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On Domination Numbers of Some Classes of Graceful Graphs

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Abstract: In this paper, we find the domination number, split domination number, non-split domination number and inverse domination number of some classes of graceful graphs.

Keywords: Graceful graphs, domination number, split domination number, non-split domination number, inverse domination number.

1. INTRODUCTION

In graph theory, a dominating set is the set of all those vertices of a graph, which have dominance property. In many areas, dominating sets play a very important role. In wireless networking, dominating sets are used to search for efficient routes with ad-hoc mobile networks. Not only this, dominating sets have been used in document summarization and in designing secure systems for electrical grids. As in a graceful graph with n vertices and m edges, $n \leq m+1$, so, it is very important to find those vertices, which dominates other vertices of the graph. For practical purposes, it is needed to find the dominating set, with minimum cardinality. In this paper, we find the domination number, inverse domination number, Split and non-split domination numbers of some classes of graceful graphs.

2. PRELIMINARIES

For graph-theoretic concepts and results used here, but not defined or explained, are fairly standard by now (and can be found in [2], [4]). However, for convenience, we recall some of the notions used in the sequel.

Definition 2.1.1: By a graph G, we mean an ordered pair (V, E), where V is a non-empty set, whose elements are called vertices and E is a set, containing unordered pair of vertices. Elements of E are called edges. V is called the vertex set of G and E is said to be the edge set of G.

As an edge is identified by a pair of vertices, those vertices are called the end vertices of that edge. Every graph is represented by a diagram, in which each vertex is represented by a point or small circle and each edge is represented by a line segment or arc, which joins its end vertices. Such a diagram of a graph, is itself called a graph. We say that an edge is incident on its end vertices.

Definition 2.1.2: Two vertices are said to be adjacent, if there is an edge between those vertices in the graph.

Definition 2.1.3: If the end vertices of an edge are the same, then such an edge is called a self-loop.

Definition 2.1.4: If any two edges have the same end vertices, then the edges are said to be parallel edges.

Definition 2.1.5: A graph, having no self-loops and parallel edges, is called a simple graph.

Definition 2.1.6: The number of edges, incident on a vertex (with self-loop counted twice) is called the degree of that vertex.

Definition 2.1.7: A graph G'= (V', E') is said to be a subgraph of a graph G= (V, E) if $V' \subseteq V$ and $E' \subseteq E$, such that each edge of E' is incident with the vertices in V'.

Definition 2.1.8: A subgraph G'= (V', E') of a graph G= (V, E) is said to be induced if E' contains all those edges of E, which are incident with the vertices of V' in G.

Definition 2.1.9: By a path in a graph, we mean a finite sequence of vertices and edges, beginning and ending with vertices such that no edge and vertex appear more than once. If the beginning and ending vertices of a path are the same, then such a path is called a cycle.

Definition 2.2.0: A graph is called connected if there is a path between its every pair of vertices. If a graph is not connected, then it is called disconnected. A maximal connected subgraph of a graph is called a component of the graph.

Definition 2.2.1: An edge in a connected graph is said to be a bridge if removal of the edge disconnects the graph.

Definition 2.2.2: A connected graph, having no cycles, is called a tree.

Definition 2.2.3: For a graph G= (V, E), a vertex labeling is a function of V to a set of labels and similarly, an edge labeling is a function of E to a set of labels.

Definition 2.2.4: In a graph G= (V, E), a subset D of vertex set V is said to be a dominating set if every vertex in V−D is adjacent to some vertex in D.

From the definition, it is clear that the vertex set of a graph is itself a dominating set. A graph may have many dominating sets. For practical purposes, we are interested to find those dominating sets, from which no vertex can be removed, without destroying its dominance property.

Definition 2.2.5: In a graph G= (V, E), a subset D of vertex set V is said to be a minimal dominating set, if no vertex can be removed from D, without destroying its dominance property.

A graph may possess more than one minimal dominating set, so, now we define the following:

Definition 2.2.6: In a graph G= (V, E), the smallest cardinality of minimal dominating sets, is said to be the domination number of the graph G. It is denoted by $\alpha(G)$.

Remark 2.1.1: If a graph has a self-loop and parallel edges, then its dominating set is unaffected by removal of self-loops and parallel edges, so, we shall consider only simple graphs.

