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Bayesian Shift Point Estimation of Exponentiated Inverted Weibull Distribution under General Entropy Loss Function

Uma Srivastava and Harish Kumar

Department of Mathematics and Statistics

DDU Gorakhpur University

Gorakhpur, U.P., India

Abstract

The aim of the article is to analyze inhomogeneous sequence data caused by the presence of an unknown change point. We assume that the sequence data are from a Exponentiated Inverted Weibull Distribution with an unknown point of change in the scale and/or shape parameters. The Bayes estimates of θ and m are derived for asymmetric loss function known as General Entropy Loss Function under natural conjugate prior distribution. We propose Bayesian methods of estimating the change point, together with the model parameters, before and after its occurrence. Extensive simulations have been conducted to show excellent agreement between the distribution of the change point under finite sample sizes. The simulations are conducted under a change in the scale parameter as well as a change in both scale or shape parameters.

Keywords: General Entropy Loss Function, natural conjugate prior distribution, Bayesian methods, change point.

1.1 Introduction

Bayesian inference is an approach to statistics in which all forms of uncertainty are expressed in terms of probability. A Bayesian approach to a problem starts with the formulation of a model that we hope is adequate to describe the situation of interest. We then formulated a *prior* distribution over the unknown parameters of the model, which is meant to capture our beliefs about the situation before seeing the data. After observing some data, we apply Bayes' Rule to obtain a *posterior* distribution for these unknowns, which takes account of both the prior and the data.

This theoretically simple process can be justified as the proper approach to uncertain inference by various arguments involving consistency with clear principles of rationality. Despite this, many people are uncomfortable with the Bayesian approach, often because they view the selection of a prior as being arbitrary and subjective. It is indeed subjective, but for this very reason it is not arbitrary. In theory there is just one correct prior, that captures the our prior beliefs. In contrast, other statistical methods are truly arbitrary, in that there are usually many methods that are equally good according to non-Bayesian criteria of goodness, with no principled way of choosing between them.

In decision theory the loss criterion is specified in order to obtain best estimator. The simplest form of loss function is squared error loss function (SELF) which assigns equal magnitudes to both positive and negative errors. However this assumption may be inappropriate in most of the estimation problems. Some time overestimation leads to many serious consequences. In such situation many authors found the asymmetric loss functions, more appropriate. There are several loss functions which are used to deal such type of problem. In this research work we have considered some of the asymmetric loss function named general entropy loss functions (GELF) suggested by Calabria and Pulcini (1996) .Such asymmetric loss functions are also studied by Parsian and Kirmani (2002), Braess and Dette (2004) and Pandya et. al. (2006).

1.2 Entropy Loss Function

In many practical situations, it appears to be more realistic to express the loss in terms of the ratio $\frac{\hat{\theta}}{\theta}$. In this case, Calabria and Pulcini (1994) points out that a useful asymmetric loss function is the Entropy loss

$$L(\delta) \propto [\delta^p - p\log_e(\delta) - 1], \qquad (1.2.1)$$

Where $\delta = \frac{\hat{\theta}}{\theta}$

and whose minimum occurs at $\hat{\theta} = \theta$ when $p > 0$, a positive error $(\hat{\theta} > \theta)$ causes more serious consequences than a negative error and vice-versa. For small $|p|$ value, the function is almost symmetric when both $\hat{\theta}$ and θ are measured in a logarithmic scale and approximately

$$L(\delta) \propto \frac{p^2}{2}\left[\log_e(\hat{\theta}) - \log_e(\theta)\right]^2$$

Also, the loss function $L(\delta)$ has been used in Dey et al (1987) and Dey and Lin (1992), in the original form having p = 1. Thus $L(\delta)$ can be
written as

$$L(\delta) = b[\delta - \log_e(\delta) - 1]\,;\; b > 0, \quad \text{where } \delta = \frac{\hat{\theta}}{\theta}$$

In a Bayesian setup, the unknown parameter is viewed as random variable. The uncertainty about the true value of parameter is expressed by a prior distribution. The parametric inference is made using the posterior distribution which is obtained by incorporating the observed data in to the prior distribution using the Bayes theorem, The first theorem of inference. Hence we update the prior distribution in the light of observed data. Thus the uncertainty about the parameter prior to the experiment is represented by the prior distribution and the same, after the experiment, is represented by the posterior distribution. The various statistical models are considered are as;

1.3 Natural Conjugate Prior (NCP)

The various prior distributions are considered for different situations, like non-informative, when no information about the parameter is available, Natural Conjugate Prior (NCP), when post and prior distribution of parameter belong to same distribution family, etc. Hence the appropriate distribution chosen is Natural Conjugate Prior. If there is no inherent reason to prefer one prior probability distribution over another, a conjugate prior is sometimes chosen for simplicity. A conjugate prior is defined as a prior distribution belonging to some parametric family, for which the resulting posterior distribution also belongs to the same family. This is an important property. Since the Bayes estimator, as well as its statistical properties (variance, confidence interval, etc.), can all be derived from the posterior distribution.

In each case we observe that the statistical analysis based on the sufficient statistic will be effective as the one based on the entire data set $\underline{x}$.

