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MATRIX THEORY OPERATORS ON PYTHAGOREAN NEUTROSOPHIC HYPERSOFT SET AND APPLICATIONS

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Abstract: Decision-making is a complex issue due to vague, imprecise, and indeterminate environment particularly when attributes are more than one and further bifurcated. Hypersoft set have gained more importance as a generalization of soft sets. Neutrosophic soft set environment cannot be used to tackle such type of issues. To solve such types of problems, the concept of Neutrosophic Hypersoft set is introduced. In this paper, the primary focus includes the analytical study of some common operators for PNHSM has been developed. Finally, Decision-making issues have been presented by establishing a new algorithm based on a score function and also it has been examined with the help of numerical example for the recruitment of teachers for government postings. The validity and implementation of definitions are verified by presenting suitable example. Using (PNHSMs) method to solve real-world problems such as decision makings like the selection of teachers, Personal Selection, Office administration and many other problems can be solved.

***Key Words*- MCDM, Uncertainty, Soft set, Neutrosophic soft set, Hypersoft set, Neutrosophic Hypersoft set, Pythagorean Neutrosophic Hypersoft Set, Pythagorean Neutrosophic Hypersoft matrices.**

1. INTRODUCTION

In decision–making, among the multi attributive and multi-objective problems, in uncertain and vague environments, it is difficult to take decision from the decision makers. Decision makers get more confused and uncertain in the above case. The idea of Fuzzy sets was introduced independently by Lotfi A. Zadeh in 1965[8]. In mathematics fuzzy sets (Uncertain sets) are sets whose elements have degrees of membership. A few times it might be hard to allot the membership values for fuzzy sets. Therefore, the idea of interval valued fuzzy sets was proposed [5] to catch the uncertainty for membership values. In some cases, like real life problems, Data combination, etc., we can consider membership as the non-membership values. Neither the fuzzy sets nor the interval valued fuzzy sets is convenient for such a circumstance. Intuitionistic fuzzy sets have been introduced by Krassimir Atanassov in 1983[7] as an extension of fuzzy set. Intuitionistic fuzzy set can just deal with the inadequate data whose elements have degrees of membership and non-membership. Intuitionistic fuzzy set can only handle incomplete information not the inderminate information. Since Intuitionistic fuzzy set can deal with the inadequate data considering both the membership and non-membership values. Smarandache [3] presented the idea of Neutrosophic set which is a scientific apparatus for taking care of issues including uncertain, indeterminacy and conflicting information. In Neutrosophic set Indeterminacy is quantified. Neutrosophic set indicate truth membership value (T), Indeterminacy membership value (I) and Falsity membership value (F).

The concept of soft sets introduced by Molodtsov [1] in 1999 to deal with uncertainty in a parametric manner. Soft set theory is a generalization of fuzzy set. A soft set is a parameterized family of sets intuitively, this is "soft" because the boundary of the set depends on the parameters. Soft set are useful in various regions including artificial insight and basic decision- making problems [11] and it serves to define various functions for various parameters. New theory of soft sets was to define mappings on soft sets, which was achieved in 2009 by the mathematicians Athar Khar and Bashir Ahmad. Soft sets have also been applied to the problem of medical diagnosis for use in medical expert systems.

Maji et al. [10] gives a Hypothetical study of soft sets which covers subset, superset of a soft set, equality of soft sets and operations on soft sets. For example, union, Intersection, AND and OR-Operations between different sets. Fuzzy soft set and Intuitionistic fuzzy soft set can just deal with partial data. Neutrosophic soft set can deal with inadequate, uncertain, inconsistence data.

Smaradache [2] extended the concept of soft sets to Hypersoft sets (HSS) by replacing function F of one parameter with a multi – parameter function defined on the cartesian product of n different attributes. Hypersoft set is more flexible than soft sets and more suitable for decision making environments. Also presented the further extension of HSS, such as crisp HSS, Fuzzy HSS, Intuitionistic fuzzy HSS, Neutrosophic HSSS. Now a days the Hypersoft set theory and its extensions rapidly progress, Many Researchers developed different operators and properties based on Hypersoft set and its extensions [13,10,11].

The idea of a Neutrosophic hypersoft set [15] (NHSS) was introduced by Smarandache in 2018 as a generalization of soft set. Dealing with multi-attributes, multi-objective problems with disjoint attributive values. While solving uncertainty problems we convert hypersoft set to Neutrosophic Hypersoft set. The matrix representation and aggregate operators of this idea were presented by Delhi and Broumi in [5]. Introduced the TOPSIS by using accuracy function and an application of MCDM is proposed. Many other novel approaches are also used by many researches in decision makings.

2. Preliminaries

Definition 2.1:[1] Soft Set

Let $\ddot{U}$ be the universal set and € be a set of parameters or attributes with respect to $\ddot{U}$. Let P ($\ddot{U}$) be the power set of $\ddot{U}$ and A⊆ €. A pair (F, A) is called a soft set over $\ddot{U}$, where F is a mapping given by F: A → $P(\ddot{U})$ .In other words, a soft set (F, A) over $\ddot{U}$ is a parameterized family of subsets of $\ddot{U}$.It is also defined as (F, A) = {F(e)∈ $P(\ddot{U}): e \in E, F(e) = \emptyset\ if\ e \notin A\}$.

Definition 2.2:[12] Neutrosophic Soft Set

Let $\ddot{U}$ be the universal set and € be a set of parameters or attributes with respect to $\ddot{U}$. Let P ($\ddot{U}$) be the set of Neutrosophic values of $\ddot{U}$ and A⊆ €. A pair (F, A) is called a Neutrosophic soft set over $\ddot{U}$, where F is a mapping given by F: A → $P(\ddot{U})$.

Definition 2.3:[2] Hypersoft Set

Let $\ddot{U}$ be the universal set and P ($\ddot{U}$) be the power set of $\ddot{U}$. Consider $l^1, l^2, l^3, \ldots l^n$ for n ≥ 1, be n well-defined attributes, whose corresponding attributive values are respectively the set $\mathcal{L}^1, L^2, L^3, \ldots \mathcal{L}^n$ with $\mathcal{L}^i \cap \mathcal{L}^j = \emptyset$,for i≠j and i,j€{1,2,3,…n}then the pair (F, $\mathcal{L}^1 \times L^2 \times L^3, \ldots L^n$) is said to be Hyper Soft Set over $\ddot{U}$ where F : $\mathcal{L}^1 \times L^2 \times L^3, \ldots L^n$ → P($\ddot{U}$).

Definition 2.4:[17] Neutrosophic Hypersoft Set

Let $\ddot{U}$ be the universal set and P ($\ddot{U}$) be the power set of $\ddot{U}$. Consider $l^1, l^2, l^3, \ldots l^n$ for n ≥ 1,be n well-defined attributes, whose corresponding attributive values are respectively the set $\mathcal{L}^1, L^2, L^3, \ldots \mathcal{L}^n$ with $\mathcal{L}^i \cap \mathcal{L}^j = \emptyset$,for i≠j and i,j €{1,2,3,…n}and their relation $\mathcal{L}^1 \times L^2 \times L^3, \ldots L^n$ = ß, then the pair (F, ß) is said to be Neutrosophic Hypersoft set (NHSS) over $\ddot{U}$ Where F : $\mathcal{L}^1 \times L^2 \times L^3, \ldots L^n$ →P($\ddot{U}$). F ($\mathcal{L}^1 \times L^2 \times L^3, \ldots, \mathcal{L}^n$) = {<x, T(F(ß)),I(F(ß)),F(F(ß))>,x∈ $\ddot{U}$} where T is the membership value of truthiness ,I is the membership value of indeterminacy and F is the membership value of falsity such that T,I,F: $\ddot{U}$ → [0,1] Also 0 ≤ T(F(ß)),I(F(ß)),F(F(ß)) ≤ 3.

Definition 2.5:[4] Pythagorean Neutrosophic Hypersoft Set

Let $\ddot{U}$ be the universal set and P($\ddot{U}$) be the power set of $\ddot{U}$. Consider $l^1, l^2, l^3, \ldots l^n$ for n ≥ 1,be n well-defined attributes ,whose corresponding attributive values are respectively the set $\mathcal{L}^1, L^2, L^3, \ldots \mathcal{L}^n$ with $\mathcal{L}^i \cap \mathcal{L}^j = \emptyset$,for i≠j and i,j€{1,2,3,…n}and their relation $\mathcal{L}^1 \times L^2 \times L^3, \ldots L^n$ = ß, then the pair (F, ß) is said to be Pythagorean Neutrosophic Hypersoft set (PNHSS)over $\ddot{U}$ Where F : $\mathcal{L}^1 \times L^2 \times L^3, \ldots L^n$ → P($\ddot{U}$) and F ($\mathcal{L}^1 \times L^2 \times L^3, \ldots L^n$) = {<x, T(F(ß)),I(F(ß)),F(F(ß))>,x∈ $\ddot{U}$}where T is the membership value of truthiness ,I is the membership value of indeterminacy and F is the membership value of falsity such that T,I,F: $\ddot{U}$ → [0,1] Also 0 ≤ $(T(F(ß))^2+(I(F(ß))^2+(F(F(ß))^2 \leq 2$.