Remark 2.1.2: If a graph is disconnected, then it is required to find dominating sets of all the components independently, so disconnected graphs do not contribute anything to dominating set, so we shall consider only connected graphs.

3. DOMINATION, INVERSE DOMINATION, SPLIT DOMINATION, NON-SPLIT DOMINATION NUMBERs OF SOME CLASSES OF GRACEFUL GRAPHS

Definition 3.1.1 ([4], [9]): A graph, which has a graceful labeling is called a *graceful graph***.** By **a** graceful labeling of a graph with m edges, we mean, labeling of its vertices with some subset of the integers from 0 to m inclusive, such that no two vertices have the same label, and each edge is uniquely identified by the absolute difference between its endpoints, such that this magnitude lies between 1 and m inclusive. Clearly, if a graph has n vertices and m edges m, then $n \leq m+1$.

Definition 3.1.2: Let G= (V, E) be a graph and D be a minimal dominating subset of V. A dominating subset D' of V – D is said to be an inverse dominating set with respect to D. Then the inverse domination number $\gamma'(G)$ is the minimum cardinality taken over all minimal inverse dominating sets.

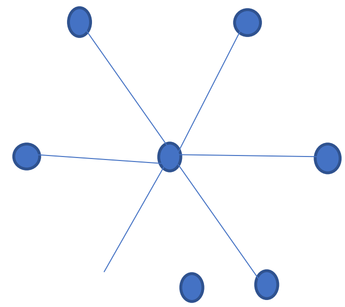
Definition 3.1.3: Let D be a dominating set in a graph G= (V, E). If the induced subgraph of G with vertex set V – D is disconnected, then we say that D is a split dominating set in G.

Definition 3.1.4: The smallest cardinality of all split dominating sets in a graph G is called the split domination number of graph G and it is denoted by $\gamma_s(G)$.

Definition 3.1.5: Let D be a dominating set in a graph G= (V, E). If the induced subgraph of G with vertex set V – D is connected, then we say that D is a non-split dominating set in G.

Definition 3.1.6: The smallest cardinality of all non-split dominating sets in a graph G is called the non-split domination number of graph G and it is denoted by $\gamma_{ns}(G)$.
If for all dominating sets D in G, V−D is always disconnected, the, we write $\gamma_{ns}(G)$= 0.

Definition 3.1.7([4], [7], [11]): The star graph S_k of order k, is a tree_on k vertices with one vertex having degree n−1 and the other having vertex degree 1.
The star graph S_k is isomorphic to the complete bipartite graph $k_{1,n-1}$ (cf. [9]).
We call the vertex of degree k-1 as centre vertex. Clearly, a star graph is graceful. The following graph is the star graph S_7.



Theorem 3.1.1: For a star graph S_k,

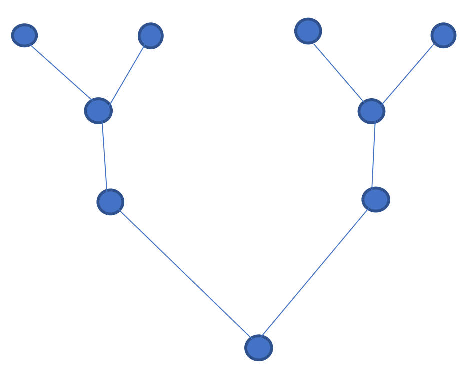
1. The domination number is 1
2. The split domination number is 1, provided k >1.
3. The non-split domination number is $k-1$.
4. Inverse domination number is 1.

Proof: Let S_k = (V, E).

1. Let the centre vertex in the star graph S_k be u. As the degree of vertex u is k−1 and other vertices are adjacent to u, so {u} is a dominating set. Clearly, it is the smallest dominating set. Thus, the domination number of S_k is 1.
2. As {u} is a dominating set, so, the induced graph, obtained by removing the vertex u, is disconnected and so {u} is a split dominating set. So, the split domination number is 1.
3. Let the vertices other than u be $u_1, u_2, \ldots u_{k-1}$. Clearly, D = $\{u_1, u_2, \ldots u_{k-1}\}$ is a dominating set. Then the induced subgraph with vertex set V−D, i.e., {u}, is connected, so D is a non-split dominating set. Here, we note that if any proper subset of D is not a dominating set, so non-split domination number is k−1.
4. First, we consider the dominating set D =$\{u_1, u_2, \ldots u_{k-1}\}$. Then V− D ={u}, which is a dominating set. So, {u} is an inverse dominating set of G w. r. to D. In fact, {u} is a minimal dominating set and hence inverse domination number of G is 1.

