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STUDY ON SOLVING OF ORDINARY DIFFERENTIAL DIFFERENCE EQUATIONS HAVING BOUNDARY LAYERS

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ABSTRACT

Delay differential equation refers to an ordinary differential equation (ODE) with a delay parameter. The term "advanced differential equation" is used to describe an ODE with a forward parameter. The term "differential-difference equation" is used to refer to an ODE that includes both delay and advance parameters. Similar to the usage of "advance" and "delay" words, "positive shift" and "negative shift" appear in the literature. When the highest order derivative is multiplied by a small parameter, the boundary layer phenomenon appears in the solution of the differential-difference equation. The boundary layer is an area where the answer is constantly evolving. These issues are typically addressed by means of perturbation techniques, such as the matched asymptotic expansions and WKB approach. Skill, insight, and experimentation are needed for these asymptotic expansions. As a result, scientists have begun to rely on numerical approaches. Due to the presence of the boundary layer, oscillatory/unsatisfactory solutions are obtained if we apply the existing numerical methods with the step-size greater than the perturbation parameter. In addition, the solution is graphically represented so that the effects of the delay and complex factors may be comprehended. It is noticed from the tables and figures that our quantitative approaches produce very good approximation to the exact solution. In addition to being theoretically simpler and easier to use, these quantitative approaches are also easily adaptable for computer implementation with only a small amount of computing effort.

Keywords: Ordinary differential equation (ODE), boundary layer, delay parameter.

1. INTRODUCTION

1.1. Differential-difference equations

An ODE with a delay parameter is often referred to as a "delay differential equation." If you take an ordinary differential equation (ODE) and add a third parameter, it becomes a differential equation (DDE) with additional complexity. Delay and advance parameters in an ODE are represented by the terms differential and difference. The phrases "advance" and "delay" can also be found in the literature under the names "positive shift" and "negative shift," respectively. When the highest-order derivative of the differential-difference equation is multiplied by a small parameter, the boundary layer phenomenon occurs. The boundary layer is defined as the region where the solution evolves at an extremely quick rate. These issues are typically addressed by means of perturbation techniques, such as matched asymptotic expansions and WKB methods.

1.2. Differential-difference equation models

Most general model of the differential-difference equation having boundary layers is given by:

$$\varepsilon w''(s) + a(s)w'(s) + b(s)w'(s-\delta) + c(s)w(s) + d(s)w(s-\delta) + e(s)w(s+\eta) = f(s) (1)$$

depending on parameters

$$w(s) = \alpha(s), -\delta \le s \le 0 \ (2)$$

$$w(s) = \beta(s), 1 \le s \le 1+\eta \ (3)$$

where a(s), b(s), c(s), d(s), e(s), f(s), α (s), β (s) are considered to be smooth on (0, 1); 0 1 is the small perturbation parameter.

1.3 Need of study

Perturbation methods like: matched asymptotic expansions and WKB methods have been extensively used to solve these problems. But these perturbation methods demand/require skill, insight and experimentation. As a result, researchers have begun to rely on numerical approaches. However, if we use the existing numerical methods with a step size larger than the perturbation parameter, we obtain oscillatory/unsatisfactory solutions due to the presence of the boundary layer. But if the step size is equal or less than the perturbation parameter then only the existing numerical methods will give better results. This takes a long time and a lot of money. Therefore, there is a high need for innovative approaches that can function with a practical step size to address these issues.

2. LITERATURE REVIEW

The Duressa Gemechis File, Year 2023 Singularly perturbed differential difference equations are researched in relation to their computational solution under both negative and positive changes to a spatial variable. Solving single perturbation equations that arise from modeling brain activity is the focus of this review, which covers the years 2012 through 2022. We considered ordinary differential equations with singular perturbations and small or large negative shifts, partial differential equations with singular perturbations and small or large negative shifts, and mixed-type partial differential-difference equations with singular perturbations. The primary objective of this survey is to identify recent advances in numerical and asymptotic methods for addressing these kinds of issues. In addition, it hopes to inspire researchers to create novel, effective approaches to resolving related classes of issues.