3. Pythagorean Neutrosophic Hypersoft Matrix (PNHSM)

In this section, we have introduced some operators with suitable examples

3.1: Pythagorean Neutrosophic Hypersoft Matrix

Let $\ddot{U} = \{u^1, u^2, \ldots u^\alpha\}$be the universal set and P ($\ddot{U}$) be the power set of $\ddot{U}$. Consider $\mathcal{L}_1, \mathcal{L}_2, \ldots \mathcal{L}_\beta$, for β ≥ 1, where β be well-defined attributes ,whose corresponding attributive values are respectively, the set $\mathcal{L}_1^a, \mathcal{L}_2^b, \ldots \mathcal{L}_\beta^z$ and their relation $\mathcal{L}_1^a \times \mathcal{L}_2^b \times \ldots \times \mathcal{L}_\beta^z$, where a,b,c,….z = 1,2,….,n ; then , the pair (F, $\mathcal{L}_1^a \times \mathcal{L}_2^b \times \ldots \times \mathcal{L}_\beta^z$) is said to be Pythagorean Neutrosophic Hypersoft set (PNHSS)over $\ddot{U}$ where F : $(\mathcal{L}_1^a \times \mathcal{L}_2^b \times \ldots \times \mathcal{L}_\beta^z)$ → P($\ddot{U}$) and is defined as F $(\mathcal{L}_1^a \times \mathcal{L}_2^b \times \ldots \times \mathcal{L}_\beta^z)$ = {<u,T_Ω(u), I_Ω(u) ,F_Ω(u)>u∈ $\ddot{U}$,Ώ ∊ $(\mathcal{L}_1^a \times \mathcal{L}_2^b \times \ldots \times \mathcal{L}_\beta^z)$}where T is the membership value of truthiness ,I is the membership value of indeterminacy and F is the membership value of falsity such that T,I,F: $\ddot{U}$ → [0,1] .Table 1 represents the tabular form of PNHSS R_Ω .If $O_{ij} = \mathfrak{X}_{\mathrm{R}_\Omega}(u^i, \mathcal{L}_j^k)$,where i = 1,2,3,…….; α ,j = 1,2,3,….,β and k = a, b, c,….,z then a matrix is defined as

$$[O_{ij}]_{\alpha\times\beta} = \begin{pmatrix} O_{11} & 0_{12} \ldots.. & O_{1\beta} \\ 0_{21} & 0_{22} \ldots.. & O_{2\beta} \\ & \ldots\ldots\ldots\ldots & \\ & \ldots\ldots\ldots\ldots & \\ 0_{\alpha 1} & O_{\alpha 2} \ldots.. & 0_{\alpha\beta} \end{pmatrix}, \text{Where } O_{ij} = (T_{\mathcal{L}_j^k}(u_i), I_{\mathcal{L}_j^k}(u_i), F_{\mathcal{L}_j^k}(u_i)),$$

$u_i \in \ddot{U}., \mathcal{L}_j^k \in \left(\mathcal{L}_1^a \times \mathcal{L}_2^b \times \ldots \times \mathcal{L}_\beta^z\right) = (\mathcal{T}_{ijk}^o, \mathcal{I}_{ijk}^o, \mathcal{F}_{ijk}^o)$.

Thus, we represent any Pythagorean Neutrosophic Hypersoft set (PNHSS) in terms of a Pythagorean Neutrosophic Hypersoft Matrix (PNHSM).

Example: Formulation of the problem. Here we consider a problem on requirement of teachers for Government postings based on their CVs received. Let $\ddot{U}$ be the set of candidates applied for teacher's requirement for government posting. Let $\ddot{U} = \{\mathbb{T}^1, \mathbb{T}^2, \mathbb{T}^3, \mathbb{T}^4, \mathbb{T}^5\}$. Also consider the set of attributes as $\mathbb{A}_1$= Qualification, $\mathbb{A}_2$= Reservation, $\mathbb{A}_3$ = Gender, $\mathbb{A}_4$ = Grade and their respective Parameters are given as

(i) $\mathbb{T}_i$ = Universal set of teachers, where i = 1,2,3,4,5
(ii) $\mathbb{A}_i$ = Attributes, where i= 1,2,3,4 and values are categorized into the following:
(iii) $\mathbb{A}_1^a$ = Qualification = $\{M.phil.(B.Ed), B.Sc.(B.Ed), M.Sc.(B.Ed), Ph.D.\}$
(iv) $\mathbb{A}_2^b$ = Reservation = {MBC, BC, SC, Others}
(v) $\mathbb{A}_3^c$ = Gender = {Male, Female}
(vi) $\mathbb{A}_4^d$ = Grade = {Very High, High, Average}

Let the function be F: $\mathbb{A}_1^a \times \mathbb{A}_2^b \times \mathbb{A}_3^c \times \mathbb{A}_4^d \rightarrow P(\ddot{U})$. Below are the tables 2 – 5 of their Pythagorean Neutrosophic values assigned by distinct decision makers. Let as assume

F (($\mathbb{A}_1^a \times \mathbb{A}_2^b \times \mathbb{A}_3^c \times \mathbb{A}_4^d$)) = F (M.Phil. (B.Ed.), BC, Female, High) = $\{\mathbb{T}^1, \mathbb{T}^2, \mathbb{T}^4, \mathbb{T}^5\}$,

F (($\mathbb{A}_1^a \times \mathbb{A}_2^b \times \mathbb{A}_3^c \times \mathbb{A}_4^d$)) = F (M.Phil. (B.Ed.), BC, Female, High)

= {<< $\mathbb{T}^1$, (M.Phil. (B.Ed.) {0.6,0.7,0.5}, BC {0.6,0.5,0.2}, F {0.7,0.5,0.2}, High {0.6,0.4,0.1})>>,

<< $\mathbb{T}^2$, (M.Phil. (B.Ed.) {0.5,0.4,0.6}, BC {0.6,0.5,0.3}, F {0.5,0.4,0.1}, High {0.5,0.6,0.2})>>,

<< $\mathbb{T}^4$, (M.Phil. (B.Ed.) {0.6,0.4,0.3}, BC {0.6,0.5,0.2}, F {0.5,0.6,0.2}, High {0.5,0.3,0.1})>>,

<< $\mathbb{T}^5$, (M.Phil. (B.Ed.) {0.5,0.7,0.2}, BC {0.5,0.4,0.2}, F {0.7,0.6,0.1}, High {0.7,0.6,0.2})>>,}.

Then a Pythagorean Neutrosophic Hypersoft set of above assumed relation in the tabular form is represented in table 6 and its matrix is defined as

Table 1: Matrix representation of PNHSS.

	$\mathcal{L}_1^a$	$\mathcal{L}_2^b$	……..	$\mathcal{L}_\beta^z$
u^1	$\mathfrak{X}_{R\Omega}(u^1, \mathcal{L}_1^a)$	$\mathfrak{X}_{R\Omega}(u^1, \mathcal{L}_2^b)$	……..	$\mathfrak{X}_{R\Omega}(u^1, \mathcal{L}_\beta^z)$
u^2	$\mathfrak{X}_{R\Omega}(u^2, \mathcal{L}_1^a)$	$\mathfrak{X}_{R\Omega}(u^2, \mathcal{L}_2^b)$	……..	$\mathfrak{X}_{R\Omega}(u^2, \mathcal{L}_\beta^z)$
……..	……..	……..	……..	……..
u^α	$\mathfrak{X}_{R\Omega}(u^\alpha, \mathcal{L}_1^a)$	$\mathfrak{X}_{R\Omega}(u^\alpha, \mathcal{L}_2^b)$	……..	$\mathfrak{X}_{R\Omega}(u^\alpha, \mathcal{L}_\beta^z)$

Table 2: Decision makers will assign Pythagorean Neutrosophic values to each candidate $\mathbb{T}_i$ against qualification.

$\mathbb{A}_1^a$(Qualification)	$\mathbb{T}^1$	$\mathbb{T}^2$	$\mathbb{T}^3$	$\mathbb{T}^4$	$\mathbb{T}^5$
$M.phil.(B.Ed)$	(0.6,0.7,0.5)	(0.5,0.4,0.6)	(0.7,0.5,0.1)	(0.6,0.4,0.3)	(0.5,0.7,0.2)
$B.Sc.(B.Ed)$	(0.6,0.5,0.3)	(0.6,0.5,0.4)	(0.5,0.3,0.1)	(0.6,0.5,0.2)	(0.5,0.6,0.3)
$M.Sc.(B.Ed)$	(0.5,0.4,0.3)	(0.5,0.3,0.2)	(0.6,0.4,0.3)	(0.5,0.6,0.2)	(0.6,0.5,0.1)
$Ph.D.$	(0.6,0.5,0.2)	(0.6,0.7,0.1)	(0.5,0.6,0.3)	(0.6,0.4,0.2)	(0.5,0.3,0.1)

Table 3: Decision makers will assign Pythagorean Neutrosophic values to each candidate $\mathbb{T}_i$ against Reservation

