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Dynamical System With Mathematical Applications

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Abstract

In this work, we study dynamical systems with mathematical applications. We deal with specific properties of dynamical systems, namely stability and optimal control. Both qualities are common in industrial economics and engineering. We have selected some important examples and geometric solutions such as B. Portfolio selection and related economic problems. We have described the corresponding differential equations and used optimal control theory to approach the problem, and we have found some geometric solutions in dynamical systems and control theory.

Definition (1.1) [1] Monoid written additively, M is a set and Φ is a function $\Phi: U \subset T \times M \to M$

With $\mathrm{I}(x) = \{t \in T: (t, x) \in U\}$, $\Phi(0, x) = x$

$$\Phi\big(t_2, \Phi(t_1, x)\big) = \Phi(t_1 + t_2, x), \text{for } t_1, t_2, t_1 + t_2 \in I(x)$$

The function $\Phi(t, x)$ is called the evolution function of the dynamical system: it associates to every point in the set M a unique image, depending on the variable t, called the evolution parameter. M is called phase space or state space, while the variable x represents an initial state of the system, we often write $\Phi_x(t) \coloneqq \Phi(t, x)$, $\emptyset^t(x) \coloneqq \emptyset(t, x)$

If we take one of the variables as constant. $\Phi_x: I(x) \to M$ is called flow through x and its graph trajectory through x. The set

$\gamma_x := \{\Phi(t,x): t \in I(X)\}$ is called orbit through x. A subset S of the state space M is called Φ-invariant if for all x in S and all t in T

$\Phi(t,x) \in S$. For S to be Φ-invariant, we require that $I(x) = T$ for all x in S. That is, the flow through x should be defined for all time for every element of S.A real dynamical system, real-time dynamical system or flow is a tuple (T, M, Φ) with T an open interval in the real numbers R, M a manifold locally diffeomorphic to Banach space, and Φ a continuous function. If $T = R$ we call the system global, if T is restricted to the non-negative real's we call the system a semi-flow. If Φ is continuously differentiable we say the system is a

Differentiable dynamical system. If the manifold M is locally diffeomorphic to R^n the dynamical system is finite-dimensional and if not, the dynamical system is infinite-dimensional. Define a discrete-time dynamical system to be a pair(M, φ), where M is a metric space and $\varphi: M \to M$ a continuous map.

Let (M, φ) and (N, Ψ) be dynamical systems and $\theta: M \to N$ be a homeomorphism, that is, θ is continuous, one-to-one, and onto, and its inverse is continuous. The homeomorphism θ is a called a topological conjugacy if $\varphi = \theta^{-1}\Psi^{\theta}$. Let A be a finite alphabet and $\sum a$ set of bi-infinite sequences on A, that is, for $s \in \Sigma$, $s = \cdots, s-2, s-1. s_0. s_1 s_2., \ldots$ where $s_n \in A$. For $\delta > 1$ define a metric on $\Sigma$by

$$d\sum(s,s') = \sum_{n=-\infty}^{\infty} \frac{d_A(s_n, s'_n)}{\delta|n|}, \qquad (1.1)$$

Where

$$d_A(a,b) = \begin{cases} 0 & a = b, \\ 1 & a \neq b. \end{cases}$$

The metric space $\Sigma$is called a shift space if there is a map σ: $\sum \to \Sigma$, such that if $s' = \sigma(s)$, then $s'_n = s_{n-1}$.The important requirement is $\Sigma$is closed under the action of σ

Let $F \subseteq C^k(R)$ be the subspace of k-times continuously differentiable scalar functions on R. On this space there are metrics

$$dF(f,f') = \int_{-\infty}^{\infty} |f(t) - f'(t)|\, \beta^{-|t|} dt,$$

For $\beta > 1$, F is a signal space, if the map $\Phi_\tau: F \to F$ where $\Phi_\tau(f)(t) = f(t+\tau), \tau \in R$, (Σ, σ)there can be constructed a dynamical system $(F, \Phi_\tau), (F, \Phi_1)$ is conjugate to (Σ, σ) , $h: \Sigma \to F$ which is a homeomorphism between (Σ, σ) and $(h(\Sigma), \Phi_1)$. the function $\emptyset_a(t) \in c^k(R), for\ a \in A$that satisfy the following conditions.

1. $\emptyset_a(t) \neq \emptyset_b(t)$ almost everywhere for a $\neq$ b,
2. $max_{a,b}|\emptyset_a(t) - \emptyset_b(t)| < G\gamma^{-|t|}, \gamma > \beta > 1$, and
3. $\min_{a \neq b} \int_{|t|<\frac{1}{2}} |\emptyset_a(t) - \emptyset_b(t)|\, dt > \int_{|t|<\frac{1}{2}} \max|\emptyset_a(t) - \emptyset_b(t)| dt$.

Where $\beta$is the same as used for the metric (3).

Define $h: \Sigma \to c^k(R)$by

$$h(s) = \sum_{n=-\infty}^{\infty} \emptyset_{s_n}(t-n). \qquad (1.2)$$

Define $F = \{f(t+\tau): f \in h(\Sigma), 0 \leq \tau < 1\}$.It is easy to check that $h(\sigma(s)) = \Phi_1(\sigma(s))$, and so, such an F is a signal space.

Corollary (1.1) [2]

There exist $K, J > 0$ such that $Kd_\Sigma(s,s') \geq d_F(h(s),h(s')) \geq Jd_\Sigma(s,s')$.

Proof

The prove there exists $K > 0$ such that $d_F(h(s),h(s')) \leq Kd_\Sigma(s,s')$. The following chain of inequalities holds:

$$d_F(h(s),h(s')) = \int_{-\infty}^{\infty} \left| \sum_{n=-\infty}^{\infty} \emptyset_{s_n}(t-n) - \emptyset_{s'_n}(t-n) \right| \beta^{-|t|} dt$$

$$\leq \int_{-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \left| \emptyset_{s_n}(t-n) - \emptyset_{s'_n}(t-n) \right| \beta^{-|t|} dt$$

$$= \int_{-\infty}^{\infty} \sum_{n=-\infty}^{\infty} d_A(s_n, s'_n) \left| \emptyset_{s_n}(t-n) - \emptyset_{s'_n}(t-n) \right| \beta^{-|t|} dt$$

$$< \int_{-\infty}^{\infty} \sum_{n=-\infty}^{\infty} d_A(s_n, s'_n) G\gamma^{-|t-n|} \beta^{-|t|} dt = G \sum_{n=-\infty}^{\infty} d_A(s_n, s'_n) \int_{-\infty}^{\infty} \gamma^{-|t-n|} \beta^{-|t|} dt$$

Where $n \geq 0$ *and* $n < 0$ separately,

$\int_{-\infty}^{\infty} \gamma^{-|t-n|} \beta^{-|t|} dt = \frac{\gamma^{-|n|} + \beta^{-|n|}}{\log \beta\gamma} + \frac{\beta^{-|n|} - \gamma^{-|n|}}{\log \frac{\gamma}{\beta}} \leq \left(\frac{2}{\log \beta\gamma} + \frac{1}{\log \frac{\gamma}{\beta}} \right) \beta^{-|n|} \equiv C\beta^{-|n|}$. Hence, putting K = GC,

$$d_F(h(s),h(s')) \leq GC \sum_{n=-\infty}^{\infty} \frac{d_A(s_n, s'_n)}{\beta^{|n|}} = Kd_\Sigma(s,s').$$

