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ON RECENT STUDIES ON THE THEORY OF GRAPH

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Abstract: Graph coloring is one of the best known, popular and extensively field of graph theory, having many applications, which are still open and studied by various mathematicians and computer scientists along the world. In this paper we present a survey of graph coloring as an important subject field of graph theory, describing various methods of the coloring, and a list of problems and associated with them. Lastly, we turn our attention to graphs, a lass of graphs, which has been found to be very interesting to study . A brief review of graph coloring methods (in Polish) was given by Kubale a more detailed one in a book by the same author. We extend this review and explore the field of graph coloring further, describing various results obtained by other authors and show some interesting applications of this field of graph theory.

Keywords: Graph coloring, vertex coloring, edge coloring, dominating energy complexity, algorithms.

1 Introduction

The study on graph theory was initiated by many great mathematicians suchas Euler who solved **Königsberg Bridge Problem**, Hamilton well know after Hamilton cycle Problem, Kirchoff known for laws of electrical circuits, Claude Berge who first introduced domination in his popular book and so on. Graph theoryis one of the widely growing area of Mathematics and finds applications in many interdisciplinary and as well as multidisciplinary areas like Algebra, Probability, Computer Science, Operation Research etc. A graph G = (V, E) can be constructed and visualized wherever there is a finite non-empty set V of objects called vertices together with a set E of unordered pairs of vertices of G, called edges. The vertex set and the edge set of G are denoted by V (G) and E(G) respectively. For standard reference books on graph theory, one can refer Bondy and Murthy [3], Chatrand and Lesnaik [4], Douglas B. West.[6], F.Harary [10], Narsingh Deo [17].

Domination in polynomial is the assignment of colors to the vertices of graph G such that adjacent vertices receive different colors. Such a coloring is called as proper vertex coloring of G. The minimum number of colors required for proper vertex coloring of G is said to be chromatic number of G denoted by χ(G). Aproper vertex coloring of G with k number of colors is said to be k−coloring of G.

Also the proper vertex coloring of G partitions the vertex set of G into independent sets. The problem of coloring a graph arises in many practical areas such as pattern matching, scheduling problems, solving Sudoku problems etc.

Although the mathematical study of dominating sets in graphs began in 1960, the subject has historical roots dating back to 1862 when de Jaenish studied the problem of determining the minimum number of queens which are necessary to cover an n x n chessboard. In 1958 Claude Berge wrote a book of graph the-ory in which he defined for the first time the concept of domination number of a graph (although he called this number as 'coefficient of external stability'). In 1962, Oystien Ore published his book on graph theory, in which he used for the first time, the names 'dominating set' and 'domination number'. A set $S \subseteq V$ of verticesin a graph $G = (V, E)$ is called a dominating set if every vertex $v \in V$ is either an element of S or is adjacent to an element of S. The minimum cardinality of dominating set of G is called domination number of G denoted by γ(G). The work on graph domination is steadily growing and finds applications in social network theory, Optimization problems, computer communication networks, radio stations, land surveying etc.

1.1 Basic Definitions

A **graph** G = (V, E) is a nonempty set of objects called **vertices** together with a prescribed set of unordered pairs of distinct vertices of G called **edges**. We represent the vertex set of G by V (G) or (V) and the edge set of G by E(G) or (E). If e = (u, v)is an edge of G, we write e = uv; we say that e joins the vertices u and v; u andv are **adjacent vertices** and the vertex u and edge e are **incident** as are v and

e. If two distinct edges e_i and e_j are incident with a common vertex, then they are **adjacent edges**.

A graph G is **finite** if both vertex set and edge set of G are finite. An edge join a vertex to itself is called a **loop**. If two or more edges join the same two verticesin a graph, then such edges are called **multiple edges**. A graph without loops and multiple edges is called a **simple** graph. We denote the number of vertices andedges in G by |V |(or n) and |E| or (m). The number n and m are referred to as the **order** and **size** of G and G is called a **(n, m)** graph. The **complement** G of Ghas V (G) as its vertex set, but two vertices are adjacent in G if and only if they are not adjacent in G. A **self complementary** graph is isomorphic to its complement. A **clique** of G is a maximal complete sub graph. A **cut vertex** of a graph is one whose removal increases the number of components, and a **bridge** is such an edge. A **cutset** of a connected graph is a collection of edges whose removal results in a disconnected graph.

2 Dominator coloring of graphs

Using the idea of graph coloring and domination, **Hedetniemi** et al.[11] introducedthe concept of dominator coloring of a graph.

Definition 2.1. The proper coloring of G such that every vertex of G dominates some color class is said to be dominator coloring of a graph. The color partition *$\{V_1, V_2, ..., V_k\}$ of G such that every $v \in V(G)$ dominates some V_i $(1 \leq i \leq k)$ is* said to be the dominator partition of G. The minimum number of colors required for dominator coloring of G is said to be dominator chromatic number of G denotedby $\chi_d(G)$.

