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Study on the Diophantine Equation $k^a + 37^b = c^2$, where $k \equiv 2(mod\ 111)$

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ABSTRACT

In this paper, we will discuss the solution of the non-linear exponential Diophantine equation $k^a + 37^b = c^2$, where k is a natural number such that $k \equiv 2(mod\ 111)$ and $a, b, c \in W$, here W is the set of whole numbers. We demonstrate the following results:
(1) If $k = 2$, then this Diophantine equation has a unique solution$(a, b, c) = (3, 0, 3)$.
(2) If $k \neq 2$ and $(k + 1)$ is not a perfect square, then this Diophantine equation has no solution.
(3) If $(k + 1)$ is a perfect square, then this Diophantine equation has a unique solution $(k, a, b, c) = (k, 1, 0, \sqrt{k + 1})$.

Keywords: Exponential Diophantine equations; Integral solutions.
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1. INTRODUCTION

The significance of the Diophantine Equation is critical in Number Theory, which is a fascinating branch of mathematics. Many other branches of mathematics are related to number theory.

In computer science, the Diophantine equation has various uses. The Diophantine equation is most commonly used in computer organization and security, random number generation, coding and cryptography, hash functions, and graphics.

Many scholars are now working on the solution of the Diophantine problem of the form $p^x + q^y = z^2$, where p, q are different primes and $x, y, z \in W$. In [1], Acu proved that there are only two solutions $\{x = 3, y = 0, z = 3\}$ and $\{x = 2, y = 1, z = 3\}$ of the Diophantine equation $2^x + 5^y = z^2$. In [2], A conjecture was formulated by Catalan that the Diophantine equation $a^x - b^y = 1$, where $a, b, x, y \in Z$, here Z is the set of integers, has unique solution $\{a = 3, b = 2, x = 2, y = 3\}$ under condition $\min\{a, b, x, y\} > 1$. In [3], Gupta and Kumar discussed the Exponential Diophantine equation $n^x + (n + 3m)^y = z^{2k}$, where n is a number of the form $6r + 1$ and $x, y, z, m, k, r \in W$. In [4], Gupta and Kumar discussed the Exponential Diophantine equation $a^u + (a + 5b)^v = c^{2w}$, where a is a number of the form $5r + 1$ and $u, v, w, b, c, r \in W$.

In [5], Gupta et al. discussed the Exponential Diophantine equation $x^\alpha + (1 + my)^\beta = z^2$ under some conditions and demonstrated that it has no solution in W. In [6], Kumar et al. discussed the equation $p^x + (p + 12)^y = z^2$ and showed that, under some conditions, it has no non-negative integer solution. In [7], Kumar et al. talked about the solution of equations $31^x + 41^y = z^2$ and $61^x + 71^y = z^2$, where x, y, z are natural numbers. In [8], Mihailescu proved the Catalan's hypothesis. In [9], Sroysang proved that the Diophantine equation $2^x + 19^y = z^2$ has a unique solution in W, which is $\{x = 3, y = 0, z = 3\}$. In [10], Sroysang proved that the Diophantine equation $5^x + 43^y = z^2$ has no solution in W. In [11], Terai shows that the equation $(12m^2 + 1)^x + (13m^2 - 1)^y = (5m)^z$ has a unique solution $\{x = 1, y = 1, z = 2\}$ under some conditions.

In [12], Viriyapong and Viriyapong find all the solutions to the Diophantine equation $n^x + 13^y = z^2$, where $n \equiv 2 (mod\ 39)$ and $n + 1$ is not a square number. In [13], Viriyapong and Viriyapong discussed the Diophantine equation $n^x + 19^y = z^2$, where $n \equiv 2 (mod\ 57)$ and prove that it has a unique solution $\{n = 2, x = 3, y = 0, z = 3\}$.

2. PRELIMINARIES

Conjecture 1([2]). (Catalan's Conjecture) The Diophantine equation $a^x - b^y = 1$, where $a, b, x, y \in Z$ has unique solution $(a, b, x, y) = (3, 2, 2, 3)$ under condition $\min\{a, b, x, y\} > 1$.

Lemma 2.1. There is no solution of the exponential Diophantine equation $1 + 37^b = c^2$ in non-negative integers, where b, c are non-negative integers.
Proof. Here we consider the following cases:
Case 1. If $b = 0$, then $c^2 = 2$, which is not possible.
Case 2. If $b = 1$, then $c^2 = 38$, which is not possible.
Case 3. If $b > 1$, then $c^2 - 37^b = 1 \Rightarrow c > 37$. Hence, by Catalan's conjecture, there is no solution in this case.

Lemma 2.2. If k is a positive integer such that $k \equiv 2 (mod\ 111)$ and a, b, c are non-negative integers, then the exponential Diophantine equation $k^a + 1 = c^2$, has
(1) Unique solution $(a, c) = (3, 3)$, if $k = 2$.
(2) No solution, if $k \neq 2$ and $(k + 1)$ is not a perfect square.
(3) Unique solution $(k, a, c) = \left(k, 1, \sqrt{k + 1}\right)$, if $(k + 1)$ is a perfect square.
Proof. Here we consider the following cases:
Case 1. If $a = 0$, then $c^2 = 2$, which is not possible.
Case 2. If $a = 1$, then $c^2 = k + 1$
Thus if $(k + 1)$ is not a perfect square, then there is no solution. However, if it is a perfect square, then there is a unique solution $(k, a, c) = \left(k, 1, \sqrt{k + 1}\right)$.
Case 3. If $a > 1$, then $c^2 - k^a = 1 \Rightarrow c > 2$. Hence, by Catalan's conjecture, there is a unique solution $(k, a, c) = (2, 3, 3)$ in this case.