Next, we prove$Jd_\Sigma(s,s') \leq d_F(h(s),h(s'))$. The following chain of inequalities holds:

$$d_F(h(s),h(s')) = \sum_m \int_{m-\frac{1}{2}}^{m+\frac{1}{2}} \left| \sum_n \emptyset_{s_n}(t-n) - \emptyset_{s'_n}(t-n) \right| \beta^{-|t|} dt$$

$$= \sum_m \int_{-\frac{1}{2}}^{\frac{1}{2}} \left| \sum_n \emptyset_{s_n}(t-n+m) - \emptyset_{s'_n}(t-n+m) \right| \beta^{-|t+m|} dt$$

$$= \sum_n \int_{-\frac{1}{2}}^{\frac{1}{2}} \left| \sum_n \emptyset_{s_{m-j}}(t+j) - \emptyset_{s'_{m-j}}(t+j) \right| \beta^{-|t+m|} dt$$

$$\geq \sum_m d_A(s_n, s'_n) \int_{-\frac{1}{2}}^{\frac{1}{2}} \left| \sum_n \emptyset_{s_{m-j}}(t+j) - \emptyset_{s'_{m-j}}(t+j) \right| \beta^{-|t+m|} dt$$

$$\geq \sum_m \frac{d_A(s_n, s'_n)}{\beta^{|m|}} \int_{-\frac{1}{2}}^{\frac{1}{2}} \left| \sum_n \emptyset_{s_{m-j}}(t+j) - \emptyset_{s'_{m-j}}(t+j) \right| \beta^{-|t|} dt$$

Where $J > 0$ for all m

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} \left| \sum_j \emptyset_{s_{m-j}}(t+j) - \emptyset_{s'_{m-j}}(t+j) \right| \beta^{-|t|} dt \geq J$$

The following reduction uses the stated third condition in the final strict inequality.

$$\int_{-\frac{1}{2}}^{\frac{1}{2}}\left|\sum_{j}\emptyset_{s_{m-j}}(t+j)-\emptyset_{s'_{m-j}}(t+j)\right|\beta^{-|t|}dt \geq \int_{-\frac{1}{2}}^{\frac{1}{2}}\left|\emptyset_{s_m}(t)-\emptyset_{s'_m}(t)\right|\beta^{-|t|}dt$$

$$-\sum_{j\neq 0}\int_{-\frac{1}{2}}^{\frac{1}{2}}\left|\emptyset_{s_{m-j}}(t-n)-\emptyset_{s'_{m-j}}(t-n)\right|\beta^{-|t|}dt \geq \min_{a\neq b}\int_{-\frac{1}{2}}^{\frac{1}{2}}\left|\emptyset_{a}(t)-\emptyset_{b}(t)\right|\beta^{-|t|}dt$$

$$-\sum_{j\neq 0}\int_{-\frac{1}{2}}^{\frac{1}{2}}\max_{a,b}\left|\emptyset_{a}(t-n)-\emptyset_{b}(t-n)\right|dt>0$$

Hence, there exists $J > 0$ such that

$$d_F\big(h(s),h(s')\big)\geq J\sum_{m}\frac{dA(s_n,s'_n)}{\beta^{|m|}}=Jd_{\Sigma}(s,s')$$

The global dynamical system (R, X, Φ) on a locally compact and topological space X, it is often useful to study the continuous extension Φ^* of Φ to the one –point compactification X^* of X. Although we lose the differential structure of the original system, we can now use compactness arguments to analyze the new system (R, X^*, Φ^*). In compact dynamical systems the limit set of any orbit is non-empty, compact and simply connected. Let us remind the definition of a continuous dynamical system. Denote by X, and with ρa metric function. A dynamical system on X is defined to be a mapping $\emptyset: R\times X \to X$; such that

1. $\emptyset(0,x)=x$ for all x, (Identical property);

2. $\emptyset(t+s,x)=\emptyset\big(t,\emptyset(s,x)\big)$ for all x ∈ X; and t, s ∈ R, (Group property);

3.$\emptyset(t,x)$ is a continuous function.

One may expect that systems with similar properties can be defined for processes with discontinuities. Present chapter is devoted to the problem of identification of such kind of systems, one of the most interesting and difficult problems for impulsive differential equations.

Theorem (1.1)[3] A function $\emptyset(t)\in pC'(T,\theta), \emptyset(t_0)=x_0$ is a solution of $\Delta x|_{t=\theta_i}=J_i(x)$. If and only if

$$\emptyset(t)=\begin{cases}x_0+\int_{t_0}^{t}f\big(s,\emptyset(s)\big)ds+\sum_{t_0\leq\theta_i<t}J_i\big(\emptyset(\theta_i)\big), & t\geq t_0,\\ x_0+\int_{t_0}^{t}f\big(s,\emptyset(s)\big)ds+\sum_{t_0\leq\theta_i<t}J_i\big(\emptyset(\theta_i)\big), & t\geq t_0.\end{cases}$$

Proof. Necessity. Let $\emptyset(t)$ be a solution of $\Delta x|_{t=\theta_i}=J_i(x)$ on T: Define a function

$$\psi(t)=\begin{cases}x_0+\int_{t_0}^{t}f\big(s,\emptyset(s)\big)ds+\sum_{t_0\leq\theta_i<t}J_i\big(\emptyset(\theta_i)\big), & t\geq t_0,\\ x_0+\int_{t_0}^{t}f\big(s,\emptyset(s)\big)ds+\sum_{t_0\leq\theta_i<t}J_i\big(\emptyset(\theta_i)\big), & t\geq t_0.\end{cases}$$

We shall check that $\psi\in pC'(T,\theta)$ and show that functions $\psi,\emptyset$ satisfy. Condition $\emptyset(t_0)=\psi(t_0)$ is obviously true. If $t\notin\theta$ and it is not an end point of T, then differentiating. $\psi(t)$ we find that $\psi'=f\big(t,\emptyset(t)\big)=\emptyset'(t),\ t\geq t_0, j\in A$ one has that

$$\psi(\theta_j) = x_0 + \int_{t_0}^{\theta_j} f(s, \emptyset(s))ds + \sum\nolimits_{t_0 \le \theta_i < \theta_j} J_i(\emptyset(\theta_i)),$$

and

$$\psi(\theta_j - h) = x_0 + \int_{t_0}^{\theta_j - h} f(s, \emptyset(s))ds + \sum\nolimits_{t_0 \le \theta_i < \theta_j - h} J_i(\emptyset(\theta_i))$$

Where $h > 0$, we obtain that

$$\psi' - (\theta_j) = \lim_{h \to 0+} \frac{\psi(\theta_j - h) - \psi(\theta_j)}{h} =$$

$$\lim_{h \to 0+} \frac{1}{h}\left[\int_{t_0}^{\theta_j - h} f(s, \emptyset(s))ds + \int_{t_0}^{\theta_j} f(s, \emptyset(s))ds\right] = f\left(\theta_j, \emptyset(\theta_j)\right) = \emptyset' - (\theta_j)$$

If α is the right end point of T and $\alpha \in T$ then similarly one can check that $\psi'\_(\alpha) = \emptyset'\_(\alpha), j \in A$ then

$$\Delta\psi|_{t=\theta_j} = \psi(\theta_j +) - \psi(\theta_j) =$$

$$\left[x_0 + \int_{t_0}^{\theta_j +} f(s, \emptyset(s))ds + \sum\nolimits_{t_0 \le \theta_i < \theta_j} J_i(\emptyset(\theta_i))\right] -$$

$$\left[x_0 + \int_{t_0}^{\theta_j} f(s, \emptyset(s))ds + \sum\nolimits_{t_0 \le \theta_i < \theta_j} J_i(\emptyset(\theta_i))\right] =$$

$$J_i(\emptyset(\theta_i)) = \Delta x|_{t=\theta_i}.$$

Thus, the conditions are verified if $t \ge t_0$.