Gera [7], gave a strict bound for dominator chromatic number of a graph,

Theorem 2.2. Let G be a connected graph. Then
$\max\{\chi(G), \gamma(G)\} \leq \chi_d(G) \leq \chi(G) + \gamma(G)$.
This bound lead to the following Questions,
Question 2.3. For what graphs does $\chi_d(G) = \chi(G)$?
Question 2.4. For what graphs does $\chi_d(G) = \gamma(G)$?
Question 2.5. For what graphs does $\chi_d(G) = \chi(G) + \gamma(G)$?
Theorem 2.6. Gera [8], proved the following result for complete bipartite graphsand complete graphs with $\chi_d(G) = \chi(G)$,

Theorem 2.7. Let G be a connected graph of order n. Then

$$\chi_d(G) = 2 \text{ if and only if } G = K_{a,b} \text{ for } a, b \in N.$$

Theorem 2.8. Let G be a connected graph of order n. Then

$$\chi_d(G) = n \text{ if and only if } G = K_n \text{ for } n \in N.$$

Theorem 2.9. Chellali and **Maffray** [5], gave a lower bound for Dominator chro-matic number of some class of graphs in terms of its domination number.

Theorem 2.10. Let G be a connected graph of order $n \geq 2$ such that either (a) G is C_4- free, or (b) G is claw free and not a C_4. Then $\chi_d(G) \geq \gamma(G) + 1$. **Corollary 2.11.** Every chordal graph of order $n \geq 2$ G satisfies $\chi_d(G) \geq \gamma(G) + 1$.**Corollary 2.12.** Every tree of order $n \geq 2$, T satisfies $\gamma(T)+1 \leq \chi_d(T) \leq \gamma(T)+2$. **Question 2.13.** Characterize trees T for which $\chi_d(T) = \gamma(T) + 1$.

Arumugam et al. [2] proved the following result,

Proposition 2.14. Let T be a tree of order n. If there exists a $\gamma-$set S in T such that $V - S$ is independent, then $\chi_d(T) = \gamma(T) + 1$.

Chellali and **Maffray** [5] also characterized split graphs with $\chi_d(T) = \gamma(T) + 1$

by the following result,

Theorem 2.15. Let G be a connected split graph, whose vertex set is partitioned into a clique Q and a stable set S such that Q is minimal. Then $\chi_d(G) = \gamma(G) + 1$if and only if every vertex of Q is a support vertex.

Arumugam et al.[2] characterized unicyclic graphs, split graphs and comple- ment of bipartite graphs with $\chi_d(G) = \chi(G)$ by the following results,

Theorem 2.16. Let G be a connected unicyclic graph. Then $\chi_d(G) = \chi(G)$ if and only if G is isomorphic to C_3 or C_4 or C_5 or the graph obtained from C_3 by attachingany number of leaves at one or two vertices of C_3.

Theorem 2.17. Let G be a split graph with split partition (K, I) and $|K| = \omega$. Then $\chi_d = \omega$ or $\omega + 1$. Furthermore $\chi_d = \omega$ if and only if there exists a dominating set D of G such that $D \subseteq K$ and every vertex v in I is nonadjacent to at least one vertex in $K - D$.

Gera et al.[8] defined safe clique partitions with the motive of finding dominatorchromatic number of the complement of the graph.

Definition 2.19. The clique partition is the partition of the vertex set $V(G)$ ofthe graph G into $\{V_1, V_2, ..., V_k\}$ such that each V_i $(1 \le i \le k)$ is a clique and the minimum value of k is said to be clique partition number $\chi(G)$. For a graph Gand clique partition $\{\bar{V_1}, V_2, ..., V_k\}$, a vertex v is said to be safe if there exists some$(1 \le i \le k)$ such that $N(v) \cap V_i = \emptyset$. The clique partition $\{V_1, V_2, ..., V_k\}$ is said to be safe clique partition if every vertex $v \in V(G)$ is safe and the minimum value of k is said to be safe clique partition number of G denoted by $\chi_d(G)$.

They proved some results on trees concerning safe clique partition and also foundthat $\chi_d(P) = \chi_d(P) = \chi_d(\bar{P}) = 5$, where P is the petersen graph.

This raised the question,

Question 2.20. For what graphs $\chi_d(G) = \chi_d(\bar{G})$? Or in particular, for what trian- gle free graphs does $\chi_d(G) = \chi_d(\bar{G})$?

Dominator coloring of Mycielskian of graphs

One of the important graph transformation is Mycielskian of a graph introduced by

Mycielskian [16].

Definition 2.21. For a graph G = (V, E), the Mycielskian of G denoted by $\mu(G)$

is the graph with vertex set $V \cup V' \cup \{u\}$ where $V' = \{x' : x \in V\}$ and is disjoint *from V, and edge set* $E' = E \cup \{xy' : xy \in E\} \cup \{x'u : x' \in V'\}$. *The vertices x and*

x' are called twins of each other and u is called the root of $\mu(G)$.

Arumugam et al.[2] also studied the dominator chromatic number of Myciel-skian of a graph and proved the following result,

Theorem 2.22. For any graph G, $\chi_d(G) + 1 \leq \chi_d(\mu(G)) \leq \chi_d(G) + 2$. Further if there exists a χ_d-coloring C of G in which every vertex v dominates a color class V_i with $v \notin V_i$, then $\chi_d(\mu(G)) = \chi_d(G) + 1$.