$\mathbb{A}_2^b$(Reservation)	$\mathbb{T}^1$	$\mathbb{T}^2$	$\mathbb{T}^3$	$\mathbb{T}^4$	$\mathbb{T}^5$
BC	(0.6,0.5,0.2)	(0.6,0.5,0.3)	(0.5,0.4,0.1)	(0.6,0.5,0.2)	(0.5,0.4,0.2)
MBC	(0.5,0.6,0.2)	(0.5,0.4,0.1)	(0.8,0.7,0.1)	(0.6,0.5,0.1)	(0.5,0.6,0.1)
SC	(0.8,0.7,0.2)	(0.7,0.6,0.1)	(0.6,0.5,0.1)	(0.5,0.4,0.2)	(0.6,0.4,0.1)
Others	(0.6,0.7,0.1)	(0.5,0.2,0.1)	(0.6,0.4,0.2)	(0.5,0.4,0.1)	(0.5,0.4,0.1)

Table 4: Decision makers will assign Pythagorean Neutrosophic values to each candidate $\mathbb{T}_i$ against Gender

$\mathbb{A}_3^c$ (Gender)	$\mathbb{T}^1$	$\mathbb{T}^2$	$\mathbb{T}^3$	$\mathbb{T}^4$	$\mathbb{T}^5$
Male	(0.5,0.4,0.7)	(0.6,0.5,0.1)	(0.7,0.5,0.2)	(0.6,0.4,0.2)	(0.5,0.4,0.1)
Female	(0.7,0.5,0.2)	(0.5,0.4,0.1)	(0.6,0.5,0.1)	(0.5,0.6,0.2)	(0.7,0.6,0.1)

Table 5: Decision makers will assign Pythagorean Neutrosophic values to each candidate $\mathbb{T}_i$ against Grade

$\mathbb{A}_4^d$ (Grade)	$\mathbb{T}^1$	$\mathbb{T}^2$	$\mathbb{T}^3$	$\mathbb{T}^4$	$\mathbb{T}^5$
Very high	(0.5,0.4,0.1)	(0.5,0.6,0.3)	(0.7,0.5,0.1)	(0.6,0.4,0.1)	(0.5,0.4,0.2)
High	(0.6,0.4,0.1)	(0.5,0.6,0.2)	(0.7,0.6,0.1)	(0.5,0.3,0.1)	(0.7,0.6,0.2)
Average	(0.7,0.6,0.1)	(0.6,0.4,0.3)	(0.5,0.4,0.1)	(0.6,0.7,0.1)	(0.6,0.4,0.2)

Table 6: The tabular form of the above relation.

	$\mathbb{A}_1^a$	$\mathbb{A}_2^b$	$\mathbb{A}_3^c$	$\mathbb{A}_4^d$
$\mathbb{T}^1$	(M.Phil., (0.6,0.7,0.5))	(BC, (0.6,0.5,0.2))	(Female, (0.7,0.5,0.2))	(High, (0.6,0.4,0.1))
$\mathbb{T}^2$	(M.Phil., (0.5,0.4,0.6))	(BC, (0.6,0.5,0.3))	(Female, (0.5,0.4,0.1))	(High, (0.5,0.6,0.2))
$\mathbb{T}^4$	(M.Phil., (0.6,0.4,0.3))	(BC, (0.6,0.5,0.2))	(Female, (0.5,0.6,0.2))	(High, (0.5,0.3,0.1))
$\mathbb{T}^5$	(M.Phil., (0.5,0.7,0.2))	(BC, (0.5,0.4,0.2))	(Female, (0.7,0.6,0.1))	(High, (0.7,0.6,0.2))

$$[O]_{4\times 4} = \begin{bmatrix} (\text{M. Phil.}, (0.6,0.7,0.5)) & (\text{BC}, (0.6,0.5,0.2)) & (\text{Female}, (0.7,0.5,0.2)) & (\text{High}, (0.6,0.4,0.1)) \\ (\text{M. Phil.}, (0.5,0.4,0.6)) & (\text{BC}, (0.6,0.5,0.3)) & (\text{Female}, (0.5,0.4,0.1)) & (\text{High}, (0.5,0.6,0.2)) \\ (\text{M. Phil.}, (0.6,0.4,0.3)) & (\text{BC}, (0.6,0.5,0.2)) & (\text{Female}, (0.5,0.6,0.2)) & (\text{High}, (0.5,0.3,0.1)) \\ (\text{M. Phil.}, (0.5,0.7,0.2)) & (\text{BC}, (0.5,0.4,0.2)) & (\text{Female}, (0.7,0.6,0.1)) & (\text{High}, (0.7,0.6,0.2)) \end{bmatrix}$$

and Let $[\mathfrak{M}]_{4\times 4}$ =

$$\begin{bmatrix} (\text{M. Phil.}, (0.5,0.4,0.3)) & (\text{BC}, (0.5,0.6,0.2)) & (\text{Female}, (0.5,0.4,0.7)) & (\text{High}, (0.7,0.6,0.1)) \\ (\text{M. Phil.}, (0.5,0.3,0.2)) & (\text{BC}, (0.5,0.4,0.1)) & (\text{Female}, (0.6,0.5,0.1)) & (\text{High}, (0.6,0.4,0.3)) \\ (\text{M. Phil.}, (0.5,0.6,0.2)) & (\text{BC}, (0.6,0.5,0.1)) & (\text{Female}, (0.6,0.4,0.2)) & (\text{High}, (0.6,0.7,0.1)) \\ (\text{M. Phil.}, (0.6,0.5,0.1)) & (\text{BC}, (0.5,0.6,0.1)) & (\text{Female}, (0.5,0.4,0.1)) & (\text{High}, (0.6,0.4,0.2)) \end{bmatrix}$$

3.2. OPERATORS OF PNHSMs

Let O = $[O_{ij}]$ and $\mathfrak{M}$ =$[\mathfrak{M}_{ij}]$ be two PNHSM, where $O_{ij} = (\mathbb{T}^o_{ijk}, \mathbb{I}^o_{ijk}, \mathbb{F}^o_{ijk})$ and $\mathfrak{M}_{ij} = (\mathbb{T}^{\mathfrak{M}}_{ijk}, \mathbb{I}^{\mathfrak{M}}_{ijk}, \mathbb{F}^{\mathfrak{M}}_{ijk})$. Then,

(i) Union of two PNHSM:

$O \cup \mathfrak{M} = \delta$, where $\mathbb{T}^s_{ijk} = \max(\mathbb{T}^o_{ijk}, \mathbb{T}^{\mathfrak{M}}_{ijk})$, $\mathbb{I}^s_{ijk} = \max(\mathbb{I}^o_{ijk}, \mathbb{I}^{\mathfrak{M}}_{ijk})$, and $\mathbb{F}^s_{ijk} = \min(\mathbb{F}^o_{ijk}, \mathbb{F}^{\mathfrak{M}}_{ijk})$.

Example: $O\cup\mathfrak{M}$ =

$$\begin{bmatrix} (\text{M. Phil.}, (0.6,0.7,0.3)) & (\text{BC}, (0.6,0.6,0.2)) & (\text{Female}, (0.7,0.5,0.2)) & (\text{High}, (0.7,0.6,0.1)) \\ (\text{M. Phil.}, (0.5,0.4,0.2)) & (\text{BC}, (0.6,0.5,0.1)) & (\text{Female}, (0.6,0.5,0.1)) & (\text{High}, (0.6,0.6,0.2)) \\ (\text{M. Phil.}, (0.6,0.6,0.2)) & (\text{BC}, (0.6,0.5,0.1)) & (\text{Female}, (0.6,0.6,0.2)) & (\text{High}, (0.6,0.7,0.1)) \\ (\text{M. Phil.}, (0.6,0.7,0.1)) & (\text{BC}, (0.5,0.6,0.1)) & (\text{Female}, (0.7,0.6,0.1)) & (\text{High}, (0.7,0.6,0.2)) \end{bmatrix}$$

(ii) Intersection of two PNHSM:

$O\cap\mathfrak{M} = \delta$, where $\mathbb{T}^s_{ijk} = \min(\mathbb{T}^o_{ijk}, \mathbb{T}^{\mathfrak{M}}_{ijk})$, $\mathbb{I}^s_{ijk} = \min(\mathbb{I}^o_{ijk}, \mathbb{I}^{\mathfrak{M}}_{ijk})$, and $\mathbb{F}^s_{ijk} = \max(\mathbb{F}^o_{ijk}, \mathbb{F}^{\mathfrak{M}}_{ijk})$.