Example (1. 1)[4] Let us consider the following system:

$$x'_1 = -\tfrac{1}{3}x_1 - 3x_2.$$

$$x'_2 = 3x_1 - \tfrac{1}{3}x_2,$$

$$\Delta x_{1|x\in\Gamma} = \left(2cos\tfrac{\pi}{6} - 1\right)x_1 - 2sin\tfrac{\pi}{6}x_2,$$

$$\Delta x_2|\mathrm{x} \in \Gamma = 2sin\frac{\pi}{6}x_1 + (2\cos\frac{\pi}{6} - 1)x_2. \quad (1.3)$$

Where G=R^2, and $\Gamma = [(x_1, x_2)|x_1 = x_2 \ ,0 < x_1]$. Let us start to

check conditions (C1)–(C6). One can easily find that $\bar{\Gamma} = [(x_1, x_2)|\sqrt{3}\ x_1 = x_2,\ 0 < x_1], \emptyset(x) = x_1 - x_2, \bar{\emptyset}(x) = \sqrt{3}\,x_1 - \mathrm{x}_2, f(x) = \left(-\tfrac{1}{3}x_1 - 3x_2, 3x_1 - \tfrac{1}{3}x_2\right),\ J(x) = (2cos\tfrac{\pi}{6}x_1 - 2sin\tfrac{\pi}{6}x_{2,}2sin\tfrac{\pi}{6}x_1 + 2cos\tfrac{\pi}{6}x_2)$ Consequently, we have that $\nabla\emptyset(x) = (1, -1) \neq 0$, so, condition (C1) is satisfied. It is seen that J, f are continuously differentiable functions and

det $[\frac{\partial J(x)}{\partial x}]$= det$\begin{pmatrix} 2cos\frac{\pi}{6} & 2sin\frac{\pi}{6} \\ 2sin\frac{\pi}{6} & 2cos\frac{\pi}{6} \end{pmatrix}$=4($cos^2\frac{\pi}{6}$+$sin^2\frac{\pi}{6}$)=4$\neq 0$, for all x it is also obvious that $\Gamma \cap \bar{\Gamma} = \emptyset$ moreover,

$$(\nabla\emptyset(x), f(x)) = \left((1, -1), \left(-\tfrac{1}{3}x_1 - 3x_2, 3x_1 - \tfrac{1}{3}x_2\right)\right) = \left(-\tfrac{10}{3}x_1 - \tfrac{8}{3}x_2\right) \neq 0$$

For all $x \in \Gamma$ the inequality $(\nabla\emptyset(x), f(x)) \neq 0$for all x $\in \bar{\Gamma}$ can be shown

Similarly. Thus, all conditions, (C1)–(C6) are fulfilled

Definition (1.2)[5] A solution x (t) = x(t,0, x_0) of (1.5) is said to be continuable to a set S $\subset R^n$as time decreases (increases) if there exists a moment $\xi \in R$ such that $\xi \leq 0(\xi \geq 0)$and x(ξ) $\in s$. The following theorems(1.2) provide sufficient conditions for the continuation of solutions of (1.5)

Lemma (1.1) [5]

There exists constant $c_3 = c_3(m) > 0$ such that for all $\Lambda^1, \Lambda^2 \in Z_m,$

$$\|H(\Lambda^1) - H(\Lambda^2)\|_{c^{(2+\alpha)((2+\alpha)/2)(\mu\times[0,T_m])}} \leq c_3\|\Lambda^1 - \Lambda^2\|_{Z_m}.$$

Proof

For $k = 1,2$ denote by $\Gamma^k$the hypersurface defined by Λ^k

Let $h(\mu,\lambda) \in C^\infty([-L,L]\times R)$ be function satisfying

$$h(\mu,\lambda) = \begin{cases} \mu, & if|\mu| \geq \frac{3}{4}L, \\ 0, & if\ \mu = \lambda \end{cases} \quad , \quad h_\mu(\mu,\lambda) > c > 0 \quad \text{for same constant c}$$

Define $Y_K = \Omega\times[0,T_m] \to \Omega\times[0,T_m] by$

$$Y_K(x,t) = \begin{cases} x, \ if\ dist(x,\Gamma_0) > \frac{3}{4}L, \\ x^0(s') + h\left(s_n, \Lambda^k(s',t)\right)N(s')\Big|_{(s',s_n)=\left(s^1(x),\ldots,s^n(x)\right)}, \ ifdist(x,\Gamma_0) \leq \frac{3}{4}L \end{cases}$$

Clearly, Y_K is a $C^{3+\alpha(3+\alpha)/2}$ diffeomorphism of $\Omega\times[0,T_m]$ which maps Q_k^i onto $\widehat{Q_0^l}$ denoting by $x = Y_k^{-1}(y,t)$the inverse function of $y = Y_K(x,t) \equiv (Y_k^1,\ldots,Y_k^n)$, $v_k(y,t) \equiv u_k(Y_k^{-1}(y,t),t)$. Then

$$\begin{cases} \frac{\partial v'_k}{\partial t} - \sum_{i,j=1}^n a_{i,j}^k \frac{\partial^2 v'_k}{\partial y_i \partial y_j} + \sum_{i=1}^n b_i^k \frac{\partial v'_k}{\partial y_i} = 0 \quad in\ \widehat{Q_0^l}. \\ u_k^1 = u_k^2 \quad on\ \Gamma_0\times[0,T_m], \\ \widehat{N^k}.\left[k^1\nabla_y u_k^1 - k^2\nabla_y u_k^2\right] = lV^k \quad on\ \Gamma_0\times[0,T_m], \\ v_k = g(Y_k^{-1}(y,t),t) \quad on\ (\Omega\times\{t=0\})\cup(\partial\Omega\times[0,T_m]), \end{cases}$$

Where $v_k = v_k^l \quad on \quad \widehat{Q_0^l}. \quad and$

$$a_{ij}^k = a_{i,j}^k(y,t) = \nabla_x Y_k^i(x,t).\nabla_x Y_k^j(x,t)\big|_{x=Y_k^{-1}(y,t)},$$

$$b_i^k = b_i^k(y,t) = \frac{\partial Y_k^i}{\partial t}(x,t) - \nabla_x Y_k^i(x,t)\big|_{x=Y_k^{-1}(y,t)},$$

$$\widehat{N^k} = \widehat{N^k}(y,t) = (N^k(x,t).\nabla_x)Y_k(x,t)\big|_{x=Y_k^{-1}(y,t)},$$

$N^k(x,t)$ is the unit vector normal to $\Gamma_t{}^k$, V^k is the normal velocity $of\ \Gamma^k$.