Definition 3.1.8([1]): An (n, k)-banana tree is a graph, obtained by connecting one leaf of each of n copies of an k-star graph with a single root vertex that is distinct from all the stars. It is denoted by $B_{n,k}$, for n≥2 and k≥ 3

The following graph is the banana tree $B_{2,4}$ (as there are 2 star graphs with 4 vertices)



We note that all banana trees are graceful (cf. [3], [8])

Theorem: 3.1.2: For a banana tree $B_{n,k}$,

1. The domination number is n+1.
2. The split domination number is n+1.
3. The non-split domination number is $k + (n-1)(k-1)$.
4. The inverse domination number is $2n$.

Proof: As $B_{n,k}$ has n copies of star graph S_k, we denote the star graphs as $S_k^1, S_k^2 \ldots, S_k^n$ and the root vertex r. Let centre of n star vertices be $c_1, c_2, \ldots, c_n$. Then

1. The set D = $\{c_1, c_2, \ldots, c_n$, r$\}$ is a minimal dominating set. In fact, D is the dominating set, with minimum cardinality. Hence $\gamma(B_{n,k}) = n + 1$.
2. As D is a minimal dominating set, with minimum cardinality, and the induced graph, obtained with V−D vertices is disconnected, so split domination number of $B_{n,k}$ is $n + 1$.
3. We note that there is exactly one vertex in each S_k, which is adjacent to the root vertex r. Let those vertices by $v_1, v_2 \ldots, v_n$. Consider the set A, which contains all k vertices of S_k^1, all vertices of S_k^i, except v_i for each $i = 2, 3, \ldots, n$. Then clearly, A is a dominating set, containing $k + (n - 1)(k - 1)$. As V−A = {r, $v_2, v_3, \ldots, v_n$ },
   and the induced subgraph with vertex set V−A is connected. We cannot find any dominating set B, having vertices less than the vertices in A such that the induced subgraph with vertex set V−B, is connected. Hence the non-split domination number of $B_{n,k} = k + (n-1)(k-1)$.
4. Consider the minimal dominating set D = $\{c_1, c_2, \ldots, c_n$, r$\}$. Then V−D is a dominating set, containing $n(k-1)$ vertices. In fact, it is a minimal dominating set. So, V−D is an inverse dominating set. Next, we consider the minimal dominating set C, having $S_k^i - c_i, - v_i$ for each $i = 1, 2, 3, \ldots, n$ and root vertex r. Then V−C consists of $\{c_1, c_2, \ldots, c_n, v_1, v_2 \ldots, v_n\}$ which is clearly a minimal dominating set, implies that V−C is an inverse dominating set. Thus, the inverse domination number of $B_{n,k}$ is $2n$.

Remark 3.1: It follows from Theorem 3.1.1 and Theorem 3.1.2, that

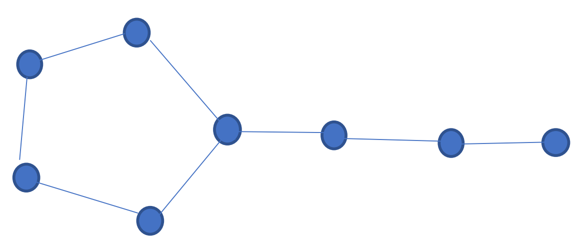
$$\gamma(B_{n,k}) = n\gamma(S_k)+1$$

$$\gamma_s(B_{n,k}) = n\gamma_s(S_k)+1$$

As the banana tree $B_{n,k}$ has n copies of star graph S_k and a root, we note that the domination and split domination numbers of banana tree and star graphs are also related.

Definition 3.1.9([5],[10]): For m$\geq 3, n \geq 1, an\ (m,n)$ −tadpole graph, is the graph obtained by joining a cycle graph C_m to a path graph P_n with a bridge. It is denoted by $T_{m,n}$. Sometimes, it is also called a kite graph or a dragon graph.

e.g., the following graph is the tadpole graph $T_{5,3}$.