Elika Kurniadi, (2022), Since 1970, there has been a proliferation of studies devoted to the Ordinary Differential Equation (ODE). As a result, a number of publications published in various journals are available. In this article, we will review the existing research on the topic of how differential equations are presented and learned in higher education. We will use the top five findings from Scopus Q1 according to the SJR provided by Scimago Journal & Country Rank to conduct a Systematic Literature Review (SLR) of the greatest scholarly articles in the field of education published in the first three months of 2017. In this study, it is essential to consider both the proposed ODE learning approach and the presentation and debate of the ODE topic in the academic literature. The comprehensive literature review identified four routes to ODE mastery: active learning, mathematical modeling, information and technology communication, and geometric and qualitative solutions.

Ordinary differential equations (ODEs) consist of the derivatives of a set of functions with only one independent variable. It is a foundational field of mathematics with applications across the natural sciences (Rasmussen & Keene, 2019) and beyond.

The active learning pedagogic approach and the use of technology in the classroom have had significant modern impacts on how ODE is taught and learned. The development of helpful tools for solving the differential equation and gaining an understanding of some of its aspects has been made possible by technological progress (Hoyles, 2018; Quinn & Aaro, 2020). The development of technology has enhanced the learning environment in the classroom.

Evidence for ODE instruction in higher education is the goal of this research. Particular areas of interest include design-based learning, student reactions to different ODE pedagogical approaches, the efficacy of innovation, and the centrality of mathematical concepts in ODE. The best way to learn from this study's findings is to first do a literature review (Xiao & Watson, 2019). Meanwhile, those reports lack a thorough literature review. In other words, there is room for further research into how ODE is taught and learned. This research may take the form of a literature review aimed at locating and analyzing existing studies on the subject.

3. LIOUVILLE GREEN TRANSFORMATION OR SOLVING DIFFERENTIAL-DIFFERENCE EQUATIONS

3.1 Introduction

Liouville Green Transformation is used to solve: (i) delay differential equations having boundary layers and (ii) Boundary-layer differential-difference equations. In both the cases, using Taylor series, the given problems are approximated by singular perturbation problems. Liouville Green Transformation is used to solve these problems. Two model instances are used to test the provided method, and our results are compared to those of the literature, including both exact and approximate solutions. The answers are visualized so that the effects of the delay and complex parameters can be comprehended.

3.2 Description of the method

3.2.1 Case-I: Boundary layer differential equations with a delay

Consider a problem:

$$\varepsilon w''(s) + a(s)w'(s-\delta) + b(s)w(s) = 0, 0 \le s \le 1 \quad (4)$$

with boundary conditions

$$w(0) = \alpha, -\delta \le s \le 0 \quad (5)$$

$$w(1) = \beta \quad (6)$$

Expansion of the Taylor series provides us

$$w'(s-\delta) \approx w'(s) - \delta w''(s) \quad (7)$$

3.2.2 Case-II: Differential-Difference Equation having boundary layers

Now consider:

$$\varepsilon w''(s) + a(s)w'(s) + b(s)w(s-\delta) + c(s)w(s) + d(s)w(s+\eta) = 0 \quad (8)$$

for 0 < s<1 and subject to the conditions

$$w(s) = \alpha(s), -\delta \le s \le 0 \quad (9)$$

$$w(s) = \beta(s), 1 \le s \le 1+\eta \quad (10)$$

3.3 Liouville Green Transformation:

Consider the Eqn

$$-\varepsilon w''(s) + f(s)w'(s) + g(s)w(s) = 0, s\epsilon[0,1] \quad (11)$$

This gives rise to the Liouville-Green transition.

$$z = \varphi(s) = \frac{1}{\varepsilon}\int f(s)ds \quad (12)$$

$$\phi(s) = \varphi'(s) = \frac{1}{\varepsilon}f(s) \quad (13)$$

$$v(z) = \phi(s)w(s) \quad (14)$$

According to Eqn above we have

$$\frac{dw}{ds} = \frac{1}{\phi(s)}\frac{dv}{dz}z'(s) - \frac{\phi'(s)}{\phi^2(s)}v(z) = \frac{\phi'(s)}{\phi(s)}\frac{dv}{dz} - \frac{\phi'(s)}{\phi^2(s)}v(z) \quad (15)$$

$$\frac{d^2w}{ds^2} = \frac{1}{\phi(s)}\left(\left(\phi^2(s)\frac{d^2v}{dz^2} + \left(\phi'' - \frac{2\varphi'(s)\phi'(s)}{\phi(s)}\right)\frac{dv}{dz}\right) - \frac{\phi''(s)}{\phi(s)} - \frac{2\varphi'^2(s)}{\phi^2(s)}v\right) \quad (16)$$