Hence, setting $w(y,t) = v_1(y,t) - v_2(y,t)$, we see that ω satisfies

$$\begin{cases} \frac{\partial\omega^l}{\partial t} - \sum_{i,j=1}^n a_{ij}^1 \frac{\partial^2\omega}{\partial y_i\partial y_j} + \sum_{i=1}^n b_i^1 \frac{\partial\omega^l}{\partial y_i} = \sum_{i,j=1}^n [a_{ij}^1 - a_{ij}^2]\frac{\partial^2 v_2^l}{\partial y_i \partial y_j} - \sum_{i=1}^n [b_i^1 - b_i^2]\frac{\partial v_2^l}{\partial y_i}] \equiv \emptyset \\ in\ Q_0^l \quad , \\ \omega^1 = \omega^2 \quad ,on\ \Gamma_0\times[0,T_m], \\ \widehat{N^1}.\left[k^1\nabla_y\omega^1 - k^2\nabla_y\omega^2\right] = l[V^1 - V^2] + \left(\widehat{N^1} - \widehat{N^2}\right).\left(k^1\nabla_y v_2^1 - k^2\nabla_y v_2^2\right) \\ \equiv \Psi \quad ,on\ \Gamma_0\times[0,T_m], \\ \omega = 0 \quad ,on\ (\Omega\times\{t=0\})\cup(\partial\Omega\times[0,T_m]), \end{cases}$$

Thus, we can now reflect ω^2 across $\Gamma_0\times[0,T_m]$

$$\|\omega\|_{C^{2+\alpha,(2+\alpha)/2}}\widehat{Q_0^1}+\|\omega\|_{C^{2+\alpha,(2+\alpha)/2}}\widehat{Q_0^2}\le C\left\{\|\omega\|_{C^{\alpha,\alpha/2}}\widehat{Q_0^1}+\|\omega\|_{C^{\alpha,\alpha/2}}\widehat{Q_0^2}+\|\Psi\|_{C^{1+\alpha,(1+\alpha)/2}(\Gamma_0\times[0,T_m])}\right\},$$

Where C depends on the $C^{\alpha,\alpha/2}$ norm of a_{ij}^k, b_i^k or, equivalently, on . $\|\emptyset\|_{C^{\alpha,\alpha/2}}(\widehat{Q_0^1})+\|\emptyset\|_{C^{\alpha,\alpha/2}}(\widehat{Q_0^2})C(m)\{\sum_{i,j=1}^n\|a_{ij}^1-a_{ij}^2\|_{C^{\alpha,\alpha/2}(\Omega\times[0,T_m])}+\sum_{i=1}^n\|[b_i^1-b_i^2]\|_{C^{\alpha,\alpha/2}(\Omega\times[0,T_m])}\le$

$$C(m)\|\Lambda^1-\Lambda^2\|_{C^{3+\alpha,(3+\alpha)/2}},$$

Definition (1.3)[6] . The space $C^{weak}([0,T];H^s(\Omega))$ denotes continuity on the interval $[0,T]$ with values in the weak topology of H^s. In other words, for any fixed $\emptyset\in H^s,(\emptyset,u(t))_s$ is a continuous scalar function on$[0,T]$. The inner product of H^s is given by

$$(u,v)_s=\sum_{\alpha\le s}\int D^\alpha u.D^\alpha v dx.$$

The Hilbert spaces we will be working on for most of the time is:

$$V^m=\{(u,v)\in H^m(\Omega)\times H^m(\Omega)\} \qquad (1.4)$$

We consider the following regularization of

$$\frac{\partial A^\epsilon}{\partial t}=\eta J_\epsilon^2\Delta A^\epsilon-A^\epsilon+\rho^\epsilon A^\epsilon+A^0, \qquad (1.5a)$$

$$\frac{\partial\rho^\epsilon}{\partial t}=J_\epsilon(J_\epsilon\Delta\rho^\epsilon)-2J_\epsilon\left[\nabla.\left(\frac{\partial\rho^\epsilon}{\partial A^\epsilon}J_\epsilon\nabla A^\epsilon\right)-\rho^\epsilon A^\epsilon+\bar{B}.\right] \qquad (1.5b)$$

The Banach space $V^2, m=2\ in(10) with\ norm\ \|(A,\rho)\|_{V^2}=\|A\|_2+\|\rho\|_2$.

Theorem (1.2)[6] (Local existence of solutions to the regularized residential burglary Model) For any $\epsilon>0$ and initial conditions $(A_0(x),\rho_0(x))\in V^2$ such that $A_0(x)>0$ there exists a solution, $(A^\epsilon,\rho^\epsilon)\in C^1([0,T_\epsilon],V^2)$, for some $T_\epsilon>0$ to the regularized system (1.5). Furthermore, the following energy estimate is satisfied:

$$\frac{d}{dt}\|A^\epsilon,\rho^\epsilon\|V^2\le c_3\|A^\epsilon,\rho^\epsilon\|^3V^2+c_2\|A^\epsilon,\rho^\epsilon\|^2V^2+c_1\|A^\epsilon,\rho^\epsilon\|V^2,$$

Where $c_1,c_2\ and\ c_3$ are constants that depend only on $\frac{1}{A^0},\epsilon\ and\ \eta$

***Proof* :**

Define the map $F^\epsilon=[F_1^\epsilon,F_2^\epsilon]:0\subseteq V^2\to X$,0 the set such that F^ϵmaps 0 toV^2 ,$i.e\ X=V^2$ define the function by:

$$F_1^\epsilon(A^\epsilon,\rho^\epsilon)=\eta J_\epsilon^2\Delta A^\epsilon-A^\epsilon+\rho^\epsilon A^\epsilon+A^0,$$

$$F_2^\epsilon(A^\epsilon,\rho^\epsilon)=J_\epsilon^2\Delta\rho^\epsilon-2J_\epsilon\left[\nabla.\left(\frac{\rho^\epsilon}{A^\epsilon}J_\epsilon\nabla A^\epsilon\right)\right]-\rho^\epsilon A^\epsilon+\bar{B}.$$

Hence, if $v^\epsilon=(A^\epsilon,\rho^\epsilon)\in V^2$ the original model reduces to an ODE in V^2

$$\frac{dv^\epsilon}{dt}=F^\epsilon(v),\qquad v^\epsilon(0)=\big(A_0(x),\rho_0(x)\big).$$

Let $v_i^\epsilon=(A_i^\epsilon,\rho_i^\epsilon)\in V^2\ (i=1,2)$ we drop ϵ for notational convenience. By definition of the V^2 -norm and F we have:

$$\|F(v_1)-F(v_2)\|_{V^2}=\|F_1(v_1)-F_1(v_2)\|_2+\|F_2(v_1)-F_2(v_2)\|_2.$$

By substituting (1.3) above and using

$\forall v\in H^m(\Omega),\gamma,k\in Z^+\cup 0, and\ 0\le\epsilon\le 1$: $\|J_\epsilon v\|_{m+\gamma}\le\frac{C_{m\gamma}}{\epsilon^\gamma}\|v\|_m$

$|J_\epsilon D^k v|_\infty \leq \frac{C_k}{\epsilon^{N/2+\gamma-k}} \|v\|_k$ and $\forall m \in Z^+ \cup 0$ there exists $C > 0$ such that for all $u, v \in L^\infty(\Omega) \cap H^m(\Omega)$:

$\|uv\|_m \leq c\{|u|_\infty \|D^m v\|_0 + \|D^m v\|_0 |v|_\infty\}, \sum_{0\leq|\alpha|\leq m} \|D^\alpha(uv) - uD^\alpha v\|_0 \leq$

$c\{|\nabla u|_\infty \|D^{m-1}\|_0 + \|D^m v\|_0 |v|_\infty\}$.

We obtain a suitable bound for F_1. Initially we have:

$\|F_1(v_1) - F_1(v_2)\|_2 \leq \eta \, \|J_\epsilon^2 \Delta(A_1 - A_2)\|_2 + \|A_1 - A_2\|_2 + \|\rho_1 A_1 - A_2\rho_2\|_2$.