Example: $O\cap\mathfrak{M}$ =

$$\begin{bmatrix} (\text{M. Phil.}, (0.5,0.4,0.5)) & (\text{BC}, (0.5,0.5,0.2)) & (\text{Female}, (0.5,0.4,0.7)) & (\text{High}, (0.6,0.4,0.1)) \\ (\text{M. Phil.}, (0.5,0.3,0.6)) & (\text{BC}, (0.5,0.4,0.3)) & (\text{Female}, (0.5,0.4,0.1)) & (\text{High}, (0.5,0.4,0.3)) \\ (\text{M. Phil.}, (0.5,0.4,0.3)) & (\text{BC}, (0.6,0.5,0.2)) & (\text{Female}, (0.5,0.4,0.2)) & (\text{High}, (0.5,0.3,0.1)) \\ (\text{M. Phil.}, (0.5,0.5,0.2)) & (\text{BC}, (0.5,0.4,0.2)) & (\text{Female}, (0.5,0.4,0.1)) & (\text{High}, (0.6,0.4,0.2)) \end{bmatrix}$$

(iii) Arithmetic Mean:

$O\oplus\mathfrak{M} = \delta$, where $\mathbb{T}^s_{ijk} = \frac{(\mathbb{T}^o_{ijk}+\mathbb{T}^{\mathfrak{M}}_{ijk})}{2}$, $\mathbb{I}^s_{ijk} = \frac{(\mathbb{I}^o_{ijk}+\mathbb{I}^{\mathfrak{M}}_{ijk})}{2}$, $\mathbb{F}^s_{ijk} = \frac{(\mathbb{F}^o_{ijk}+\mathbb{F}^{\mathfrak{M}}_{ijk})}{2}$

Example: $O\oplus\mathfrak{M}$ =

$$\begin{bmatrix} (\text{M. Phil.}, (0.55,0.55,0.4)) & (\text{BC}, (0.55,0.55,0.2)) & (\text{Female}, (0.6,0.45,0.45)) & (\text{High}, (0.65,0.5,0.1)) \\ (\text{M. Phil.}, (0.5,0.35,0.4)) & (\text{BC}, (0.55,0.45,0.2)) & (\text{Female}, (0.55,0.45,0.1)) & (\text{High}, (0.55,0.5,0.25)) \\ (\text{M. Phil.}, (0.55,0.5,0.25)) & (\text{BC}, (0.6,0.5,0.15)) & (\text{Female}, (0.55,0.5,0.2)) & (\text{High}, (0.55,0.5,0.1)) \\ (\text{M. Phil.}, (0.55,0.6,0.15)) & (\text{BC}, (0.5,0.5,0.15)) & (\text{Female},, (0.6,0.5,0.1)) & (\text{High}, (0.65,0.5,0.2)) \end{bmatrix}$$

(iv) Weighted Arithmetic Mean:

$O \odot^w \mathfrak{M} = \delta$, where $\mathbb{T}^s_{ijk} = \frac{(w^1\mathbb{T}^o_{ijk}+w^2\mathbb{T}^{\mathfrak{M}}_{ijk})}{w^1+w^2}$, $\mathbb{I}^s_{ijk} = \frac{(w^1\mathbb{I}^o_{ijk}+w^2\mathbb{I}^{\mathfrak{M}}_{ijk})}{w^1+w^2}$, $\mathbb{F}^s_{ijk} = \frac{(w^1\mathbb{F}^o_{ijk}+w^2\mathbb{F}^{\mathfrak{M}}_{ijk})}{w^1+w^2}$. $w^1, w^2 > 0$.

(v) Geometric Mean:

$O\odot\mathfrak{M} = \delta$, where $\mathbb{T}^s_{ijk} = \sqrt{\mathbb{T}^o_{ijk}.\mathbb{T}^{\mathfrak{M}}_{ijk}}$, $\mathbb{I}^s_{ijk} = \sqrt{\mathbb{I}^o_{ijk}.\mathbb{I}^{\mathfrak{M}}_{ijk}}$, $\mathbb{F}^s_{ijk} = \sqrt{\mathbb{F}^o_{ijk}.\mathbb{F}^{\mathfrak{M}}_{ijk}}$

Example:
$O\odot\mathfrak{M}$ =

$$\begin{bmatrix} (\text{M. Phil.}, (0.54,0.52,0.38)) & (\text{BC}, (0.54,0.54,0.2)) & (\text{Female}, (0.59,0.44,0.37)) & (\text{High}, (0.64,0.48,0.1)) \\ (\text{M. Phil.}, (0.5,0.34,0.34)) & (\text{BC}, (0.54,0.44,0.17)) & (\text{Female}, (0.54,0.44,0.1)) & (\text{High}, (0.54,0.48,0.24)) \\ (\text{M. Phil.}, (0.54,0.48,0.24)) & (\text{BC}, (0.6,0.5,0.14)) & (\text{Female}, (0.54,0.48,0.2)) & (\text{High}, (0.54,0.45,0.1)) \\ (\text{M. Phil.}, (0.54,0.59,0.14)) & (\text{BC}, (0.5,0.48,0.14)) & (\text{Female}, (0.59,0.48,0.1)) & (\text{High}, (0.64,0.48,0.2)) \end{bmatrix}$$

(vi) Weighted Geometric Mean:

$$O \odot^w \mathfrak{M} = \delta \text{ , where } \mathbb{T}^s_{ijk} = \sqrt[w^1+w^2]{\left(\mathbb{T}^o_{ijk}\right)^{w^1}.\left(\mathbb{T}^{\mathfrak{M}}_{ijk}\right)^{w^2}},$$

$$\mathbb{I}^s_{ijk} = \sqrt[w^1+w^2]{\left(\mathbb{I}^o_{ijk}\right)^{w^1}.\left(\mathbb{I}^{\mathfrak{M}}_{ijk}\right)^{w^2}}, \quad \mathbb{F}^s_{ijk} = \sqrt[w^1+w^2]{\left(\mathbb{F}^o_{ijk}\right)^{w^1}.\left(\mathbb{F}^{\mathfrak{M}}_{ijk}\right)^{w^2}}, w^1, w^2 > 0.$$

(vii) Harmonic Mean:

$O\oslash\mathfrak{M} = \delta$, where $\mathbb{T}^s_{ijk} = \frac{2\mathbb{T}^o_{ijk}\mathbb{T}^{\mathfrak{M}}_{ijk}}{\mathbb{T}^o_{ijk}+\mathbb{T}^{\mathfrak{M}}_{ijk}}$, $\mathbb{I}^s_{ijk} = \frac{2\mathbb{I}^o_{ijk}\mathbb{I}^{\mathfrak{M}}_{ijk}}{\mathbb{I}^o_{ijk}+\mathbb{I}^{\mathfrak{M}}_{ijk}}$, $\mathbb{F}^s_{ijk} = \frac{2\mathbb{F}^o_{ijk}\mathbb{F}^{\mathfrak{M}}_{ijk}}{\mathbb{F}^o_{ijk}+\mathbb{F}^{\mathfrak{M}}_{ijk}}$

Example: $O\oslash\mathfrak{M}$ =

$$\begin{bmatrix} (\text{M. Phil.}, (0.54,0.50,0.375)) & (\text{BC}, (0.54,0.54,0.2)) & (\text{Female}, (0.58,0.44,0.31)) & (\text{High}, (0.64,0.48,0.1)) \\ (\text{M. Phil.}, (0.5,0.34,0.3)) & (\text{BC}, (0.54,0.44,0.15)) & (\text{Female}, (0.54,0.44,0.1)) & (\text{High}, (0.54,0.48,0.24)) \\ (\text{M. Phil.}, (0.54,0.48,0.24)) & (\text{BC}, (0.6,0.5,0.13)) & (\text{Female}, (0.54,0.48,0.2)) & (\text{High}, (0.54,0.42,0.1)) \\ (\text{M. Phil.}, (0.54,0.58,0.13)) & (\text{BC}, (0.5,0.48,0.13)) & (\text{Female}, (0.58,0.48,0.1)) & (\text{High}, (0.64,0.48,0.2)) \end{bmatrix}$$

(viii) Weighted Harmonic Mean:

O $\odot^{w} \mathfrak{M} = \delta$, where

$$\mathbb{T}_{ijk}^{s} = \frac{w^1+w^2}{\left(\frac{w^1}{\mathbb{T}_{ijk}^{o}}\right)+\left(\frac{w^2}{\mathbb{T}_{ijk}^{\mathfrak{M}}}\right)};\ \mathbb{I}_{ijk}^{s} = \frac{w^1+w^2}{\left(\frac{w^1}{\mathbb{I}_{ijk}^{o}}\right)+\left(\frac{w^2}{\mathbb{I}_{ijk}^{\mathfrak{M}}}\right)};\ \mathbb{F}_{ijk}^{s} = \frac{w^1+w^2}{\left(\frac{w^1}{\mathbb{F}_{ijk}^{o}}\right)+\left(\frac{w^2}{\mathbb{F}_{ijk}^{\mathfrak{M}}}\right)},\ w^1, w^2 > 0.$$

Proposition 1. Let O = $[O_{ij}]$ and $\mathfrak{M}$ =$[\mathfrak{M}_{ij}]$ be two PNHSM, where $O_{ij} = \left(\mathbb{T}_{ijk}^{o}, \mathbb{I}_{ijk}^{o}, \mathbb{F}_{ijk}^{o}\right)$ and $\mathfrak{M}_{ij} = \left(\mathbb{T}_{ijk}^{\mathfrak{M}}, \mathbb{I}_{ijk}^{\mathfrak{M}}, \mathbb{F}_{ijk}^{\mathfrak{M}}\right)$. Then,

(i) $(O \cup \mathfrak{M})^t = O^t \cup \mathfrak{M}^t$
(ii) $(O \cap \mathfrak{M})^t = O^t \cap \mathfrak{M}^t$
(iii) $(O \oplus \mathfrak{M})^t = O^t \oplus \mathfrak{M}^t$
(iv) $(O \oplus^w \mathfrak{M})^t = O^t \oplus \mathfrak{M}^t$
(v) $(O \odot \mathfrak{M})^t = O^t \odot \mathfrak{M}^t$
(vi) $(O \odot^w \mathfrak{M})^t = O^t \odot^w \mathfrak{M}^t$
(vii) $(O \oslash \mathfrak{M})^t = O^t \oslash \mathfrak{M}^t$
(viii) $(O \oslash^w \mathfrak{M})^t = O^t \oslash^w \mathfrak{M}^t$