Table 3.1. The largest possible errors in Problem ε =3.1 for a given grid size N and a given value of

$\delta\downarrow$ $N\rightarrow$	10^2	10^3	10^4	10^5
		Proposed method		
0.01	1.3798e-04	1.3907e-05	1.3887e-06	1.5767e-06
0.03	1.0765e-04	1.0849e-05	1.0600e-06	3.1523e-06
0.06	6.1798e-05	6.2273e-06	6.3164e-07	3.1727e-06
0.08	3.0995e-05	3.1233e-06	3.3537e-07	1.5918e-06
		Results by [167]		
0.01	0.01172504	0.00122562	1.2310e-004	1.2280e-05
0.03	0.01505997	0.00158944	1.5984e-004	1.5998e-05
0.06	0.02575368	0.00281263	2.8397e-004	2.8449e-05
0.08	0.04781066	0.00562948	5.7357e-004	5.7485e-05

Table 3.2. Maximum mistakes in solving Problem 3.2 with ε= 0.01 for various values of and grid sizes N

$\delta \downarrow$ $N \rightarrow$	10^2	10^3	10^4	10^5
Proposed method				
0.001	1.5388e-04	1.5487e-05	1.5676e-06	1.5953e-06
0.003	1.1972e-04	1.2049e-05	1.2321e-06	1.5919e-06
0.006	6.8442e-05	6.8883e-06	7.2214e-07	1.5895e-06
0.008	3.4231e-05	3.4452e-06	3.6130e-07	1.7706e-09
Results by [167]				
0.001	0.09073569	0.01228700	0.00127926	1.28459e-04
0.003	0.10803507	0.01562216	0.00164450	1.65330e-04
0.006	0.12777968	0.02630926	0.00287019	2.89704e-04
0.008	0.10040449	0.04833890	0.00568876	5.79477e-04

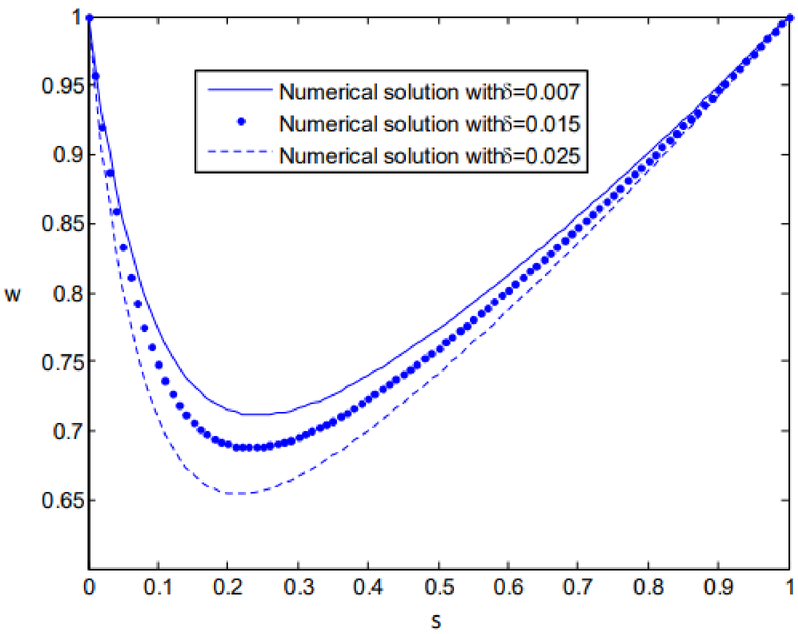

Figure 3.1. Solution to problem 3.1 with and without δ= 0.01.