Using

$\forall s > \frac{N}{2}, H^m(\Omega)$ is a Banach algebra. That is, there exists $c > 0$ such that for all $\|uv\|_s \leq C\|u\|_s \|v\|_s$. by

$\|\rho_1 A_1 - A_2\rho_2\|_2 \leq \|\rho_2\|_2 \|A_1 - A_2\|_2 + \|A_2\|_2 \|\rho_1 - \rho_2\|_2$. (1.6)

Using (1.6) we easily obtain the final estimate for F_1:

$$\|F_1(v_1) - F_1(v_2)\|_2 \leq \left(\frac{\eta}{\epsilon^2} + 1 + \|\rho_2\|_2\right) \|A_1 - A_2\|_2 + \|A_2\|_2 \|\rho_1 - \rho_2\|_2. \quad (1.7)$$

If we define the open set $\left\{(u, v) \in V^2 : \left|\frac{1}{u}\right|_\infty < K_1, \|u_2\|_2 < L_1, \ \|v_2\|_2 < L_2\right\}$

If $v_1, v_2 \in 0$ then

$\|F_1(v_1) - F_1(v_2)\|_2 \leq \tilde{C}_1 \|A_1 - A_2\|_2 + \tilde{C}_2 \|\rho_1 - \rho_2\|_2,$ (1.8)

Where

$\tilde{C}_1 = \frac{K_1}{\epsilon^3}\Big(\|\rho_2\|_2 + K_1\|A_1\|_1 |\rho_1|_\infty + K_1\|A_2\|_2 \|\rho_2\|_2 + K_1^2 \|A_2\|_2^2 \|\rho_2\|_2 +$

$\frac{K_1^3}{\epsilon} \|A_1\|_1 \|A_2\|_2 \|\rho_2\|_2 + \frac{K_1}{\epsilon^2} |\rho_1|_\infty + \|\rho_2\|_2\Big),$

$$\tilde{C}_2 = \frac{1}{\epsilon^2} + \|A_1\|_2 + \frac{C_1^2}{\epsilon^3} \|A_2\|_2 \|A_1\|_2 (1 + K\|A_2\|_1 + K_1\|A_1\|_1).$$

$\tilde{C}_1$ *and* $\tilde{C}_2$ depend only on $\|A_i\|_2, \|\rho_i\|_2$, ϵ and K_1 for $i = 1,2$. Combining (1.7) , and (1.8) gives

$$\|F_1(v_1) - F_1(v_2)\|_{V^2} \leq C(\eta, L_1, L_1, K_1, \epsilon)\|A_1 - A_2\|_2 + C(L_1, L_1, K_1, \epsilon)\|\rho_1 - \rho_2\|_2$$

1.1 Linear Homogeneous and Nonhomogeneous Systems

We start discussion of linear impulsive systems with the following differential

Equation:

$$x' = A(t)x.$$

$$\Delta x|_{t=\theta_i} = B_i x$$

Where $(t, x) \in R \times R^n, \theta_i$, $i \in Z$ is a B-sequence, such that $|\theta_i| \to \infty \; as |i| \to \infty$.

We suppose that the entries of n × n matrix A (t) are from $\text{þ}c(R, \theta)$, real valued

n × n matrices B_i, I $\in Z$, satisfy

Det $(\tau + B_i) \neq 0$,

where τ is the identical n × n matrix?

1.2 Laplace transform Method solution of Fractional ordinary Differential Equations

In 1819 Lacroix developed the formula for the n^{th} derivative $f(t) = x^m$,where m is a positive integer. then

$$D^n y = \frac{m!}{(m-n)!} x^{(m-n)},$$

Replacement of factorial symbol by the gamma function gives:

$$D^n y = \frac{\Gamma(m+1)}{\Gamma(m-n+1)} x^{m-n} \qquad (1.9)$$

Now(1.29)is defined for either n integers, or not (arbitrary number) (Debnath and Bhatta.2009). Liouville's first formula:

For any integer n, we have

$$D^n e^{ax} = a^n e^{ax} \qquad (1.10)$$

Liouville replaced n by an arbitrary number α (rational, irrational or complex), it is clear that the R.H.S of (1.10)is well defined; in this case, he obtained the following formula.

$$D^n e^{ax} = a^n e^{ax}$$

This formula is called first Liouville's formula. In series expansion of $f(x)$

, Liouville formula is given by

$$D^a f(x) = \sum_{n=0}^{\infty} c_n a_n^a \, e^{a_n x} \qquad (1.11)$$

Liouville's second formula:

Liouville formulated another definition (second form) of a fractional derivative based on the gamma function to extend Lacroix's formula (Loke Nath and Bhatt a, 2009).

$$D^a x^{-\beta} = \frac{(-1)^\alpha}{\Gamma(\beta)} \Gamma(\beta+\alpha) x^{-\beta-\alpha} \qquad (1.12)$$

Formula (1.11), (1.12)is called the Liouville's second definition of fractional derivative. We note that the Liouville derivative of a constant (when$\beta = 0$) is zero, but the derivative of a constant function to Lacroix's formula is

$$D^a 1 = \frac{x^{-\alpha}}{\Gamma(1-\alpha)} \neq 0 \qquad (1.13)$$

This led to a discrepancy between the two definitions of fractional derivative. But mathematicians prefer Liouville's definition. The idea of fractional derivatives or fractional integral can be described in different ways. Now by considering a linear homogenous n^{th} -order ordinary differential equation (initial value problem),

$$D^n y = 0, y^{(k)}(a) = o, \quad 0 \leq k \leq n-1$$

The solution is the fundamental set $\{1, x, x^2 \ldots \ldots \ldots, x^{n-1}\}$,

$$\therefore y = \sum_{n=0}^{\infty} c_r x^r$$

Now we must derive the Riemann-Liouville formula, that is by seeking the solution of the following in homogeneous ordinary differential equation

$$D^n y = f(x), \qquad D^{(k)}(0) = 0, \quad k = 0,1,2,\ldots\ldots\ldots$$

To get the solution of the above problem we use the Laplace transform method as follows,

$\pounds D^n\{y\} = \pounds\{f(x)\} \Rightarrow \bar{y}(s) = s^{-n} \bar{f}(s)$, where $\bar{y} = \pounds[y] and\ \bar{f} = \pounds[f]$

$\therefore y(x) = \pounds^{-1} s^{-n} \bar{f}(s)$. By using the convolution theorem, we get the solution as follows:

$$y(x) = \frac{1}{\Gamma(n)}\int_a^x (x-t)^{n-1} f(t)dt, \qquad (1.14)$$

Formula ($D^n y = \frac{m!}{(m-n)!}x^{(m-n)}$) is called Riemann-Liouville formula. Replacement of n by a number α gives the Riemann-Liouville fractional integral.

