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SOLUTION OF LAPLACE EQUATION INVOLVING GENERALIZED HYPERGEOMETRIC FUNCTIONS

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ABSTRACT

In the idea of generalized hypergeometric functions, classical summation theorems which includes the ones of Gauss, Gauss second, Kummer, Bailey, Dixon, Watson, Whipple, Saalschütz and Dougall play a key role. Applications of the above-cited classical summation theorems are well-known. The aim of this paper is to obtain the solution of Laplace equation in terms of Fox's H-function of one variable.

Keywords: Fox's H-function, Laplace Equation, Boundary Value Problem.

1. INTRODUCTION:

Charles Fox [2] introduced a more general function which is well-known inthe literature as Fox's H-function or the H-function. This function is defined and represented by means of the following Mellin-Barnes type of contour integral:

$$H^{m,n}_{p,q}\left[x\,\middle|\,{(a_j,a_j)_{1,p} \atop (b_j,\beta_j)_{1,q}}\right] = \frac{1}{2\,\omega}\int_{-\omega\infty}^{+\omega\infty} \theta(s)\, x^s ds \qquad (1)$$

where $\omega = \sqrt{(-1)}$,

$$\theta(s) = \frac{\prod_{j=1}^{m} (b_j-\beta_j s) \prod_{j=1}^{n} (1-a_j+a_j s)}{\prod_{j=m+1}^{q} (1-b_j+\beta_j s) \prod_{j=n+1}^{p} (a_j-a_j s)}$$

x is not equal to zero and an empty product is interpreted as unity; p, q, m, n are integers satisfying $0 \le m \le q$, $0 \le n \le p$, α_j (j = 1, …., p), β_j (j = 1, …, q) are positive numbers and a_j (j = 1, …, p), b_j (j = 1, …, q) are complex numbers. L is asuitable contour of Barnes type such that poles of $\Gamma(b_j - \beta_j s)$ (j = 1, …, m) lie on the right side of the contour and those of $\Gamma(1 - a_j + \alpha_j s)$ (j = 1, …, n) lie on the lefthand of the contour. These assumptions for the H-function will be adhered to through out this research work.

The behavior of the H-function has been given by Braakasma [1, p. 279, (6.5) and p. 246, (2.16)]:

$$H^{m,n}_{p,q}\left[x\,\middle|\,{(a_j,a_j)_{1,p} \atop (b_j,\beta_j)_{1,q}}\right] = O(|x|^{\alpha}) \text{ for small } x,$$

where $\sum_{j=1}^{p} \alpha_j - \sum_{j=1}^{q} \beta_j \le 0$ and $\alpha = \min R(b_h/\beta_h)$ (h = 1, .., m)

and

$$H^{m,n}_{p,q}\left[x\,\middle|\,{(a_j,a_j)_{1,p} \atop (b_j,\beta_j)_{1,q}}\right] = O(|x|^{\beta}) \text{ for large } x,$$

where $\sum_{j=1}^{n}\alpha_j - \sum_{j=n+1}^{p}\alpha_j + \sum_{j=1}^{m}\beta_j - \sum_{j=m+1}^{q}\beta_j \equiv A > 0$ (2)

$\sum_{j=1}^{p}\alpha_j - \sum_{j=1}^{q}\beta_j < 0$ and

$|\arg x| < \frac{1}{2}A\pi$ and $\beta = \max R[(a_j - 1)/\alpha_j]$ $(j = 1, \ldots, n)$.

The Laplace equation is often encountered in heat and mass transfer theory, fluid mechanics, elasticity, electrostatics and other areas of mechanics and physics. Since Fox's H-function is believed to be quite general in nature because it includes a number of well known elementary functions as its particular cases. Evidently, therefore, our results would apply to a wide variety of useful functions (or products of several such functions) occurring frequently in mathematicalphysics and engineering.