Proof:

(i)
$$\begin{aligned} (O \cup \mathfrak{M})^t &= \left[\left(\max\left(\mathbb{T}_{ijk}^{o}, \mathbb{T}_{ijk}^{\mathfrak{M}}\right), \max\left(\mathbb{I}_{ijk}^{o}, \mathbb{I}_{ijk}^{\mathfrak{M}}\right), \min\left(\mathbb{F}_{ijk}^{o}, \mathbb{F}_{ijk}^{\mathfrak{M}}\right)\right)\right]^t \\ &= \left[\left(\max\left(\mathbb{T}_{jki}^{o}, \mathbb{T}_{jki}^{\mathfrak{M}}\right), \max\left(\mathbb{I}_{jki}^{o}, \mathbb{I}_{jki}^{\mathfrak{M}}\right), \min\left(\mathbb{F}_{jki}^{o}, \mathbb{F}_{jki}^{\mathfrak{M}}\right)\right)\right] \\ &= \left[\left(\mathbb{T}_{jki}^{o}, \mathbb{I}_{jki}^{o}, \mathbb{F}_{jki}^{o}\right)\right] \cup \left[\left(\mathbb{T}_{jki}^{\mathfrak{M}}, \mathbb{I}_{jki}^{\mathfrak{M}}, \mathbb{F}_{jki}^{\mathfrak{M}}\right)\right] \\ &= \left[\left(\mathbb{T}_{ijk}^{o}, \mathbb{I}_{ijk}^{o}, \mathbb{F}_{ijk}^{o}\right)\right]^t \cup \left[\left(\mathbb{T}_{ijk}^{\mathfrak{M}}, \mathbb{I}_{ijk}^{\mathfrak{M}}, \mathbb{F}_{ijk}^{\mathfrak{M}}\right)\right]^t \\ &= O^t \cup \mathfrak{M}^t . \end{aligned}$$

(ii)
$$\begin{aligned} (O \cap \mathfrak{M})^t &= \left[\left(\min\left(\mathbb{T}_{ijk}^{o}, \mathbb{T}_{ijk}^{\mathfrak{M}}\right), \min\left(\mathbb{I}_{ijk}^{o}, \mathbb{I}_{ijk}^{\mathfrak{M}}\right), \max\left(\mathbb{F}_{ijk}^{o}, \mathbb{F}_{ijk}^{\mathfrak{M}}\right)\right)\right]^t \\ &= \left[\left(\min\left(\mathbb{T}_{jki}^{o}, \mathbb{T}_{jki}^{\mathfrak{M}}\right), \min\left(\mathbb{I}_{jki}^{o}, \mathbb{I}_{jki}^{\mathfrak{M}}\right), \max\left(\mathbb{F}_{jki}^{o}, \mathbb{F}_{jki}^{\mathfrak{M}}\right)\right)\right] \\ &= \left[\left(\mathbb{T}_{jki}^{o}, \mathbb{I}_{jki}^{o}, \mathbb{F}_{jki}^{o}\right)\right] \cap \left[\left(\mathbb{T}_{jki}^{\mathfrak{M}}, \mathbb{I}_{jki}^{\mathfrak{M}}, \mathbb{F}_{jki}^{\mathfrak{M}}\right)\right] \\ &= \left[\left(\mathbb{T}_{ijk}^{o}, \mathbb{I}_{ijk}^{o}, \mathbb{F}_{ijk}^{o}\right)\right]^t \cap \left[\left(\mathbb{T}_{ijk}^{\mathfrak{M}}, \mathbb{I}_{ijk}^{\mathfrak{M}}, \mathbb{F}_{ijk}^{\mathfrak{M}}\right)\right]^t \\ &= O^t \cap \mathfrak{M}^t . \end{aligned}$$

Remaining parts are proved in a similar way.

Proposition 2. Let O = $[O_{ij}]$ and $\mathfrak{M}$ =$[\mathfrak{M}_{ij}]$ be two Upper triangular PNHSM, where $O_{ij} = \left(\mathbb{T}_{ijk}^{o}, \mathbb{I}_{ijk}^{o}, \mathbb{F}_{ijk}^{o}\right)$ and $\mathfrak{M}_{ij} = \left(\mathbb{T}_{ijk}^{\mathfrak{M}}, \mathbb{I}_{ijk}^{\mathfrak{M}}, \mathbb{F}_{ijk}^{\mathfrak{M}}\right)$. Then, $(O \cup \mathfrak{M})$, $(O \cap \mathfrak{M})$, $(O \oplus \mathfrak{M})$, $(O \oplus^w \mathfrak{M})$, $(O \odot \mathfrak{M})$, $(O \odot^w \mathfrak{M})$ are all upper triangular PNHSM and vice versa.

Theorem 1. Let O = $[O_{ij}]$ and $\mathfrak{M}$ =$[\mathfrak{M}_{ij}]$ be two PNHSM, where $O_{ij} = \left(\mathbb{T}_{ijk}^{o}, \mathbb{I}_{ijk}^{o}, \mathbb{F}_{ijk}^{o}\right)$ and $\mathfrak{M}_{ij} = \left(\mathbb{T}_{ijk}^{\mathfrak{M}}, \mathbb{I}_{ijk}^{\mathfrak{M}}, \mathbb{F}_{ijk}^{\mathfrak{M}}\right)$. Then,

(i) $(O \cup \mathfrak{M})^\circ = O^\circ \cap \mathfrak{M}^\circ$
(ii) $(O \cap \mathfrak{M})^\circ = O^\circ \cup \mathfrak{M}^\circ$
(iii) $(O \oplus \mathfrak{M})^\circ = O^\circ \oplus \mathfrak{M}^\circ$
(iv) $(O \oplus^w \mathfrak{M})^\circ = O^\circ \oplus \mathfrak{M}^\circ$
(v) $(O \odot \mathfrak{M})^\circ = O^\circ \odot \mathfrak{M}^\circ$
(vi) $(O \odot^w \mathfrak{M})^\circ = O^\circ \odot^w \mathfrak{M}^\circ$
(vii) $(O \oslash \mathfrak{M})^\circ = O^\circ \oslash \mathfrak{M}^\circ$
(viii) $(O \oslash^w \mathfrak{M})^\circ = O^\circ \oslash^w \mathfrak{M}^\circ$

Proof:

$$\begin{aligned} (i)\ (O \cup \mathfrak{M})^\circ &= \left[\left(\max\left(\mathbb{T}_{ijk}^{o}, \mathbb{T}_{ijk}^{\mathfrak{M}}\right), \max\left(\mathbb{I}_{ijk}^{o}, \mathbb{I}_{ijk}^{\mathfrak{M}}\right), \min\left(\mathbb{F}_{ijk}^{o}, \mathbb{F}_{ijk}^{\mathfrak{M}}\right)\right)\right]^\circ \\ &= \left[\left(\min\left(\mathbb{F}_{ijk}^{o}, \mathbb{F}_{ijk}^{\mathfrak{M}}\right), 1-\max\left(\mathbb{I}_{ijk}^{o}, \mathbb{I}_{ijk}^{\mathfrak{M}}\right), \max\left(\mathbb{T}_{ijk}^{o}, \mathbb{T}_{ijk}^{\mathfrak{M}}\right)\right)\right] \\ &= \left[\left(\min\left(\mathbb{F}_{ijk}^{o}, \mathbb{F}_{ijk}^{\mathfrak{M}}\right), \min\{(1-\mathbb{I}_{ijk}^{o}, 1-\mathbb{I}_{ijk}^{\mathfrak{M}})\}, \max\left(\mathbb{T}_{ijk}^{o}, \mathbb{T}_{ijk}^{\mathfrak{M}}\right)\right)\right] \\ &= \left(\mathbb{F}_{ijk}^{o}, 1-\mathbb{I}_{ijk}^{o}, \mathbb{T}_{ijk}^{o}\right) \cap \left(\mathbb{F}_{ijk}^{\mathfrak{M}}, 1-\mathbb{I}_{ijk}^{\mathfrak{M}}, \mathbb{T}_{ijk}^{\mathfrak{M}}\right) \\ &= \left(\mathbb{T}_{ijk}^{o}, \mathbb{I}_{ijk}^{o}, \mathbb{F}_{ijk}^{o}\right)^\circ \cap \left(\mathbb{T}_{ijk}^{\mathfrak{M}}, \mathbb{I}_{ijk}^{\mathfrak{M}}, \mathbb{F}_{ijk}^{\mathfrak{M}}\right)^\circ \\ &= O^\circ \cap \mathfrak{M}^\circ . \end{aligned}$$

Remaining parts are proved in a similar way.