Tadpole graphs are graceful when m≡ or 2 (mod 4) (cf. [3], [6]).

Before finding the domination number of a tadpole graph, we find the domination number of cycle graph C_m of m-vertices and path graph P_n.

Theorem 3.1.3: For a cycle graph C_m, m ≥ 3,

1. $\gamma(C_m) = \left\lceil \frac{m}{3} \right\rceil$, where $\lceil x \rceil$ denotes the ceiling function of x, i.e., its value is the least integer, greater than or equal to x.
2. $\gamma_s(C_3) = 0$ and $\gamma_s(C_m) = \left\lceil \frac{m}{3} \right\rceil$ for m≥ 4
3. $\gamma_{ns}(C_m) = \left\lceil \frac{m}{3} \right\rceil$, for m =3, 4 and $\gamma_{ns}(C_m) = 0$.
4. $\gamma'(C_m) = \left\lceil \frac{m}{3} \right\rceil$.

Proof: Let the C_m cycle be $v_1, v_2 \ldots\ldots, v_m$.

1. Clearly, each vertex v_i is adjacent to exactly two vertices v_{i-1} and v_{i+1} for i =1, 2,…, m, (here $v_{m+1} = v_1$ *and* $v_{1-1} = v_m$) so, while constructing a minimal dominating set, we need to take only one vertex from three consecutive vertices in the cycle, i.e., if we take v_2, then we need to take next vertex v_5, and then next vertex is v_8 and so on. If m is a multiple of 3, then the minimal dominating set will contain $m/3$ vertices. If m is not a multiple of 3, i.e., it is of the form 3k+1 or 3k+2 for some natural number k, then we need to take k+1 vertices to form a minimal dominating set. Hence the domination number of C_m is $\left\lceil \frac{m}{3} \right\rceil$.
2. If m=3, then by Part 1, the minimal dominating set is $\{v_2\}$ and the induced subgraph of C_3 with vertex set V−$\{v_2\}$ is connected and so, $\gamma_s(C_3) = 0$.
   If m = 4, $\{v_2, v_4\}$ is a minimal dominating set and the induced subgraph of C_4 with vertex set V−$\{v_{2,}v_4\}$ is disconnected. Hence $\gamma_s(C_4)$ =2=$\left\lceil \frac{m}{3} \right\rceil$.
   If m≥5, then removal of any minimal dominating set leaves the graph disconnected and so, $\gamma_s(C_m) = \left\lceil \frac{m}{3} \right\rceil$ for m≥ 5. Hence $\gamma_s(C_m) = \left\lceil \frac{m}{3} \right\rceil$ for m≥ 4.
3. If m =3, then it follows from Part 2, that $\gamma_{ns}(C_3) = 1$, i.e., $\gamma_{ns}(C_3) = \left\lceil \frac{m}{3} \right\rceil$.
   If m = 4, then the removal of minimal dominating set $\{v_2, v_3\}$ leaves the graph connected and so, $\gamma_{ns}(C_3) = 2$, i.e., $\gamma_{ns}(C_4) = \left\lceil \frac{m}{3} \right\rceil$.
   It follows from Part 2, that if m≥5, then removal of any minimal dominating set leaves the graph disconnected and so, $\gamma_{ns}(C_m) = 0$, for m≥ 5.
4. As m >$\left\lceil \frac{m}{3} \right\rceil$, i.e., the cardinality of a minimal dominating set D is always less than the cardinality of vertices in C_m, so, the set V−D will contain a minimal dominating set with cardinality $\left\lceil \frac{m}{3} \right\rceil$. Hence $\gamma'(C_m) = \left\lceil \frac{m}{3} \right\rceil$.

The following theorem directly follows from Theorem 3.1.3.

Theorem 3.1.4: The domination number of a path graph P_n is $\left\lceil \frac{n}{3} \right\rceil$, for n≥ 2.