In this paper, we shall make application of the following modified form of the integral [3, p.372, (1)]:

$$\int_0^L \{\sin(\pi x/L)\}^{\omega-1}\sin(n\pi x/L)\,dx = \frac{L\sin\frac{1}{2}n\pi\,\Gamma(\omega)}{2^{\omega-1}\Gamma\{\frac{1}{2}(\omega \pm n+1)\}}, \quad (3)$$

where n is any integer and $\mathrm{Re}(\omega) > 0$.

2. INTEGRAL:

The integral involving the H-function of one variable to be evaluated is:

$$\int_0^L (\sin(\pi x/L))^{\omega-1}\sin(n\pi x/L)\, H_{p,q}^{m,l}\left[z(\sin(\pi x/L))^{\lambda}\,\middle|\,\begin{matrix}(a_j,\alpha_j)_{1,p}\\(b_j,\beta_j)_{1,q}\end{matrix}\right]dx$$

$$= 2^{1-\omega} L\sin\tfrac{1}{2}n\pi\, H_{p+1,q+2}^{m,l+1}\left[z\,2^{-\lambda}\,\middle|\,\begin{matrix}(1-\omega,\lambda),(a_j,\alpha_j)_{1,p}\\(b_j,\beta_j)_{1,q},(1/2-\omega/2 \pm n/2,\lambda/2)\end{matrix}\right], \quad (4)$$

where $\lambda \geq 0$ and $\mathrm{Re}(\omega) > 0$, $|\arg z| < \frac{1}{2}M\pi$, M is given by:

$\sum_{j=1}^{l}\alpha_j - \sum_{j=n+1}^{p}\alpha_j + \sum_{j=1}^{m}\beta_j - \sum_{j=m+1}^{q}\beta_j \equiv M > 0$

Proof of (4):

Replace the H- function by its equivalent contour integral as given in (1), change the order of integration, evaluate the inner integral with the help of (3) and finally interpret it with (1), to get (4).

3. FIRST BOUNDARY VALUE PROBLEMFOR THE LAPLACE EQUATION:

The two-dimensional Laplace equation has the following form:

$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = 0 \quad (5)$$

in the Cartesian coordinate system.

Under the domain $0 \le x \le a$, $0 \le y \le b$, a rectangle is considered. Boundaryconditions are prescribed:
$w = f_1(y)$ at $x = 0$, $w = f_2(y)$ at $x = a$, $w = f_3(x)$ at $y = 0$, $w = f_4(x)$ at $y = b$.
The solution of equation (5) is given as

$$w(x, y) = \sum_{n=1}^{\infty} A_n \sinh[n\pi(a - x)/b] \sin(n\pi y/b)$$

$$+ \sum_{n=1}^{\infty} B_n \sinh(n\pi x/b) \sin(n\pi y/b)$$

$$+ \sum_{n=1}^{\infty} C_n \sin(n\pi x/a) \sinh[n\pi(b - y)/a]$$

$$+ \sum_{n=1}^{\infty} D_n \sin(n\pi x/a) \sinh(n\pi y/a) \qquad (6)$$

where the coefficients A_n, B_n, C_n and D_n are expressed as

$$A_n = (2/\lambda_n) \int_0^b f_1(\xi) \sin(n\pi\xi/b)\, d\xi, \qquad (7)$$

$$B_n = (2/\lambda_n) \int_0^b f_2(\xi) \sin(n\pi\xi/b)\, d\xi, \qquad (8)$$

$$C_n = (2/\mu_n) \int_0^a f_3(\xi) \sin(n\pi\xi/a)\, d\xi, \qquad (9)$$

$$D_n = (2/\mu_n) \int_0^a f_4(\xi) \sin(n\pi\xi/a)\, d\xi, \qquad (10)$$

$$\lambda_n = b \sinh(n\pi a/b), \qquad (11)$$

and $$\mu_n = a \sinh(n\pi b/a). \qquad (12)$$

4. SOLUTION IN TERMS OF H-FUNCTION:

Now choose

$$f_1(\xi) = f_2(\xi) = \sin(\pi\xi/b)^{\omega - 1} H_{p,q}^{m,l}\left[z(\sin(\pi\xi/b))^{\lambda} \middle| \begin{matrix} (a_j, \alpha_j)_{1,p} \\ (b_j, \beta_j)_{1,q} \end{matrix}\right] \qquad (13)$$