Theorem 2. Let O = $[O_{ij}]$ and $\mathfrak{M}$ =$[\mathfrak{M}_{ij}]$ be two PNHSM, where $O_{ij} = (\mathbb{T}^{o}_{ijk}, \mathbb{I}^{o}_{ijk}, \mathbb{F}^{o}_{ijk})$ and $\mathfrak{M}_{ij} = (\mathbb{T}^{\mathfrak{M}}_{ijk}, \mathbb{I}^{\mathfrak{M}}_{ijk}, \mathbb{F}^{\mathfrak{M}}_{ijk})$. Then,

(i) $(O \cup \mathfrak{M}) = (\mathfrak{M} \cup O)$
(ii) $(O \cap \mathfrak{M}) = (\mathfrak{M} \cap O)$
(iii) $(O \oplus \mathfrak{M}) = (\mathfrak{M} \oplus O)$
(iv) $(O \oplus^{w} \mathfrak{M}) = (\mathfrak{M} \oplus^{w} O)$
(v) $(O \odot \mathfrak{M}) = (\mathfrak{M} \odot O)$
(vi) $(O \odot^{w} \mathfrak{M}) = (\mathfrak{M} \odot^{w} O)$
(vii) $(O \oslash \mathfrak{M}) = (\mathfrak{M} \oslash O)$
(viii) $(O \oslash^{w} \mathfrak{M}) = (\mathfrak{M} \oslash^{w} O)$

Proof:

(i) $$(O \cup \mathfrak{M}) = [(\max(\mathbb{T}^{o}_{ijk}, \mathbb{T}^{\mathfrak{M}}_{ijk}), \max(\mathbb{I}^{o}_{ijk}, \mathbb{I}^{\mathfrak{M}}_{ijk}), \min(\mathbb{F}^{o}_{ijk}, \mathbb{F}^{\mathfrak{M}}_{ijk}))]$$
$$= [(\max(\mathbb{T}^{\mathfrak{M}}_{ijk}, \mathbb{T}^{o}_{ijk}), \max(\mathbb{I}^{\mathfrak{M}}_{ijk}, \mathbb{I}^{o}_{ijk}), \min(\mathbb{F}^{\mathfrak{M}}_{ijk}, \mathbb{F}^{o}_{ijk}))]$$
$$= (\mathbb{T}^{\mathfrak{M}}_{ijk}, \mathbb{I}^{\mathfrak{M}}_{ijk}, \mathbb{F}^{\mathfrak{M}}_{ijk}) \cup (\mathbb{T}^{o}_{ijk}, \mathbb{I}^{o}_{ijk}, \mathbb{F}^{o}_{ijk})$$
$$= (\mathfrak{M} \cup O).$$

Remaining parts are proved in a similar way.

Theorem 3. Let O = $[O_{ij}]$ and $\mathfrak{M}$ =$[\mathfrak{M}_{ij}]$, and $\mathbb{N}$ =$[\mathbb{N}_{ij}]$be PNHSM , where $O_{ij} = (\mathbb{T}^{o}_{ijk}, \mathbb{I}^{o}_{ijk}, \mathbb{F}^{o}_{ijk})$ and $\mathfrak{M}_{ij} = (\mathbb{T}^{\mathfrak{M}}_{ijk}, \mathbb{I}^{\mathfrak{M}}_{ijk}, \mathbb{F}^{\mathfrak{M}}_{ijk})$, $\mathbb{N}_{ij} = (\mathbb{T}^{\mathbb{N}}_{ijk}, \mathbb{I}^{\mathbb{N}}_{ijk}, \mathbb{F}^{\mathbb{N}}_{ijk})$ Then,

(i) $(O \cup \mathfrak{M}) \cup \mathbb{N} = O \cup (\mathfrak{M} \cup \mathbb{N})$
(ii) $(O \cap \mathfrak{M}) \cap \mathbb{N} = O \cap (\mathfrak{M} \cap \mathbb{N})$
(iii) $(O \oplus \mathfrak{M}) \oplus \mathbb{N} \neq O \oplus (\mathfrak{M} \oplus \mathbb{N})$
(iv) $(O \odot \mathfrak{M}) \odot \mathbb{N} \neq O \odot (\mathfrak{M} \odot \mathbb{N})$
(v) $(O \oslash \mathfrak{M}) \oslash \mathbb{N} \neq O \oslash (\mathfrak{M} \oslash \mathbb{N})$

Proof:

(i) $$(O \cup \mathfrak{M}) \cup \mathbb{N} = [(\max(\mathbb{T}^{o}_{ijk}, \mathbb{T}^{\mathfrak{M}}_{ijk}), \max(\mathbb{I}^{o}_{ijk}, \mathbb{I}^{\mathfrak{M}}_{ijk}), \min(\mathbb{F}^{o}_{ijk}, \mathbb{F}^{\mathfrak{M}}_{ijk}))] \cup [(\mathbb{T}^{\mathbb{N}}_{ijk}, \mathbb{I}^{\mathbb{N}}_{ijk}, \mathbb{F}^{\mathbb{N}}_{ijk})]$$
$$= [(\max(\mathbb{T}^{o}_{ijk}, \mathbb{T}^{\mathfrak{M}}_{ijk}, \mathbb{T}^{\mathbb{N}}_{ijk}), \max(\mathbb{I}^{o}_{ijk}, \mathbb{I}^{\mathfrak{M}}_{ijk}, \mathbb{I}^{\mathbb{N}}_{ijk}), \min(\mathbb{F}^{o}_{ijk}, \mathbb{F}^{\mathfrak{M}}_{ijk}, \mathbb{F}^{\mathbb{N}}_{ijk})]$$
$$= (\mathbb{T}^{o}_{ijk}, \mathbb{I}^{o}_{ijk}, \mathbb{F}^{o}_{ijk}) \cup [(\max(\mathbb{T}^{\mathfrak{M}}_{ijk}, \mathbb{T}^{\mathbb{N}}_{ijk}), \max(\mathbb{I}^{\mathfrak{M}}_{ijk}, \mathbb{I}^{\mathbb{N}}_{ijk}), \min(\mathbb{F}^{\mathfrak{M}}_{ijk}, \mathbb{F}^{\mathbb{N}}_{ijk}))]$$
$$= (\mathbb{T}^{o}_{ijk}, \mathbb{I}^{o}_{ijk}, \mathbb{F}^{o}_{ijk}) \cup ((\mathbb{T}^{\mathfrak{M}}_{ijk}, \mathbb{I}^{\mathfrak{M}}_{ijk}, \mathbb{F}^{\mathfrak{M}}_{ijk}) \cup (\mathbb{T}^{\mathbb{N}}_{ijk}, \mathbb{I}^{\mathbb{N}}_{ijk}, \mathbb{F}^{\mathbb{N}}_{ijk}))$$
$$= O \cup (\mathfrak{M} \cup \mathbb{N}).$$

Remaining parts are proved in a similar way.

Theorem 4. Let O = $[O_{ij}]$ and $\mathfrak{M}$ =$[\mathfrak{M}_{ij}]$, and $\mathbb{N}$ =$[\mathbb{N}_{ij}]$be PNHSM , where $O_{ij} = (\mathbb{T}^{o}_{ijk}, \mathbb{I}^{o}_{ijk}, \mathbb{F}^{o}_{ijk})$ and $\mathfrak{M}_{ij} = (\mathbb{T}^{\mathfrak{M}}_{ijk}, \mathbb{I}^{\mathfrak{M}}_{ijk}, \mathbb{F}^{\mathfrak{M}}_{ijk})$, $\mathbb{N}_{ij} = (\mathbb{T}^{\mathbb{N}}_{ijk}, \mathbb{I}^{\mathbb{N}}_{ijk}, \mathbb{F}^{\mathbb{N}}_{ijk})$ Then,

(i) $O \cap (\mathfrak{M} \oplus \mathbb{N}) = (O \cap \mathfrak{M}) \oplus (O \cap \mathbb{N})$
(ii) $(O \oplus \mathfrak{M}) \cap \mathbb{N} = (O \cap \mathbb{N}) \oplus (\mathfrak{M} \cap \mathbb{N})$
(iii) $O \cup (\mathfrak{M} \oplus \mathbb{N}) = (O \cup \mathfrak{M}) \oplus (O \cup \mathbb{N})$
(iv) $(O \oplus \mathfrak{M}) \cup \mathbb{N} = (O \cup \mathbb{N}) \oplus (\mathfrak{M} \cup \mathbb{N})$

Proof :

(i) $$O \cap (\mathfrak{M} \oplus \mathbb{N}) = (\mathbb{T}^{o}_{ijk}, \mathbb{I}^{o}_{ijk}, \mathbb{F}^{o}_{ijk}) \cap \left[\frac{(\mathbb{T}^{\mathfrak{M}}_{ijk}+\mathbb{T}^{\mathbb{N}}_{ijk})}{2}, \frac{(\mathbb{I}^{\mathfrak{M}}_{ijk}+\mathbb{I}^{\mathbb{N}}_{ijk})}{2}, \frac{(\mathbb{F}^{\mathfrak{M}}_{ijk}+\mathbb{F}^{\mathbb{N}}_{ijk})}{2}\right]$$

$$= \left[\left(min\left(\mathbb{T}^{o}_{ijk}, \frac{(\mathbb{T}^{\mathfrak{M}}_{ijk}+\mathbb{T}^{\mathbb{N}}_{ijk})}{2}\right), min\left(\mathbb{I}^{o}_{ijk}, \frac{(\mathbb{I}^{\mathfrak{M}}_{ijk}+\mathbb{I}^{\mathbb{N}}_{ijk})}{2}\right), max\left(\mathbb{F}^{o}_{ijk}, \frac{(\mathbb{F}^{\mathfrak{M}}_{ijk}+\mathbb{F}^{\mathbb{N}}_{ijk})}{2}\right)\right)\right]$$