Theorem: 3.1.5: For a tadpole graph $T_{m,n}$, where m≥ 3 and n≥1,

1. $\gamma(T_{m,n}) = \left\lceil \frac{m}{3} \right\rceil + \left\lceil \frac{n}{3} \right\rceil$, if $n \geq 2$ and $\gamma(T_{m,1}) = \left\lceil \frac{m}{3} \right\rceil$.
2. $\gamma_s(T_{m,n}) = \left\lceil \frac{m}{3} \right\rceil + \left\lceil \frac{n}{3} \right\rceil$, if $n \geq 2$ and $\gamma_s(T_{m,1}) = \left\lceil \frac{m}{3} \right\rceil$.
3. $\gamma_{ns}(T_{m,1}) = 2$, for m= 3, 4 and $\gamma_{ns}(T_{5,1}) = 3$, otherwise 0.
4. $\gamma'(T_{m,1}) = \left\lceil \frac{m}{3} \right\rceil$ and $\gamma'(T_{m,n}) = \left\lceil \frac{m}{3} \right\rceil + \left\lceil \frac{n}{3} \right\rceil$, if $n \geq 2$

Proof: The tadpole graph $T_{m,n}$ has a cycle C_m of m vertices, a path P_n of n vertices and an edge between a vertex of C_m $and\ a\ vertex\ of\ P_n$. Let the C_m cycle be $v_1, v_2 \ldots\ldots, v_m$, where $v_1 = v_m$, path P_n be $p_1, p_2 \ldots\ldots, p_n$ and an edge (v_2, p_1).

1. To construct a minimal dominating set, we select vertices $v_2, v_5, v_{8\ldots\ldots}$(as we did in Theorem 3.1.3). If n =1, then as p_1 is adjacent to v_2, so there is no need to take vertex p_1 in the dominating set. So, from Theorem 3.1.3, it follows that if $n = 1$, then the domination number of $T_{m,n}$ is $\left\lceil \frac{m}{3} \right\rceil$.

   From Theorem 3.1.3 and Theorem 3.1.4, it follows that the domination number of $T_{m,n}$ is $\left\lceil \frac{m}{3} \right\rceil + \left\lceil \frac{n}{3} \right\rceil$, if $n \geq 2$.

2. From the construction of the minimal dominating set, say D in Part-1, it is clear that V−D is disconnected and so the domination number and the split domination number of $T_{m,n}$ are the same.
3. If m = 3 and n = 1, then clearly, the set B= $\{v_1, p_1\}$ is a minimal dominating set. Then the induced subgraph of $T_{3,1}$ with vertex set V$-B$ =$\{v_2, v_3\}$, is connected and so, non-split domination number of $T_{3,1}$ is 2.

   If m = 4 and n = 1, then clearly, the set C= $\{v_4, p_1\}$ is a minimal dominating set. Then the induced subgraph of $T_{4,1}$ with vertex set V−C = $\{v_1, v_2, v_3\}$, is connected and so, the non-split domination number of $T_{4,1}$ is 2.

   If m = 5 and n = 1, then clearly, the set D= $\{v_4, v_5, p_1\}$ is a minimal dominating set (as v_4 is adjacent to v_3, v_5 is adjacent to v_1 and p_1 is adjacent to v_2). Then the induced subgraph of $T_{5,1}$ with vertex set V−D = $\{v_1, v_2, v_3\}$, is connected and so, the non-split domination number of $T_{5,1}$ is 3.

   If m≥ 6 or n≥ 2, then we cannot construct a dominating set E, such that V−E is connected. So, the non-split domination number of $T_{m,n}$ is zero, if m≥ 6 or n≥ 0.
4. As m +n >$\left\lceil \frac{m}{3} \right\rceil + \left\lceil \frac{n}{3} \right\rceil$, i.e., the cardinality of a minimal dominating set D is always less than the cardinality of vertices in $T_{m,n}$, so, the set V−D will still contain a minimal dominating set with cardinality $\left\lceil \frac{m}{3} \right\rceil + \left\lceil \frac{n}{3} \right\rceil$. Thus, the inverse domination number and domination number for a tadpole graph are the same. Hence $\gamma'(T_{m,1}) = \left\lceil \frac{m}{3} \right\rceil$ and $\gamma'(T_{m,n}) = \left\lceil \frac{m}{3} \right\rceil + \left\lceil \frac{n}{3} \right\rceil$, if $n \geq 2$

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