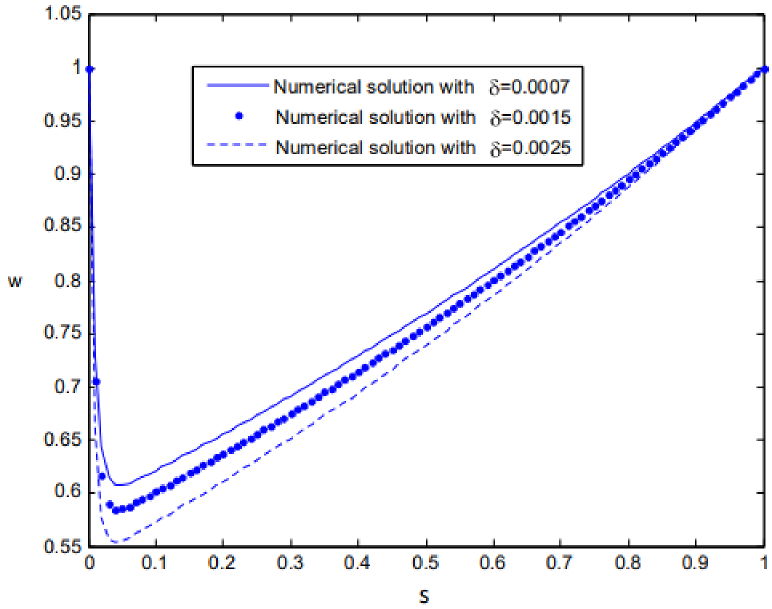

Figure 3.2. Problem 3.1, solved for δ = 0.001 and other values

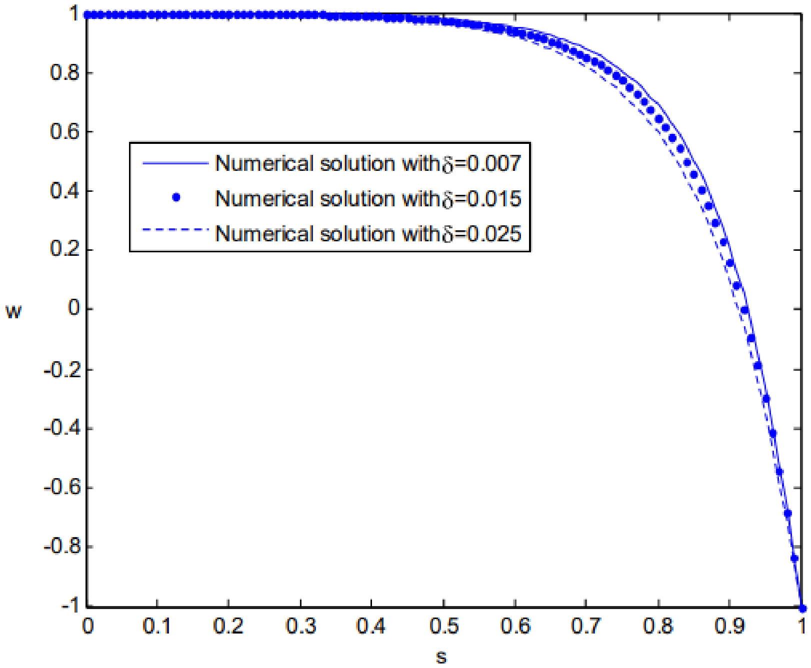

Figure 3.3 Calculation of the solution to Issue 3.2 with a parameter value of = 0.001 and variable δ.