Question 2.23. Characterize graphs G for which $\chi_d(\mu(G)) = \chi_d(G) + 1$.

The following question was solved by the result,

Theorem 2.24. Given a graph G, $\chi_d(\mu(G)) = \chi_d(G) + 1$ if and only if for some

χ_d—coloring C of G:

(i) each vertex v dominates some color class V_i with $v \notin V_i$;

(ii) a vertex v is a solitary vertex and C contains a spare color class V_i which does not contain any vertex of N(v).

3 Dominating Energy of a Graph

Professor Chandrashekar Adiga et al [1], defined the minimum covering energy, $E_C(G)$ of a graph which depends on its particular minimum cover C. Motivated by this, we introduced minimum dominating energy of a graph $E_D(G)$ and computed minimum dominating energies of a star graph, complete graph, crown graph cocktail graphs. Upper and lower bounds for $E_D(G)$ are established.

The concept of energy of a graph was introduced by I. Gutman [12] in the year 1978. Let G be a graph with n vertices and m edges and let $A = (a_{ij})$ be the adjacency matrix of the graph. The eigenvalues $\lambda_1, \lambda_2, , \lambda_n$ of A, assumed in non increasing order, are the eigenvalues of the graph G. As A is real symmetric, the eigenvaluesof G are real with sum equal to zero. The energy E(G) of G is defined to be the sum of the absolute values of the eigenvalues of G. i.e $E(G) = \sum_{i=1}^{n} |\lambda i|$

Minimum Dominating Energy

let G be a simple graph of order n with vertex set $V=\{v_1,v_2,\ldots,v_n\}$ and edge set E. A subset D of V is called a dominating set of G if every vertex of V−D is adjacent to some vertex in D. Any dominating set with minimum cardinality is called a minimumdominating set. Let D be a minimum dominating set of a graph G. The minimum dominating matrix of G is the nxn matrix defined by $A_D(G) := (a_{ij})$,

Let G be a simple graph with vertex set V={ $v_1, v_2, ..., v_n$} edge set E and D={$u_1, u_2, ..., u_k$} be a minimum dominating set. If $\lambda_1, \lambda_2, , , , , \lambda_n$ are the eigen values of minimum dominating matrix

$A_D(G) = \sum_{i=1}^{n} |\lambda i|$

Example 3.1. The possible minimum dominating sets for the following graph G are:

i) $D_1 = \{v_1, v_5\}$;

ii) $D_2 = \{v_2, v_5\}$

iii) $D_3 = \{v_2, v_6\}$.

Theorem 3.2. The minimum dominating energy of Cocktail party graph $K_{(nX2)}$ is

$$(2n-3) + \sqrt{4n^2 + 4n - 9}$$

Theorem 3.3. For n ≤ 2, the minimum dominating energy of Star graph $K_{1,n-1}$ is equal to $\sqrt{4n-3}$

Theorem 3.4. Let G be a graph with a minimum dominating set D.If the minimum dominating energy $E_D(G)$ is a rational number, then $E_D(G) = |D| (\text{mod} 2)$.

4 The Maximal Domination in Graphs

Definition 4.1. A set D of vertices in a graph G = (V, E) is a dominating set of Gif every vertex in V − D is adjacent to some vertex in D. The domination numberγ(G) of G is the minimum cardinality of a dominating set. Let G be a graph with pvertices and q edges.

A dominating set D of a graph G = (V, E) is a maximal dominating set if V − D

is not a dominating set of G. the maximal domination number $\gamma_m(G)$ of G is the nimum cardinality of a maximal dominating set. V.R. Kulli [14].

A dominating set D of a graph G is minimal if for each vertex v in D, D_v is not adominating set of G.

Theorem 4.2. [1] If G is a graph with no isolated vertex, then for any minimaldominating set D of G, V − D is also a dominating set of G.

Theorem 4.3. [2] A minimal dominating set D of a graph G is a maximal domi nating set if and only if G contains an isolated vertex.

Theorem 4.4. [3] Let G be a graph with no isolated vertex. Then $\gamma_m(G) \leq \alpha(G)+1$.

Theorem 4.5. [4] For any graph G, $\gamma_m(G) = p$ if and only if $G = K_p$ or $G = \overline{K_p}$, where $\overline{K_p}$ is the compliment of K_p

Theorem 4.6. [5] For any graph G, $\frac{1}{2}(2q - p(p-3)) \leq \gamma_m(G)$ **Theorem 4.7.**

[6] For any graph G, $p - q + \delta(G) \leq \gamma_m(G)$ **Theorem 4.8.** [7] For any graph G,

$\gamma_m(G) \leq \gamma(G) + \delta(G)$ **Theorem 4.9.** [8] For any graph G, $\gamma_g(G) \leq \gamma_m(G)$

5 Research Problem

The problem of maximal domination and bounds on the concepts will be studied.

Furthermore, the relationship between other domination parameter will be discussed. Also I am interested to introduce new concept of domination in graphs.

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