and

$$f_3(\xi) = f_4(\xi) = \sin(\pi\xi/a)^{\omega-1} H^{m,l}_{p,q}\left[z(\sin(\pi\xi/a))^{\lambda} \,\Big|\, \begin{matrix}(a_j,\alpha_j)_{1,p}\\(b_j,\beta_j)_{1,q}\end{matrix}\right] \quad (14)$$

Combining (7), (8) and (13) and making the use of integral (4), we derive

$$A_n = B_n = 2^{2-\omega}(b/\lambda_n)\sin \tfrac{1}{2} n\pi\, H^{m,l+1}_{p+1,q+2}\left[z\, 2^{-\lambda} \,\Big|\, \begin{matrix}(1-\omega,\lambda),(a_j,\alpha_j)_{1,p}\\(b_j,\beta_j)_{1,q},(1/2-\omega/2 \pm n/2, \lambda/2)\end{matrix}\right], \quad (15)$$

Similarly from (9), (10) and (14) and making the use of integral (4), we derive

$$C_n = D_n = 2^{2-\omega}(a/\mu_n)\sin \tfrac{1}{2} n\pi\, H^{m,l+1}_{p+1,q+2}\left[z\, 2^{-\lambda} \,\Big|\, \begin{matrix}(1-\omega,\lambda),(a_j,\alpha_j)_{1,p}\\(b_j,\beta_j)_{1,q},(1/2-\omega/2 \pm n/2, \lambda/2)\end{matrix}\right], \quad (16)$$

Putting the value of A_n, B_n, C_n and D_n from (15) and (16) in (6), we get the following required solution of the Laplace equation in terms of Fox's H-function:

$$w(x,y) = \Big\{ \sum_{n=1}^{\infty} 2^{2-\omega}(b/\lambda_n)\sin \tfrac{1}{2} n\pi \sinh[n\pi(a-x)/b]\sin(n\pi y/b)$$

$$+ \sum_{n=1}^{\infty} 2^{2-\omega}(b/\lambda_n)\sin \tfrac{1}{2} n\pi \sinh(n\pi x/b)\sin(n\pi y/b)$$

$$+ \sum_{n=1}^{\infty} 2^{2-\omega}(a/\mu_n)\sin \tfrac{1}{2} n\pi \sin(n\pi x/a)\sinh[n\pi(b-y)/a)]$$

$$+ \sum_{n=1}^{\infty} 2^{2-\omega}(a/\mu_n)\sin \tfrac{1}{2} n\pi \sin(n\pi x/a)\sinh(n\pi y/a) \Big\} \times$$

$$\times\; H^{m,l+1}_{p+1,q+2}\left[z\, 2^{-\lambda} \,\Big|\, \begin{matrix}(1-\omega,\lambda),(a_j,\alpha_j)_{1,p}\\(b_j,\beta_j)_{1,q},(1/2-\omega/2 \pm n/2, \lambda/2)\end{matrix}\right], \quad (17)$$

where $\lambda \geq 0$ and Re $(\omega) > 0$, $|\arg z| < \tfrac{1}{2} M\pi$, M is given by:

$$\sum_{j=1}^{l} \alpha_j - \sum_{j=n+1}^{p} \alpha_j + \sum_{j=1}^{m} \beta_j - \sum_{j=m+1}^{q} \beta_j \equiv M > 0$$

5. SPECIAL SOLUTION:

The importance of the H-function lies largely from the possibility of expressing by means of the H-symbols a great many of special functions appearing in applied mathematics, physical sciences and statistics. So that each of the solutions given in (17) becomes a master or key solution from which a very large number of solutions can be derived for Meijer's G-function, Generalized Hypergeometric function, Bessel, Legendre, Whittaker functions, theircombinations and many other functions.

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