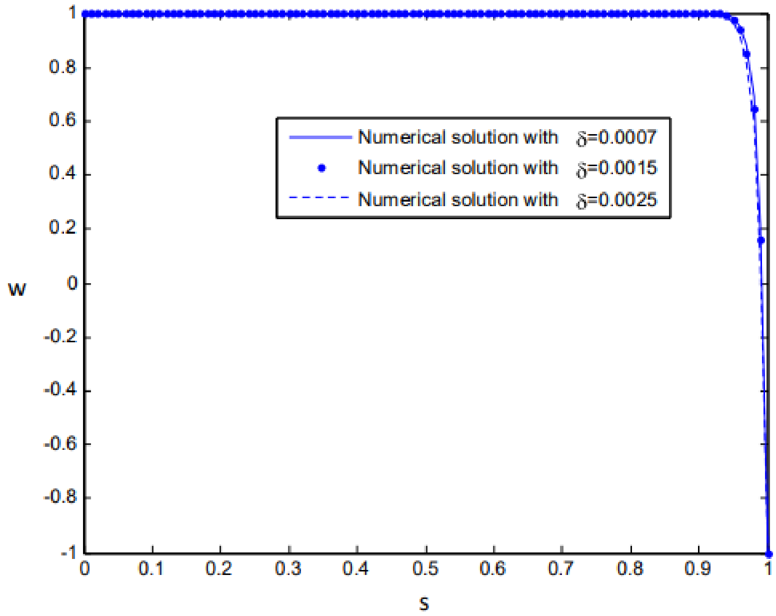

Figure 3.4. The solution to Issue 3.2 with = 0.001 and various values of δ.

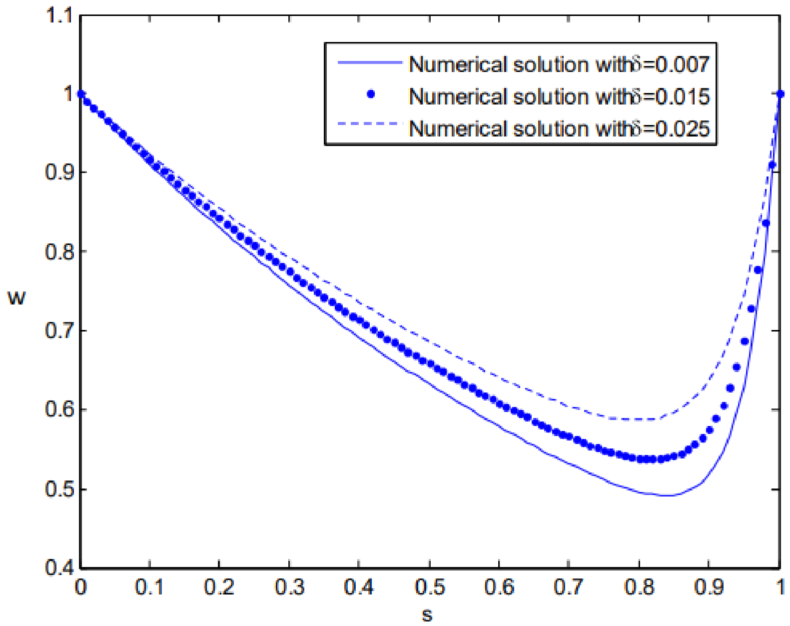

Figure. 3.5. Problem 3.3, solved for = 0.01 and other values of δ.

4. FOURTH ORDER FINITE DIFFERENCE METHOD FOR SOLVING DELAY DIFFERENTIAL EQUATIONS

4.1 Introduction

Using Taylor's method, we can amplify the delay/deviating term, transforming the original delay differential equation into a single perturbed two-point boundary value issue. The next step could be to use the Liouville Green transformation to convert the problem into a periodic perturbation of a two-point boundary value problem. The fourth-order finite difference method provides an effective solution to the problem. Following the steps outlined here, we apply them to four model problems with varying values of and then evaluate our results against both issued solutions and exact solutions. The answer is also illustrated graphically to aid in comprehending the effect of the factors.

4.2 Description of the method

Consider the problem:

$$\varepsilon w''(s) + a(s)w'(s-\delta) + b(s)w(s) = 0, 0 \le s \le 1 \ (17)$$

with boundary conditions

$$w(0) = \alpha, -\delta \le s \le 0 \ (18)$$

$$w(1) = \beta \ (19)$$

Taylor series expansion gives us

$$w'(s-\delta) \approx w'(s) - \delta w''(s) (20)$$

4.3 Liouville Green Transformation:

$$-\varepsilon w''(s) + f(s)w'(s) + g(s)w(s) = 0, s\epsilon[0,1] \ (21)$$

This gives rise to the Liouville-Green transition.