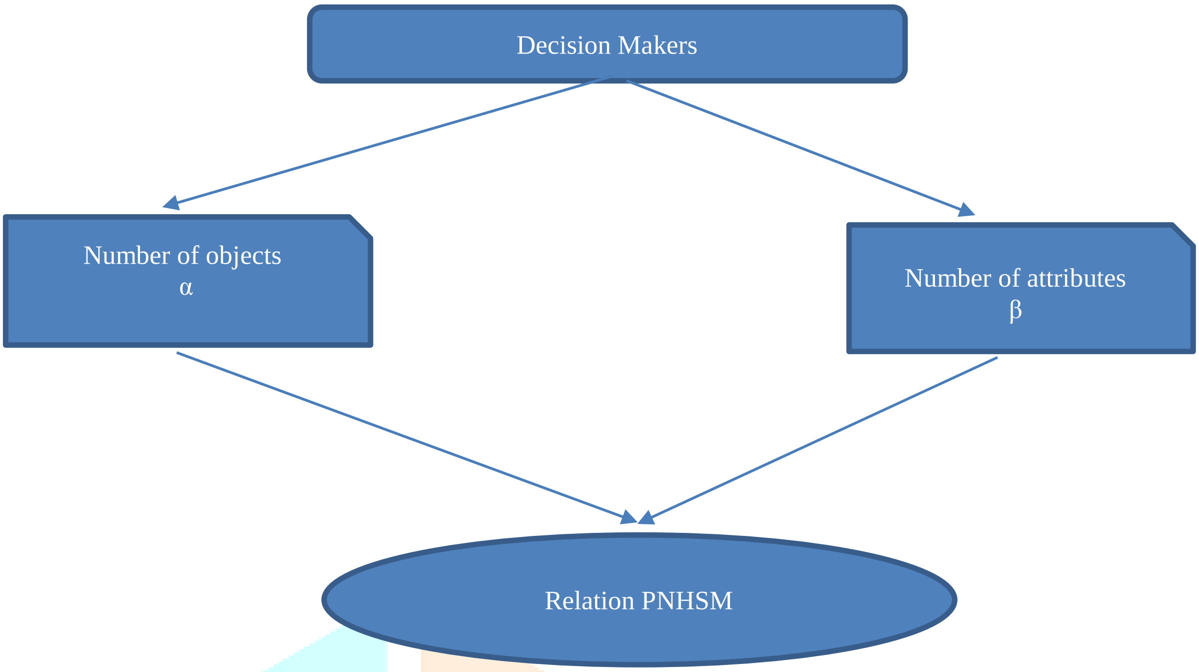
$$= \left[\left(\begin{array}{c} min\left(\frac{(\mathbb{T}^{o}_{ijk}+\mathbb{T}^{\mathfrak{M}}_{ijk})}{2}, \frac{(\mathbb{T}^{o}_{ijk}+\mathbb{T}^{\mathbb{N}}_{ijk})}{2}\right), min\left(\frac{(\mathbb{I}^{o}_{ijk}+\mathbb{I}^{\mathfrak{M}}_{ijk})}{2}, \frac{(\mathbb{I}^{o}_{ijk}+\mathbb{I}^{\mathbb{N}}_{ijk})}{2}\right), \\ max\left(\frac{(\mathbb{F}^{o}_{ijk}+\mathbb{F}^{\mathfrak{M}}_{ijk})}{2}, \frac{(\mathbb{F}^{o}_{ijk}+\mathbb{F}^{\mathbb{N}}_{ijk})}{2}\right) \end{array}\right)\right]$$

$$= [(\min(\mathbb{T}^{o}_{ijk}, \mathbb{T}^{\mathfrak{M}}_{ijk}), \min(\mathbb{I}^{o}_{ijk}, \mathbb{I}^{\mathfrak{M}}_{ijk}), \max(\mathbb{F}^{o}_{ijk}, \mathbb{F}^{\mathfrak{M}}_{ijk})] \oplus [(\min(\mathbb{T}^{o}_{ijk}, \mathbb{T}^{\mathbb{N}}_{ijk}), \min(\mathbb{I}^{o}_{ijk}, \mathbb{I}^{\mathbb{N}}_{ijk}), \max(\mathbb{F}^{o}_{ijk}, \mathbb{F}^{\mathbb{N}}_{ijk})]$$
$$= [(\mathbb{T}^{o}_{ijk}, \mathbb{I}^{o}_{ijk}, \mathbb{F}^{o}_{ijk}) \cap (\mathbb{T}^{\mathfrak{M}}_{ijk}, \mathbb{I}^{\mathfrak{M}}_{ijk}, \mathbb{F}^{\mathfrak{M}}_{ijk})] \oplus [(\mathbb{T}^{o}_{ijk}, \mathbb{I}^{o}_{ijk}, \mathbb{F}^{o}_{ijk}) \cap (\mathbb{T}^{\mathbb{N}}_{ijk}, \mathbb{I}^{\mathbb{N}}_{ijk}, \mathbb{F}^{\mathbb{N}}_{ijk})]$$
$$= (O \cap \mathfrak{M}) \oplus (O \cap \mathbb{N}).$$

Remaining parts are proved in a similar way.

4. Pythagorean Neutrosophic Hypersoft Matrix (PNHSM) in Decision-Making Using Score Function

Let us consider some decision makers wants to select from α number of objects with corresponding β number of attributes whose corresponding parameters form a relation like PNHSM. Each decision makes various Pythagorean Neutrosophic values to these attributes. Corresponding to these Pythagorean Neutrosophic values for the needed relation,we get PHNSM of order α×β. From this PNHSM, we calculate values matrices which help to get a score matrix. Finally, we calculate the total score of each object from the score matrix.



Definition 4.1:

Let O = $[O_{ij}]$ be the PNHSM of order α×β, where $O_{ij} = (\mathbb{T}^o_{ijk}, \mathbb{I}^o_{ijk}, \mathbb{F}^o_{ijk})$; then, the value of matrix O is denoted as $\mathbb{v}(O)$, and it is defined as $\mathbb{v}(O) = [\mathbb{V}_{ij}{}^{O}]$ of order α×β, where $\mathbb{V}_{ij}{}^{O} = \mathbb{T}^o_{ijk} - \mathbb{I}^o_{ijk} - \mathbb{F}^o_{ijk}$ .The score of two PHNSM O = $[O_{ij}]$ and $\mathfrak{M}$ =$[\mathfrak{M}_{ij}]$ of order α×β is given as $\mathcal{S}(O, \mathfrak{M})$ = $\mathbb{v}(O) + \mathbb{v}(\mathfrak{M})$ and $\mathcal{S}(O, \mathfrak{M})$ = $[\mathcal{S}_{ij}]$, where $\mathcal{S}_{ij}$ = $\mathbb{V}_{ij}{}^{O} + \mathbb{V}_{ij}{}^{\mathfrak{M}}$. The total score of each object in universal set is $\left|\sum_{j=1}^{n} \mathcal{S}_{ij}\right|$.

Step 1: Develop a PNHSM as defined in section 3.1.
Step 2: Find the value matrix from PNHSM. Let O = $[O_{ij}]$ be the PNHSM of order α×β, where $O_{ij} = (\mathbb{T}^o_{ijk}, \mathbb{I}^o_{ijk}, \mathbb{F}^o_{ijk})$; then, the value of matrix O is denoted as $\mathbb{v}(O)$, and it is defined as $\mathbb{v}(O) = [\mathbb{V}_{ij}{}^{O}]$ of order α×β, where $\mathbb{V}_{ij}{}^{O} = \mathbb{T}^o_{ijk} - \mathbb{I}^o_{ijk} - \mathbb{F}^o_{ijk}$.
Step 3: Calculate the score matrix with the help of value matrices. The score of 2 PNHSM O = $[O_{ij}]$ and $\mathfrak{M}$ =$[\mathfrak{M}_{ij}]$ of order α×β is given as $\mathcal{S}(O, \mathfrak{M})$ = $\mathbb{v}(0) + \mathbb{v}(\mathfrak{M})$ and $\mathcal{S}(O, \mathfrak{M})$ = $[\mathcal{S}_{ij}]$, where $\mathcal{S}_{ij}$ = $\mathbb{V}_{ij}{}^{O} + \mathbb{V}_{ij}{}^{\mathfrak{M}}$.
Step 4: Calculate the total score from the score matrix. The total score of each object in the universal set is $\left|\sum_{j=1}^{n} \mathcal{S}_{ij}\right|$.
Step 5: Find the optimal solution by selecting an object of maximum score from the total score matrix.