Lemma 2.3. If a is an odd positive integer, then
$2^a \equiv 2, 5, 6, 8, 13, 14, 15, 17, 18, 19, 20, 22, 23, 24, 29, 31, 32, 35 (mod\ 37)$.
Proof. By induction, we will show that

$$2^{2n-1} \equiv 2, 5, 6, 8, 13, 14, 15, 17, 18, 19, 20, 22, 23, 24, 29, 31, 32, 35 (mod\ 37)$$

for all $n \in N$.
For $n = 1$, we have $2^1 \equiv 2 (mod\ 37)$. Therefore, this result is true for $n = 1$.
Assuming that this result holds true for $n = m$, then

$$2^{2m-1} \equiv 2, 5, 6, 8, 13, 14, 15, 17, 18, 19, 20, 22, 23, 24, 29, 31, 32, 35 (mod\ 37)$$

Thus

$$2^{2m+1} \equiv 8, 20, 24, 32, 15, 19, 23, 31, 35, 2, 6, 14, 18, 22, 5, 13, 17, 29 (mod\ 37)$$

Thus, the statement is true for $n = m + 1$.

3. MAIN RESULTS

Theorem 3.1. If k is a positive integer such that $k \equiv 2 (mod\ 111)$ and a, b, c are non-negative integers, then for the exponential Diophantine equation $k^a + 37^b = c^2$, we prove the following results:
(1) If $k = 2$, then there is a unique solution $(a, b, c) = (3, 0, 3)$.
(2) If $k \neq 2$ and $(k + 1)$ is not a perfect square, then there is no solution.
(3) If $(k + 1)$ is a perfect square, then there is only one solution $(k, a, b, c) = \left(k, 1, 0, \sqrt{k + 1}\right)$.
Proof. We have $k \equiv 2 (mod\ 111)$
$\Rightarrow k \equiv 2 (mod\ 3)$ and $k \equiv 2 (mod\ 37)$.
By lemma 2.1 and 2.2, we need to consider $a \geq 1$ and $b \geq 1$.
Here we consider two cases:
Case 1: If a is even, then $k^a \equiv 1 (mod\ 3)$. Also $37^b \equiv 1 (mod\ 3)$.
Thus, we get $k^a + 37^b \equiv 2 (mod\ 3) \Rightarrow c^2 \equiv 2 (mod\ 3)$.
However, for any integer c, we have $c^2 \equiv 0, 1 (mod\ 3)$.
This is a contradiction. Hence, there is no solution in this case.

Case 2: If a is odd, then $k^a \equiv 2^a (mod\,37)$

$\Rightarrow k^a \equiv 2, 5, 6, 8, 13, 14, 15, 17, 18, 19, 20, 22, 23, 24, 29, 31, 32, 35 (mod\,37)$

$\Rightarrow k^a + 37^b \equiv 2, 5, 6, 8, 13, 14, 15, 17, 18, 19, 20, 22, 23, 24, 29, 31, 32, 35 (mod\,37)$

$\Rightarrow c^2 \equiv 2, 5, 6, 8, 13, 14, 15, 17, 18, 19, 20, 22, 23, 24, 29, 31, 32, 35 (mod\,37)$.

However, for any integer c, we have

$c^2 \equiv 0, 1, 3, 4, 7, 9, 10, 11, 12, 16, 21, 25, 26, 27, 28, 30, 33, 34, 36 (mod\,37)$.

This is a contradiction. Hence, there is no solution in this case.

4. SOME SPECIAL CASES

Corollary 4.1. The non-linear Diophantine equation $2^a + 37^b = c^2$, where a, b, c are non-negative integers, has a unique solution $(a, b, c) = (3, 0, 3)$ in non-negative integers.

Proof. Since $2 \equiv 2 (mod\,111)$, therefore, by theorem 3.1, $(a, b, c) = (3, 0, 3)$ is the unique solution of the Diophantine equation $2^a + 37^b = c^2$.

Corollary 4.2. The non-linear Diophantine equation $113^a + 37^b = c^2$, where a, b, c are non-negative integers, has no non-negative integer solution.

Proof. Because $113 \equiv 2 (mod\,111)$ and $113 + 1 = 114$, which is not a perfect square, therefore the exponential Diophantine equation $113^a + 37^b = c^2$ has no solution in non-negative integers, according to theorem 3.1.

Corollary 4.3. The non-linear Diophantine equation $224^a + 37^b = c^2$, where a, b, c are non-negative integers, has a unique solution $(a, b, c) = (1, 0, 15)$ in non-negative integers.

Proof. Because $224 \equiv 2 (mod\,111)$ and $224 + 1 = 225$, which is a perfect square, therefore the exponential Diophantine equation $224^a + 37^b = c^2$ has a unique solution $(a, b, c) = (1, 0, 15)$ in non-negative integers, according to theorem 3.1.

5. CONCLUSION

In this paper, the authors discussed the non-negative integer solution of the non-linear exponential Diophantine equation $k^a + 37^b = c^2$, where k is a positive integer such that $k \equiv 2 (mod\,111)$ and a, b, c are non-negative integers. The authors proved that for $k = 2$, this Diophantine equation has a unique solution $(a, b, c) = (3, 0, 3)$. If $k \neq 2$ and $(k + 1)$ is not a perfect square, then this Diophantine equation has no solution. Also, if $(k + 1)$ is a perfect square, then this Diophantine equation has a unique solution $(k, a, b, c) = \left(k, 1, 0, \sqrt{k+1}\right)$.

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