$$z = \varphi(s) = \frac{1}{\varepsilon}\int f(s)ds \ \ (22)$$

$$\phi(s) = \varphi'(s) = \frac{1}{\varepsilon}f(s) \ (23)$$

$$v(z) = \ \phi(s)w(s) \ \ (24)$$

According to Eqn above we have

$$\frac{dw}{ds} = \frac{1}{\phi(s)}\frac{dv}{dz}z'(s) - \frac{\phi'(s)}{\phi^2(s)}v(z) = \frac{\phi'(s)}{\phi(s)}\frac{dv}{dz} - \frac{\phi'(s)}{\phi^2(s)}v(z) \ \ (25)$$

$$\frac{d^2w}{ds^2} = \frac{1}{\phi(s)}\left(\left(\phi^2(s)\frac{d^2v}{dz^2} + \left(\phi'' - \frac{2\varphi'(s)\phi'(s)}{\phi(s)}\right)\frac{dv}{dz}\right) - \frac{\phi''(s)}{\phi(s)} - \frac{2\varphi'^2(s)}{\phi^2(s)}v\right) (26)$$

Table 4.1: Maximum absolute errors of Problem 4.1 for ε =10-3

δ /h	10^{-2}		10^{-3}		10^{-4}	
	Present method	Upwind method	Present method	Upwind method	Present method	Upwind method
0.0ε	0.002853	1.85e+094	0.00067097	0.232912	0.00091355	0.012377
0.3ε	0.002853	1.74e+094	0.00067097	0.232753	0.00091355	0.012373
0.6ε	0.002853	1.63e+094	0.00067097	0.232594	0.00091355	0.012370
0.9ε	0.002853	1.53e+094	0.00067097	0.232436	0.00091355	0.012367

Table 4.2: Maximum absolute errors of Problem 4.1 for ε =10-5

δ /h	10^{-3}	10^{-4}	10^{-5}
0.0ε	0.00028558	2.8573e-05	6.7186e-06
0.3ε	0.00028558	2.8573e-05	6.7186e-06
0.6ε	0.00028558	2.8573e-05	6.7186e-06
0.9ε	0.00028558	2.8573e-05	6.7186e-06

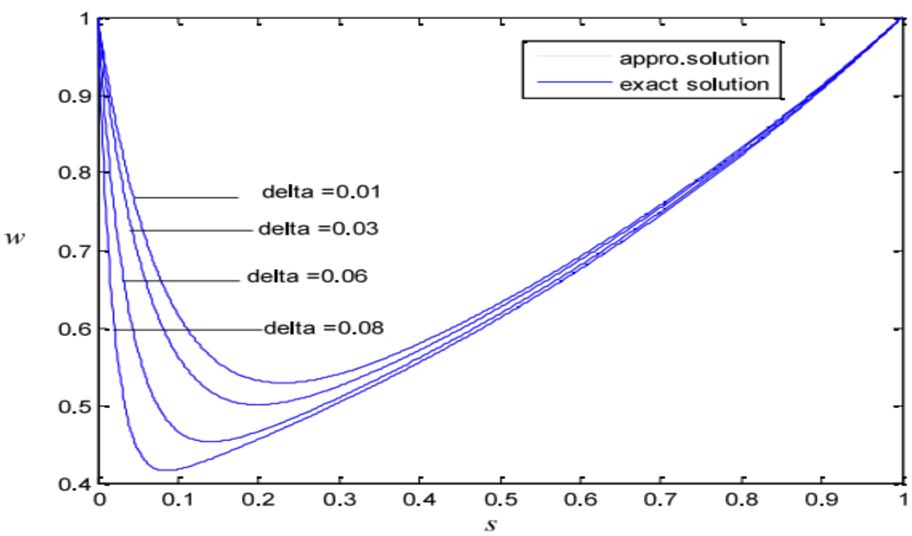

Figure. 4.1. Solution the problem 4.1 with = 0.1 and varying values δ of o(ε).