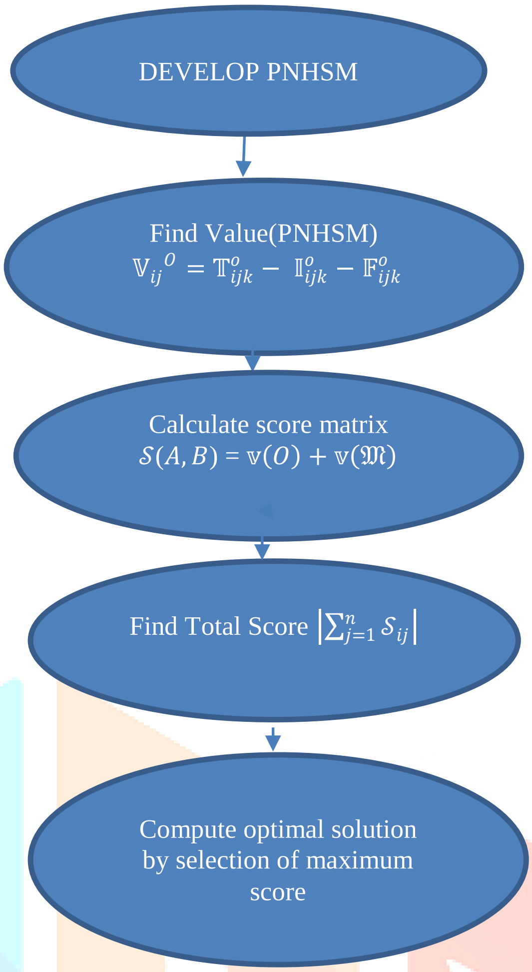


FIGURE 1: Pictorial representation of the proposed algorithm.

4. 1. Numerical Example:

In example 1, four candidates $\{\mathbb{T}^1, \mathbb{T}^2, \mathbb{T}^4, \mathbb{T}^5\}$are under shortlisted on the basis of assumed relation.
i.e, (M.Phil., BC, Female, High).
A jury of two members {A, B}is set for the selection of shortlisted candidates. These jury members give their valuable opinion in the form of PNHSSs as

A= F (M.Phil. (B.Ed.), BC, Female, High)

= {<< $\mathbb{T}^1$, M. Phil. (B.Ed.) {0.6,0.7,0.5}, BC {0.6,0.5,0.2}, F {0.7,0.5,0.2}, High {0.6,0.4,0.1}>>,

<< $\mathbb{T}^2$, M.Phil. (B.Ed.) {0.5,0.4,0.6}, BC {0.6,0.5,0.3}, F {0.5,0.4,0.1}, High {0.5,0.6,0.2}>>,

<< $\mathbb{T}^4$, M.Phil. (B.Ed.) {0.6,0.4,0.3}, BC {0.6,0.5,0.2}, F {0.5,0.6,0.2}, High {0.5,0.3,0.1}>>,

<< $\mathbb{T}^5$, M.Phil. (B.Ed.) {0.5,0.7,0.2}, BC {0.5,0.4,0.2}, F {0.7,0.6,0.1}, High {0.7,0.6,0.2}>>,}

B= F (M.Phil. (B.Ed.), BC, Female, High)

= {<< $\mathbb{T}^1$, M.Phil. (B.Ed.) {0.5,0.4,0.3}, BC {0.5,0.6,0.2}, F {0.5,0.4,0.7}, High {0.7,0.6,0.1}>>,

<< $\mathbb{T}^2$, M.Phil. (B.Ed.) {0.5,0.3,0.2}, BC {0.5,0.4,0.1}, F {0.6,0.5,0.1}, High {0.6,0.4,0.3}>>,

<< $\mathbb{T}^4$, M.Phil. (B.Ed.) {0.5,0.6,0.2}, BC {0.6,0.5,0.1}, F {0.6,0.4,0.2}, High {0.6,0.7,0.1}>>,

<< $\mathbb{T}^5$, M.Phil. (B.Ed.) {0.6,0.5,0.1}, BC {0.5,0.6,0.1}, F {0.5,0.4,0.1}, High {0.6,0.4,0.2}>>,}

Let us apply the above defined algorithm for the calculation of total score.

Step I: (constructing PNHSM) the above two PNHSSs are given in the form of PNHSMs as

$$[A]=\begin{bmatrix}(\text{M. Phil.},(0.6,0.7,0.5)) & (\text{BC},(0.6,0.5,0.2)) & (\text{Female},(0.7,0.5,0.2)) & (\text{High},(0.6,0.4,0.1))\\ (\text{M. Phil.},(0.5,0.4,0.6)) & (\text{BC},(0.6,0.5,0.3)) & (\text{Female},(0.5,0.4,0.1)) & (\text{High},(0.5,0.6,0.2))\\ (\text{M. Phil.},(0.6,0.4,0.3)) & (\text{BC},(0.6,0.5,0.2)) & (\text{Female},(0.5,0.6,0.2)) & (\text{High},(0.5,0.3,0.1))\\ (\text{M. Phil.},(0.5,0.7,0.2)) & (\text{BC},(0.5,0.4,0.2)) & (\text{Female},(0.7,0.6,0.1)) & (\text{High},(0.7,0.6,0.2))\end{bmatrix}$$

and Let $$[B]=\begin{bmatrix}(\text{M. Phil.},(0.5,0.4,0.3)) & (\text{BC},(0.5,0.6,0.2)) & (\text{Female},(0.5,0.4,0.7)) & (\text{High},(0.7,0.6,0.1))\\ (\text{M. Phil.},(0.5,0.3,0.2)) & (\text{BC},(0.5,0.4,0.1)) & (\text{Female},(0.6,0.5,0.1)) & (\text{High},(0.6,0.4,0.3))\\ (\text{M. Phil.},(0.5,0.6,0.2)) & (\text{BC},(0.6,0.5,0.1)) & (\text{Female},(0.6,0.4,0.2)) & (\text{High},(0.6,0.7,0.1))\\ (\text{M. Phil.},(0.6,0.5,0.1)) & (\text{BC},(0.5,0.6,0.1)) & (\text{Female},(0.5,0.4,0.1)) & (\text{High},(0.6,0.4,0.2))\end{bmatrix}$$

Step II: Calculation of the value matrices of PNHSMs in step I:

$$[\mathbb{v}(A)]=\begin{bmatrix}(\text{M. Phil.},(-0.6)) & (\text{BC},(-0.1)) & (\text{Female},(0)) & (\text{High},(0.1))\\ (\text{M. Phil.},(-0.5)) & (\text{BC},(-0.2)) & (\text{Female},(0)) & (\text{High},(-0.3))\\ (\text{M. Phil.},(-0.1)) & (\text{BC},(-0.1)) & (\text{Female},(-0.3)) & (\text{High},(0.1))\\ (\text{M. Phil.},(-0.4)) & (\text{BC},(-0.1)) & (\text{Female},(0)) & (\text{High},(-0.1))\end{bmatrix}$$

$$[\mathbb{v}(B)]=\begin{bmatrix}(\text{M. Phil.},(-0.2)) & (\text{BC},(-0.3)) & (\text{Female},(-0.6)) & (\text{High},(0))\\ (\text{M. Phil.},(0)) & (\text{BC},(0)) & (\text{Female},(0)) & (\text{High},(-0.1))\\ (\text{M. Phil.},(-0.3)) & (\text{BC},(0)) & (\text{Female},(0)) & (\text{High},(-0.2))\\ (\text{M. Phil.},(0)) & (\text{BC},(-0.2)) & (\text{Female},(0)) & (\text{High},(0))\end{bmatrix}$$

Step III: Calculation of the score matrix

$$[\mathcal{S}(A,B)]=\begin{bmatrix}(\text{M. Phil.},(-0.8)) & (\text{BC},(-0.4)) & (\text{Female},(-0.6)) & (\text{High},(0.1))\\ (\text{M. Phil.},(-0.5)) & (\text{BC},(-0.2)) & (\text{Female},(0)) & (\text{High},(-0.4))\\ (\text{M. Phil.},(-0.4)) & (\text{BC},(-0.1)) & (\text{Female},(-0.3)) & (\text{High},(-0.1))\\ (\text{M. Phil.},(-0.4)) & (\text{BC},(-0.3)) & (\text{Female},(0)) & (\text{High},(-0.1))\end{bmatrix}$$

Step IV: Calculation of the score matrix:

$$\text{Total Score}=\begin{bmatrix}1.7\\1.1\\0.9\\0.8\end{bmatrix}$$

Step V: The candidate $\mathbb{T}^1$ will be selected for recruitment of teacher for Government posting as the total score of $\mathbb{T}^1$ is highest among the rest of the total score of candidates.

5.Result:

The proposed algorithm for PNHSM is valid and applicable for practical situation. Real-world problems and results are compared with PNHSM algorithm. Pictorial representation of the ranking of the proposed algorithm are given in fig 1. It could be a more efficient technique.

Conclusion

In this paper we have introduced some aggregate operators on Pythagorean Neutrosophic Hypersoft matrices (PNHSMs)such as union, intersection, Arithmetic mean, Weighted Arithmetic mean, Geometric mean, Weighted Geometric mean, Harmonic mean, Weighted harmonic mean. The validity and Implementation of the proposed operators are verified by presenting the suitable examples. Moreover, we have proposed the concept of the score function. Also, Decision-making problem (Recruitment of teachers for Government postings) has been made with the score matrix's assistance. Finally, we compared the result with existing procedure, and proved the purposed technique is more optimal. Using PNHSM method real – world problems such as decision makings in Personal selection, Office administration and many other problems can be solved.

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