{1/2} e^{\frac{i2\pi kx}{L}}}{\sqrt{L}}.$$

If $\mathcal{Y}^0 \in (H_p^1)'$ is written under the form $\mathcal{Y}^0(x) = \sum_{k\in\mathbb{Z}^{\alpha_k e_k}}(x)$, then the solution $\mathcal{Y}$ of (5.3.1) can be written under the form.

$$\mathcal{Y}(t,x) = \sum_{k\in\mathbb{Z}} \alpha_k e^{i\lambda_k t} e_k\,(x).$$

One easily proves (using integrations by parts, see for example [Cor07, Section 2.7, page 101]) that for every $\varphi \in \mathcal{D}(A)$,

$$b(\varphi) = -(\Delta^{-1}\varphi)'(0),$$

so that

$$b = \delta_L' \circ \Delta^{-1},$$

where $\Delta^{-1} := -(-\Delta^{-1})$ is the inverse of the Dirichlet-Laplace operator. We have

$$|b_k| = |e_k'(L)|/k^2 \simeq 1.$$

One can apply directly theorem 5.1.2 and Theorem 5.1.3 with $k = 3$ and $R = \frac{8\pi^3}{L^3}$ to obtain :

Theorem 5.31. Equation (5.3.1) is null controllable and the cost of fast controls C_T verifies

$$C_T \lesssim e^{\frac{K}{\sqrt{T}}}$$

For every $K > \frac{8}{3^{5/4}} L^{3/2}$. Moreover, the power of 1/T involved in the exponential is optimal.

5.3.2 Linear KdV equations controlled on the boundary: the case of Dirichlet boundary conditions with a boundary control on the derivative of the state

In this section, we consider the following controlled linearized KdV equation posed on (0, T) × (0, L):

$$\begin{cases} \mathcal{Y}_t + \mathcal{Y}_x + \mathcal{Y}_{xxx} = 0 & in\ (0,T) \times (0,L), \\ \mathcal{Y}(t,0) = 0 & in\ (0,T), \\ \mathcal{Y}(t,L) = 0 & in\ (0,T), \\ \mathcal{Y}_x(t,L) = u(t) + \mathcal{Y}_x(t,0) & in\ (0,L), \end{cases} \quad \textbf{(5.3.2)}$$

With initial condition $\mathcal{Y}^0 \in L^2(0,L)$ and control $h \in L^2(0,T)$.

However, the problem is that the steady-state operator associated to (5.3.3) with the given boundary condition is neither self-adjoint nor skew-adjoint, so we cannot apply directly the results presented before. That is why we have to change a little bit the system (5.3.2) studied in [CC09].

To be able ro apply theorem 5.1.1 or Theorem 5.1.2, we have to study the sequence of eigenvalues $(\lambda_n)_{n\geq 1}$. One has the following result:

Lemma 5.3.1. $(\lambda_n)_{n\in\mathbb{Z}}$ *is regular and one has*

$$\lambda_n = \frac{8\pi^3 n^3}{L^3} + O(n^2) \quad \textbf{(5.3.4)}$$

as $n \longrightarrow \pm\infty$.

Proof of Lemma 5.3.1. This is an immediate consequence of [CC09, Proposition 1], which gives exactly (5.3.4) and proves that each eigenspace is of dimension 1, which implies the regularity of $(\lambda_n)_{n\in\mathbb{Z}}$ because of the asymptotic behavior given by (5.3.4).

From now on, we call e_k one of the unitary eigenvector (for the H^{-1} -norm) corresponding to the eigenvalue $i\lambda_k$. We fix an initial condition $\mathcal{Y}^0 := \sum_{k\in\mathbb{Z}} \alpha_k e_k \in H^{-1}(0,L)$. As in the previous Subsection, we have for every $\varphi \in \mathcal{D}(A)$,

$$b(\varphi) = -(\Delta^{-1}\varphi)'(0),$$

So that

$$b = \delta_L' \circ \Delta^{-1},$$

and

$$|b_k| = |e_k'(L)|/k^2.$$

To apply Theorem 5.1.1, we just need to ensure that

Lemma 5.3.2.

$$b_k \simeq 1.$$

Proof of Lemma 5.3.2. $b_k \neq 0$ is a consequence of [Ros97, Lemma 3.5] (because $L \notin \mathcal{N}$) and [CC09, Lemma 3.1] gives immediately that $|e_k'(0)|$ is equivalent as $k \longrightarrow \infty$ to $2\pi\sqrt{3}k^2/L^{3/2}$ (because in Lemma 3.1 of [CC09] the eigenvectors are normalized in the L^2-norm and here in the H^{-1}-norm so the behavior of their norm as $k \longrightarrow \infty$ has to be multiplied by k), so we finally have $b_k \simeq 1$.