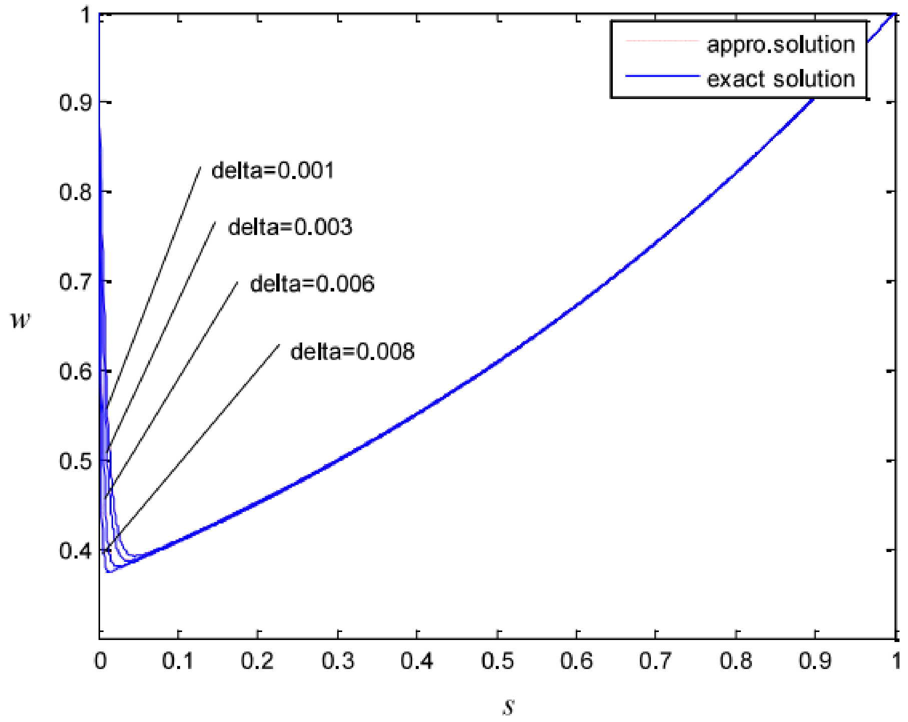

Figure. 4.2. Solution the problem 4.1 with = 0.01 and varying values δ of o(ε).

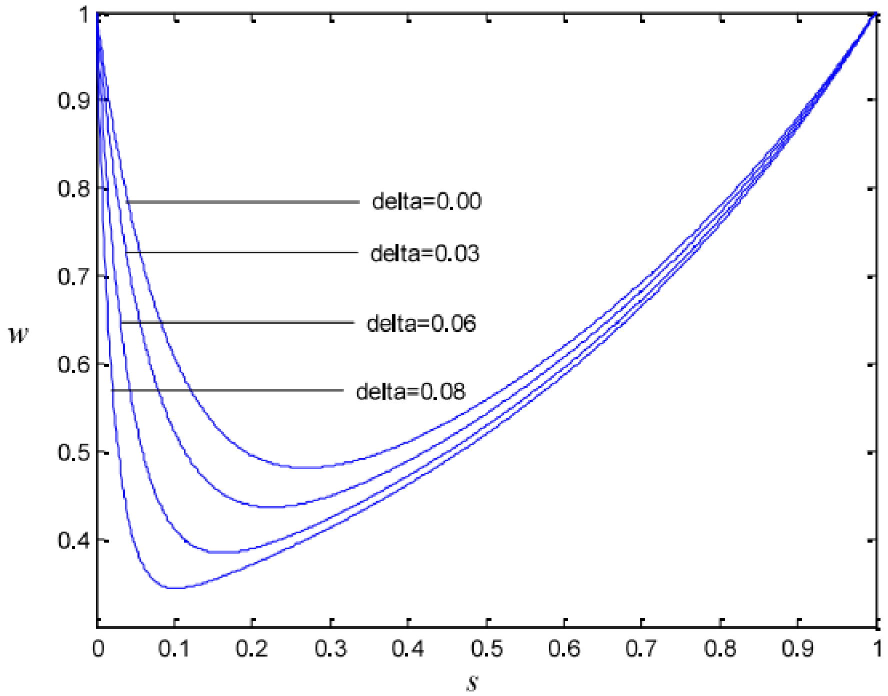

Figure. 4.3. Solution the problem 4.2 with = 0.1 and varying values δ of o(ε).