$D^{-\alpha} f(x) \equiv a^{D^{-\alpha}} x\, f(x) = \frac{1}{\Gamma(n)}\int_a^x (x-t)^{\alpha-1} f(t)dt, Re\ \alpha > 0$

Where $a^{D^{-\alpha}} x\ (or D^{-\alpha}\ )$ is called the Riemann Liouville integral operator

(Debnath and D Bhatta.2009; carl and Tom 2000). $if\ a = 0$ in the resulting formula is called Riemann fractional integral and if a= $-\infty$, is called Liouville fractional integral.The fractional derivative is given by replacing α by $-\alpha$ in The Riemann fractional integral is given by,

$$D^{-\alpha} f(x) \equiv 0^{D^{-\alpha}} x\, f(x) = \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1} f(t)dt, Re\ \alpha > 0 \qquad (1.15)$$

The formula (1.12) is of convolution type, then its Laplace transform is given by

$$£[D^{-\alpha} f(x)]] = \frac{1}{\Gamma(\alpha)} £[(x^{\alpha-1}) * f(x) = \frac{1}{\Gamma(\alpha)}\frac{\Gamma(\alpha)}{s^a}\overline{f(s)} = \frac{\overline{f(s)}}{s^a} \qquad (1.16)$$

The Laplace transform of fractional derivative of order α is given by Lokenath 2003:

$$£\{D^a x(t)\} = s^a \bar{x}(\mathrm{s}) - \sum_{k=0}^{n-1} s^k [D^{(\alpha-k-1)} x(0)] = s^a \bar{x}(\mathrm{s}) - \sum_{k=0}^{n-1} c_k\, s^k$$

Where $(n-1) < \alpha \le n$ and $c_k = D^{(\alpha-k-1)} x(0)$.

The special function of Mintage- Leffler function is defined by Haubold et al. (Humboldt et al 2009):

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k+\beta)} \qquad (1.17)$$

Where, $\beta \in \emptyset, Re(\alpha), Re(\beta) > 0. If\ \beta = 1\ then\ we\ have,$

$$E_{\alpha,1}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k+1)}$$

Some examples:

(i) $E_{0,1}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(1)} = \sum_{k=0}^{\infty} z^k$ $(ii)\ E_{1,1}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k+1)} = e^z$

$(iii) E_{1,2}(\mathrm{z}) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k+2)} = \frac{e^z-1}{z}$ $(iv) E_{1,0}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k)} = ze^z$

The function E(t,α, a) is used to solve differential equations of fractional order which is defined by:

$$E(t,\alpha,a) = t^\alpha \sum_{k=0}^{\infty} \frac{(at)^k}{\Gamma(k+\alpha+1)} = t^\alpha E_{1,\alpha+1}(at) \qquad (1.18)$$

Theorem(1.3)[8]:

$$\mathrm{E}(\mathrm{t},\alpha,a) = \frac{1}{\Gamma(\alpha)}\int_0^t \xi^{\alpha-1} e^{a(t-\xi)} d\xi$$

Proof: We start by the integral in the left-hand side of

$$\frac{1}{\Gamma(\alpha)}\int_0^t \xi^{\alpha-1} e^{a(t-\xi)} d\xi = \frac{1}{\Gamma(\alpha)}\int_0^t \xi^{\alpha-1}\left(\sum_{k=0}^{\infty}\frac{a^k(t-\xi)^k}{k!}\right) d\xi = \frac{1}{\Gamma(\alpha)}$$

$$\sum_{k=0}^{\infty} [a^k t^k (\int_0^t \xi^{\alpha-1} (1-\tfrac{\xi}{t})^k) d\xi] = \frac{1}{\Gamma(\alpha)}\sum_{k=0}^{\infty} \left[\frac{a^k t^k}{k!} I\right]$$

To complete the proof, we have to calculate the integral given by I, where

$$\mathrm{I} = \int_0^t \xi^{\alpha-1} (1-\tfrac{\xi}{t})^k d\xi$$

$Let\ u = \frac{\xi}{t}, then\ d\xi = tdu,$ as $\xi = 0\ then\ u = 0, and\ as\ \xi = t\ then\ u = 1$

by substituting into (iii)we get

I=$\int_o^1 t^{\alpha-1} u^{\alpha-1}(1-u)^k tdu = t^\alpha \int_0^1 u^{\alpha-1}(1-u)^k\, du = t^\alpha B(\alpha, k+1) =$

$$\frac{t^\alpha \Gamma(\alpha)\Gamma(k+1)}{\Gamma(k+\alpha+1)}$$

Substituting of (ii) into $(D^n y = \frac{m!}{(m-n)!} x^{(m-n)})$ yields

$$\frac{1}{\Gamma(\alpha)}\int_0^t \xi^{\alpha-1} e^{a(t-\xi)} d\xi = \frac{1}{\Gamma(\alpha)}\sum_{k=0}^{\infty}\frac{a^k t^k}{k!}\ \frac{t^\alpha \Gamma(\alpha) k!}{\Gamma(k+\alpha+1)} = t^\alpha \sum_{k=0}^{\infty}\frac{(at)^k}{\Gamma(k+\alpha+1)}\ t^\alpha E_{1,\alpha+1}(at) = E(t,\alpha,a)$$

1.3 Fractional ordinary differential equations:

The general form of fractional linear ordinary differential equations of order (n, q) is given by

$$[D^{\alpha n} + a_{n-1}D^{(n-1)\alpha} + \cdots + a_0 D^0]x(t) = h(t), \qquad t \geq 0 \qquad (1.19)$$

Where $\alpha = \frac{1}{q}.If\ q = 1, then\ \alpha = 1$

and equation (1.19) is a simple ordinary differential equation

Of order n. symbolically equation (1.39)can be expressed as

$$f(D^\alpha)x(t) = h(t) \qquad (1.20)$$

Where $D^\alpha \equiv \left[D^{\alpha n} + a_{n-1}D^{(n-1)\alpha} + \cdots + a_0 D^0\right]\ and$

$a_0, a_1, \ldots \ldots \ldots a_{n-1}$ are functions of the independent variable t.

Assume that the coefficients $a_0, a_1, \ldots \ldots \ldots a_{n-1}$are functions of the equation (1.43).They applying the Laplace transformation with respect to both sides of (1.43) we obtain,

$\pounds\{[D^{\alpha n} + a_{n-1}D^{(n-1)\alpha} + \cdots + a_0 D^0]x(t)\} = \pounds\{h(t)\}$ $\Rightarrow\pounds\{[D^{\alpha n}]x(t)\}+ a_{n-1}\pounds\{\left[D^{(n-1)\alpha}\right]x(t)\} + \cdots + a_0\pounds\{[D^0]x(t)\Rightarrow[s^{n\alpha}\bar{x}(s) - \sum_{k=1}^{n} c_k s^{nk}] + a_{n-1}\left[s^{\alpha(n-1)}\bar{x}(s) - \sum_{k=1}^{n-1} c_{k-1}s^k\right] + \cdots + a_1 s^\alpha \bar{x}(s) + a_0\bar{x}(s) = \bar{h}(s)\Rightarrow[s^{\alpha n} + a_{n-1}s^{\alpha(n-1)} + \cdots + a_1 s^\alpha + a_0]\bar{x}$ (s)=$\emptyset(s)\Rightarrow$

$$\bar{x}(s) = \frac{\emptyset(s)}{[s^{\alpha n} + a_{n-1}s^{\alpha(n-1)} + \cdots + a_1 s^\alpha + a_0]} \qquad (1.21)$$

If the equation of order (n,2), then $\alpha = \frac{1}{2}$, and formula (1.44)becomes

$$\bar{x}(s) = \frac{\emptyset(s)}{\left[s^{\frac{n}{2}} + a_{n-1}s^{\frac{(n-1)}{2}} + \cdots + a_1 s^{\frac{1}{2}} + a_0\right]} \qquad (1.22)$$