Applying Theorem 5.1.2, we obtain directly the following Theorem:

Theorem 5.3.2. Let $L \notin \mathcal{N}$. Then equation (5.3.2) is null controllable and the cost of fast controls C_T verifies.

$$V_T \leq 2^{\frac{K}{\sqrt{T}}}$$

for every $K > \frac{8}{3^{5/4}} L^{3/2}$. Morever, the power of $1/T$ involved in the exponential is optimal.

Remark 5.3.1. Using [GG09, Remark 1.3] one can also add a term of diffusion $-\mathcal{Y}_{xx}$ in equation (5.3.2) and obtain the same upper bound as in Theorem 5.3.2.

5.3.3 Anomalous diffusion equation in one dimension

Let us first consider the $1-D$ Laplace operator Δ in the domain $D(\Delta) := H_0^1(0,L)$ with state space $H := H^{-1}(0,L)$. It is well-known that $-\Delta : D(\Delta) \rightarrow \mathrm{H}^{-1}(0,L)$ is a definite positive operator with compact resolvent, the $k-th$ eigenvalue is

$$\lambda_k = \frac{k\pi}{L},$$

One of the corresponding normed (on H) is

$$e_k(x) := \frac{\sqrt{2}(1 + k\pi/L)\sin(k\pi x\,/\,L)}{\sqrt{L}}.$$

Thanks to the continuous functional calculus for positive self-adjoint operators, one can define any positive power of $-\Delta$. Let us consider here some $\gamma > 1/2$ and let us call $\Delta^\gamma := -(-\Delta)^\gamma$. The domain of Δ^γ, that we denote H_γ, is the completion of $C_0^\infty(0,L)$ for the norm.

$$\|\psi\|_\gamma := \left(\sum_{k\in\mathbb{N}^*} (1 + \lambda_k^\gamma)|< e_k;\, \psi > H|^2\right)^{1/2}.$$

We now consider the following equation on $(0,T) \times (0,L)$:

$$\begin{cases} \mathcal{Y}_t = \Delta^\gamma \mathcal{Y} & in\ (0,T) \times (0,L), \\ \mathcal{Y}(0,.) = \mathcal{Y}^0 & in\ (0,L), \end{cases} \qquad \textbf{(5.3.5)}$$

This kind of equation can modelize anomaly fast or slow diffusion (see for example [MK04]).

We now consider the following controlled equation on $(0,T) \times (0,L)$, that we write under the abstract form

$$\begin{cases} \mathcal{Y}_t = \Delta^\gamma \mathcal{Y} + bu & in\ (0,T) \times (0,L), \\ \mathcal{Y}(0,.) = \mathcal{Y}^0 & in\ (0,L), \end{cases} \qquad \textbf{(5.3.6)}$$

where for every $\varphi \in \mathcal{D}(A)$,

$$b(\varphi) = -(\Delta^{-1}\varphi)'(0),$$

i.e.

$$b := \delta_0' \, o \, \Delta^{-1} \in D((-\Delta)^{\gamma})'$$

and $u \in L^2(0,T)$. If $\gamma \in \mathbb{N}^*$, one can observe, using integrations by parts, that b corresponds to a boundary control on the left side on the $\gamma - 1 - th$ derivative of $\mathcal{Y}$, so that b van be considered as a natural extension of the boundary controls in the case of non-entire γ (this kind of controls has already been introduced in [Mil06c, Section 3.3] to give results about the control of fractional diffusion equations with $\gamma \leq 1/2$).

We see that

$$b_k = |e_k'(L)|/k^2 \simeq 1.$$

If $\mathcal{Y}^0 \in H$, then there exists a unique solution of (5.3.6) in the space $C^0([0,T],H)$ (because b is admissible for the semigroup). To our knowledge, the controllability of anomalous diffusion equations with such a control operator and $\gamma \geq 1$ has never been studied before.

Applying directly Theorem 5.1.1 and Theorem 5.1.3, we obtain:

Theorem 5.3.3. Assume $\gamma \geq 1$. Then Equation (5.3.6) is null controllable with continuous controls. Moreover, the cost of the control in L^∞ norm, still denoted C_T here, is such that

$C_T \lesssim e^{\frac{K}{T^{1/(2\gamma-1)}}}$ for every $K > 3(\alpha - 1)2^{1/(2\gamma-1)}L^{2\gamma/(2\gamma-1)}/((4\gamma\sin(\pi/(4\gamma)))^{2\gamma/(2\gamma-1)})$.

Moreover, the power of 1/T involved in the exponential is optimal.

Conclusion

In short, in this work we are interested in the cost of border control in small time of a certain number of equations for the associated space operator is self-adjoint or anti-autoadjoint with compact resolving and having eigenvalues behaving polynomially using the method of moment. Results are derived for linearized Kortewez-of- Varies, fractional diffusion, and fractional Schrodinger equations. In addition, we give some extensions to our investigation.

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