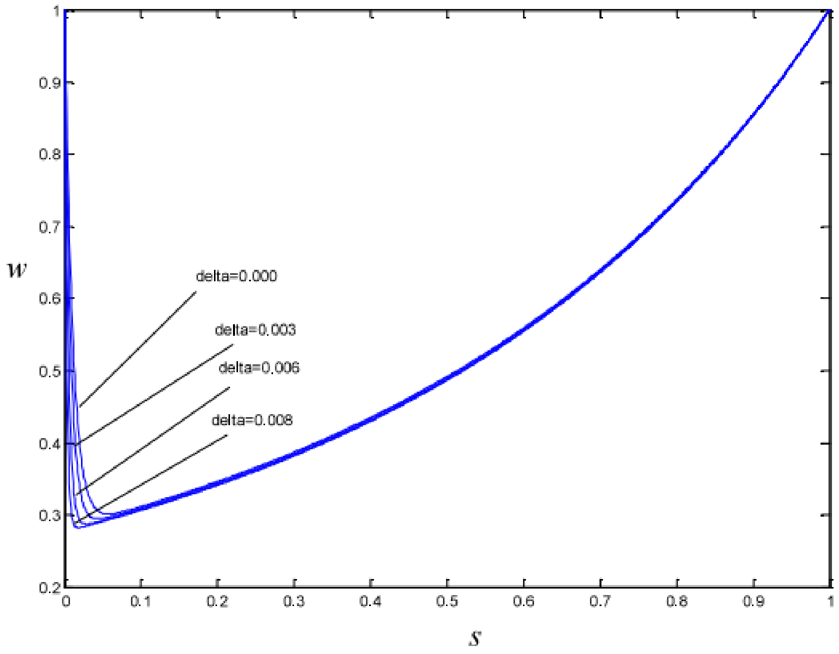

Figure 4.4. Solution the problem 4.2 with = 0.01 and varying values δ of o(ε).

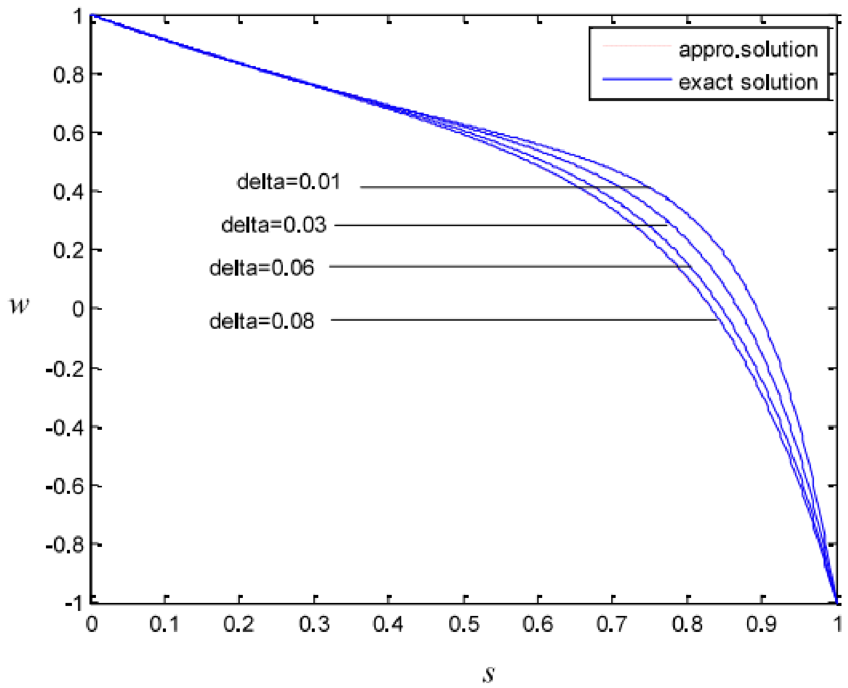

Figure 4.5. Solution the problem 4.3 with = 0.1 and varying values δ of o(ε).

CONCLUSIONS

In a nut shell, the quantitative approaches described in this study to solve differential-difference equations having boundary layers are simpler and easier than the conventional methods. In fact, the proposed methods are non-asymptotic and do not depend upon any lengthy series expansions. All of the quantitative methods discussed here are tested by applying them to a variety of model situations with varying delays, advances, and perturbation parameters. We tabulate and compare our answers to those found in the literature and/or exact solutions. The answer is also presented graphically to aid in comprehending the effects of the delay and complex parameters. It is noticed from the tables that our quantitative approaches produce very good approximation to the exact solution. In addition to being theoretically simpler and easier to use, these quantitative approaches are also easily adaptable for computer implementation with only a small amount of computing effort.

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