Assume that the R.H.S of (1.45)will be factorized and expressed as,

$$\bar{x}(s) = \sum_{r=1}^{k}\frac{\emptyset_r(s)}{\left(\sqrt{s}-\beta_r\right)^{m_r}} = \sum_{r=1}^{k}\frac{\emptyset_r(s)\left(\sqrt{s}-\beta_r\right)^{m_r}}{\left(s-{\beta_r}^2\right)^{m_r}} = \sum_{r=1}^{k}\sum_{i=0}^{m_r}\frac{\emptyset_r(s)\binom{m_r}{i}\left(\sqrt{s}\right)^{m_r}(\beta_r)^{m_r-i}}{\left(s-{\beta_r}^2\right)^{m_r}}$$

$$= \sum_{r=1}^{k}\sum_{i=0}^{m_r}\frac{\omega_i}{s^{\gamma_i}\left(s-{\beta_r}^2\right)^{m_r}} \qquad (1.23)$$

Where

$$m_1 + m_2 + \cdots + m_k = n, \omega_i = const\ \ and\ s^{\gamma_i} = \left(\emptyset_r(s)\left(\sqrt{s}\right)^{m_r}\right)^{-1}.$$

Example (1.2) [9]: consider the following fractional ordinary differential equation with variable coefficients

$tD^{\alpha}(t) + D^{\alpha-1}x(t) + tx(t) = 0, \qquad x(0) = 1, \quad 1 < \alpha \le 2 \qquad (1.24)$

The application of Laplace transform gives

$$-\frac{d}{ds} = £[D^{\alpha}(t)] + £[D^{\alpha-1}x(t)] + £[tx(t)] = 0 \qquad \Rightarrow$$

$$-\frac{d}{ds} = \left[s^{\alpha}\bar{x}(s) - \sum_{k=0}^{1} s^k D^{\alpha-k-1}x(0)\right] + \left[s^{\alpha-1}\bar{x}(s) - \sum_{k=0}^{0} s^k D^{\alpha-k-2}x(0)\right]$$

$-\frac{d\bar{x}(s)}{ds} = 0 \qquad \Rightarrow$

$-\frac{d}{ds} = [s^{\alpha}\bar{x}(s) - D^{\alpha-1}x(0) - sD^{\alpha-2}x(0)] + [s^{\alpha-1}\bar{x}(s) - D^{\alpha-2}x(0)] - \frac{d\bar{x}(s)}{ds} = 0 \qquad \Rightarrow$

$-s^{\alpha}\frac{d\bar{x}(s)}{ds} - \alpha s^{\alpha-1}\bar{x}(s) + D^{\alpha-2}x(0) + s^{\alpha-1}\bar{x}(s) - D^{\alpha-2}x(0) - \frac{d\bar{x}(s)}{ds} = 0 \qquad \Rightarrow (1+s^{\alpha})\frac{d\bar{x}(s)}{ds} = (1-\alpha)s^{\alpha-1}\bar{x}(s)$

$\frac{d\bar{x}(s)}{\bar{x}(s)} = \frac{(\alpha-1)s^{\alpha-1}}{(1+s^{\alpha})}ds \Rightarrow ln\bar{x}(s) = \frac{(1-\alpha)}{\alpha}\ ln(1+s^{\alpha}) + lnc$

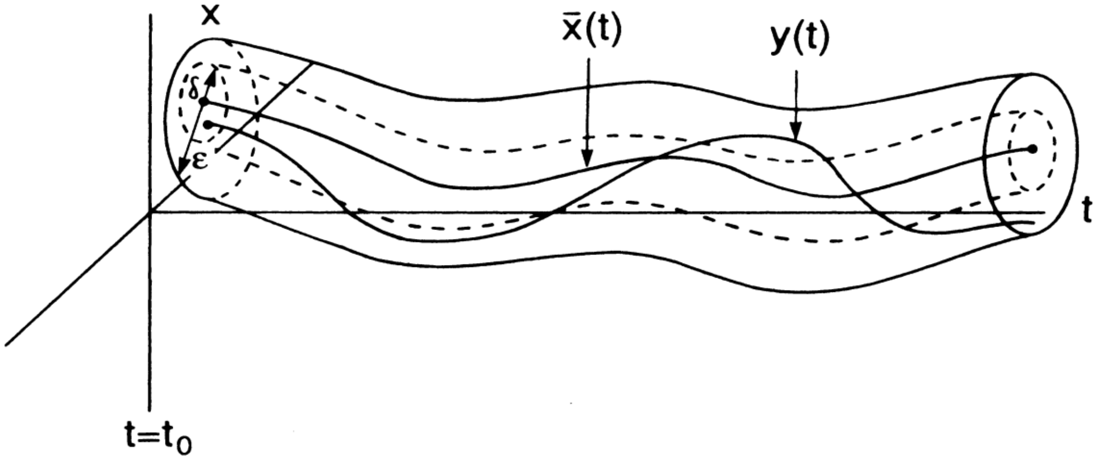
$$\therefore x(t) = £^{-1}\left[c(1+s^{\alpha})^{\frac{(1-\alpha)}{\alpha}}\right] \qquad (1.25)$$

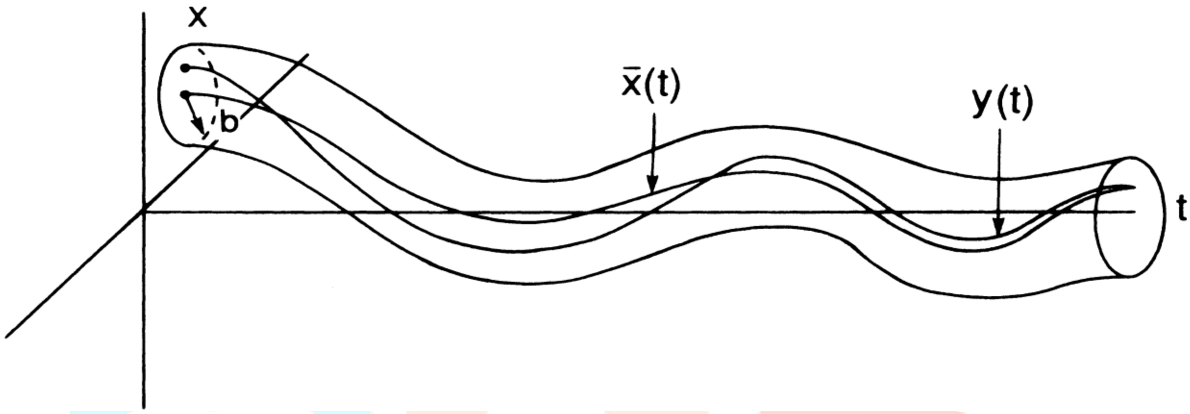
As special case takes $\alpha = 2$, then equation (1.25)becomes

$tD^2(t) + Dx(t) + tx(t) = 0, \qquad x(0) = 1$,then from (1.48) we have

$x(t) = £^{-1}\left[\frac{c}{\sqrt{(1+s^2)}}\right] = cJ_0(t)$.

Definition (1.4) [10] (Lyapunov Stability**.)** $\bar{x}(t)$ is said to be stable (or Lyapunov stable) if, given $\varepsilon > 0$, there exists *a* $\delta = \delta(\varepsilon) > 0$ such that, for any other solution, y(t), of (1.48) satisfying $|\bar{x}(t_0) - y(t_0)| < \delta$ (where $|.|$ is a norm on R^n), then $|\bar{x}(t) - y(t)| < \varepsilon$ *for* $t > t_0$, $t_0 \in R$. We remark that a solution which is not stable is said to be unstable.





1.4 Stability

An equilibrium solution, i.e.,$\bar{x}(t) = \bar{x}$, then $D\, x$(t, s) $f(\bar{x})$ is a matrix of (1.47) with constant entries, and the solution through the point

$$y^0 \in R^n$$

of $t = 0$ can immediately be written as

$$y(t) = e^{\mathrm{Df}(\bar{x}(t))} y^0.$$

Thus, y(t) is asymptotically stable if all Eigen values of $D\, f(\bar{x})$ have negative real part

Example (1.3) [11] (Stability and Eigen values of Time-Dependent Jacobians). For a general time, dependent solution $\bar{x}(t)$ it might be tempting to infer stability properties of this solution from the Eigen values of the Jacobian $D\, f(\bar{x}(t))$. The following example from Hale [1980] shows this can lead to wrong answers. Consider the following linear vector field with time-periodic coefficients

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \mathrm{A(t)} \begin{pmatrix} x_1 \\ \mathrm{x}_2 \end{pmatrix}$$

Where

$$\mathrm{A(t)}=\begin{pmatrix} -1 + \frac{3}{2}\, cos^2 t & 1 - \frac{3}{2} costsint \\ -1 - \frac{3}{2} costsint & -1 - \frac{3}{2} sin^2 t \end{pmatrix}$$

$= \left(-1+\frac{3}{2}\cos^2 t\right)\left(-1-\frac{3}{2}\sin^2 t\right)-\left(1-\frac{3}{2}cost sint\right)\left(-1-\frac{3}{2}cost sint\right)=\left(1+\frac{3}{2}\sin^2 t-\frac{3}{2}\cos^2 t-\frac{9}{4}\cos^2 t \sin^2 t\right)-1-\frac{3}{2}+\frac{3}{2}+\frac{9}{4} = \left(1-\frac{3}{2}(\cos^2 t-\sin^2 t)-\frac{9}{4}\cos^2 t\sin^2 t\right)-1+\frac{9}{4}$

The Eigen values of $A(t)$ are found to be independent of t and are given by

$$\lambda_1(t)=\frac{-1+i\sqrt{7}}{4} \quad , \qquad \lambda_2=\frac{-1-i\sqrt{7}}{4}$$

$$V_1(t)=\begin{pmatrix}-cost\\ sint\end{pmatrix}e^{\frac{t}{2}} \qquad V_2=\begin{pmatrix}sint\\ cost\end{pmatrix}e^{-t}$$

Hence, the solutions are unstable and of saddle type, a conclusion that does not

follow from the eigen values of $A(t)$.

$\{\ldots,A^{-n}y_0,\ldots,A^{-1}y_0,y_0,Ay_0,\ldots,A^n y_0,\ldots\}$ or the infinite sequence (if the map is C^r, $r\geq 1$, but noninvertible) $\{y_0,Ay_0,\ldots,A^n y_0\}$.

Recall that when the nominal feedback system is internally stable, the nominal performance condition is $\|W_1S\|_\infty<1$ and the robust stability condition is $\|W_2T\|_\infty<1$. If P is perturbed to $(1+\Delta W_2)P, S$ is perturbed to

$$\frac{1}{1-(1+\Delta W_2)L}=\frac{S}{1+\Delta W_2 T}$$

Clearly, the robust performance condition should therefore be

$$\|W_2T\|_\infty<1 \; and \; \left\|\frac{W_1S}{1+\Delta W_2T}\right\|_\infty<1, \qquad \forall\Delta.$$

Here Δ must be allowable. The next theorem gives a test for robust performance in terms of the function

$$s\mapsto |W_1(s)S(s)|+|W_2(s)T(s)|.$$

Which is denoted $|W_1S|+|W_2T|$.

Theorem (1.4) [7] A necessary and sufficient condition for robust performance is

$$\||W_1S|+|W_2T|\|_\infty<1. \qquad (1.26)$$

Proof ($\Leftarrow$) Assume (1.50), or equivalently

$$\|W_2T\|_\infty \; and \; \left\|\frac{W_1S}{1+\Delta W_2T}\right\|_\infty<1. \qquad (1.27)$$

Fix Δ. In what follows, functions are evaluated at an arbitrary point $j\omega$, but this is suppressed to simplify notation. We have

$$1=|1+\Delta W_2T-\Delta W_2T|\leq|1+\Delta W_2T|+|W_2T|$$

and therefore

$$1-|W_2T|\leq|1+\Delta W_2T|$$

This implies that

$$\left\|\frac{W_1S}{1-|W_2T|}\right\|_\infty\geq\left\|\frac{W_1S}{1+\Delta W_2T}\right\|_\infty.$$

This and (1.28) yield

$$\left\|\frac{W_1 S}{1+\Delta W_2 T}\right\|_\infty < 1$$

(⇒) Assume that

$$\|W_2 T\|_\infty < 1 \quad and \quad \left\|\frac{W_1 S}{1-|W_2 T|}\right\|_\infty < 1, \quad \forall\Delta \tag{1.28}$$

Pick a frequency ω where

$$\frac{|W_1 S|}{1-|W_2 T|}$$

is maximum. Now pick Δ so that

$$1-|W_2 T| = |1+\Delta W_2 T|$$

The idea here is that $\Delta(j\omega)$ should rotate $W_2(j\omega)T(j\omega)$ so that $\Delta(j\omega)W_2(j\omega)T(j\omega)$ is negative real. The details of how to construct such an allowable Δ are omitted. Now we have

$$\left\|\frac{W_1 S}{1-|W_2 T|}\right\|_\infty = \frac{|W_1 S|}{1-|W_2 T|}$$
$$= \frac{|W_1 S|}{|1+\Delta W_2 T|} \le \left\|\frac{W_1 S}{1+\Delta W_2 T}\right\|_\infty$$

So, from this and (1.28) there follows (1.50).

Test (1.26) also has a nice graphical interpretation. For each frequency ω, construct two closed disks: one with center -1, radius $|W_1(j\omega)|$, the other with center $L(j\omega)$, radius $|W_2(j\omega)L(j\omega)|$. More generally, let's say that robust performance level α is achieved if

$$\|W_2 T\|_\infty < 1 \;\; and \;\; \left\|\frac{W_1 S}{1+\Delta W_2 T}\right\|_\infty < \alpha, \qquad \forall\Delta.$$

Noting that at every frequency

$$\max_{|\Delta|<1}\left|\frac{W_1 S}{1+\Delta W_2 T}\right| = \frac{|W_1 S|}{1-|W_2 T|}$$

We get that the minimum α equals

$$\left\|\frac{W_1 S}{1-|W_2 T|}\right\|_\infty . \tag{1.29}$$

We allow Δ to satisfy $\|\Delta\|_\infty < \beta$. Application of Theorem (1.6) shows that internal stability is robust if $\|\beta W_2 T\|_\infty < 1$. let's say that the uncertainty level β is permissible if

$$\|\beta W_2 T\|_\infty < 1 \;\; and \;\; \left\|\frac{W_1 S}{1+\Delta W_2 T}\right\|_\infty < 1, \;\; \forall\Delta.$$

Again, noting that

$$\max_{|\Delta|<1}\left|\frac{W_1 S}{1+\beta\Delta W_2 T}\right| = \frac{|W_1 S|}{1-\beta|W_2 T|},$$

we get that the maximum β equals

$$\left\| \frac{W_2 S}{1 - |W_1 T|} \right\|_\infty^{-1}$$

Now we turn briefly to some related problems.

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