As in frequentist framework, sufficient statistic plays an important role in Bayesian inference in constructing a family of prior distributions known as Natural Conjugate Prior (NCP). The family of prior distributions $g(\theta)$, $\theta \in \Omega$, is called a natural conjugate family if the corresponding posterior distribution belongs to the same family as $g(\theta)$. De Groot(1970) has outlined a simple and elegant method of constructing a conjugate prior for a family of distributions $f(x|\theta)$ which admits a sufficient statistic.

One of the fundamental problems in Bayesian analysis is that of the choice of prior distribution $g(\theta)$ of θ. The non informative and natural conjugate prior distributions are which in practice, Box and Tiao(1973) and Jeffrey(1961) provide a thorough discussion on non informative priors.

Both De Groot(1970) and Raffia & Schlaifer (1961) provide proof that when a sufficient statistics exist a family of conjugate prior distributions exists.

The most widely used prior distribution of θ is the inverted Gamma distribution with the parameters ‘a’ and ‘b’ (> 0) with p.d.f. given by

$$g(\theta) = \begin{cases} \frac{b^a}{\Gamma a}\theta^{-(\alpha+1)}e^{-b/\theta} ; \ \theta > 0 ; \ (a, b) > 0, \\ 0 \qquad , \text{otherwise.} \end{cases} \qquad (1.3.1)$$

The main reason for general acceptability is the mathematical tractability resulting from the fact that the inverted Gamma distribution is conjugate prior of θ Raffia & Schlaifer (1961), Bhattacharya (1967) and others have found that the inverted Gamma can also be used for practical reliability applications. .

In this paper the Bayesian estimation of change point ‘m’ and scale parameter $'\theta'$ of Exponentiated Inverted Weibull distribution is done by using General Entropy Loss Function (GELF) and Natural conjugate Prior distribution as Inverted Gamma prior. The sensitivity analysis of Bayesian estimates of change point and the parameters of the distributions have been done by using R-programming.

1.4 Exponentiated Inverted Weibull Distribution

The inverted Weibull distribution is one of the most popular probability distribution to analyze the life time data with some monotone failure rates. (Khan et al. 2008) Explained the flexibility of the three parameters inverted Weibull distribution and its interested properties. Exponentiated (generalized) Inverted Weibull Distribution is a generalization to the Inverted c through adding a new shape parameter $\lambda \in \mathcal{R}^+$ by exponentiation to distribution function F,the new distribution function F^{λ}. E.K.al-hussaini et al.(2010)explained that the cumulative distribution function is flexible to monotone and non-monotone failure rates. G.S.mudholka et al (1995) introduced the Exponentiated Weibull Distribution as generalization of the statndard Weibull Distribution, the applied the new distribution as a suitable model to the bus-motor failure time data. M.N.Nasar et al.(2003) reviewed the Exponentiated Weibull Distribution with new measures.

R.D.Gupta et al.(2001) studied the exponentiated exponential distribution in details as an alternative distribution to Weibull distribution and gamma distribution. S.Nadarajah et al (2005) discussed in details the moments of the exponentiated Weibull distribution and compared exponentiated Weibull distribution with two parameters Weibull distribution and gamma distribution with respect to failure rate as well as some basic properties with data analysis. G.S.Mudholkar et at (1996) applied the exponentiatced Weibull distribution to the flood data with some properties. R.jiang et al (1999) introduced a graphical analysis as approach to study the parameters characterization of the exponentiated Weibull distribution.

Recently the two parameter Exponentiated Inverted Weibull Distribution (EIW) distribution has been proposed by Flaih et al. (2012). The two parameter EIW distribution has the following probability density function

$$f(x) = \theta\beta x^{-(\beta+1)}\left(e^{-\theta x}\right)^{-\beta}; \quad x > 0, (\beta > 0, \theta > o) \tag{1.4.1}$$

And the distribution function

$$F(x) = \left(e^{-x^{-\beta}}\right)^{\theta}; \quad x > 0 \tag{1.4.2}$$

Also, the reliability function of the EIW distribution with two shape parameters θ and β are given by

$$R(t) = 1 - \left(e^{-t^{-\beta}}\right)^{\theta}; \quad t > 0 \tag{1.4.3}$$

1. 5 Bayesian Estimation of Change Point in Exponentiated Inverted Weibull Distribution under General Entropy Loss Function (GELF)

A sequence of independent lifetimes $x_1, x_2, \dots \dots . x_m, x_{(m+1)}, \dots \dots \dots x_n$ $(n \geq 3)$ were observed from Exponentiated Inverted Weibull Distribution with parameter β, θ . But it was found that there was a change in the system at some point of time 'm' and it is reflected in the sequence after 'x_m' which results change in a sequence as well as parameter value. The Bayes estimate of θ and 'm'are derived for symmetric and asymmetric loss function under inverted gamma prior as natural conjugate prior.

1. 5. 1 Likelihood, Prior, Posterior and Marginal

Let $x_1, \dots \dots \dots \dots . x_n$, (n≥ 3) be a sequence of observed discrete life times . First let observations $x_1, \dots \dots \dots \dots . x_n$ have come from Exponentiated Inverted Weibull Distribution with probability density function as

$$\mathrm{f}(\mathrm{x}, \beta, \theta) = \theta\beta x^{-(\beta+1)}\left(e^{-\theta x}\right)^{-\beta}; \quad (\mathrm{x}, \beta, \theta > 0) \tag{1.5.1.1}$$

Let 'm' is change point in the observation which breaks the distribution in two sequences as

$$(x_1, x_2, \dots \dots \dots . . x_m) \ \& \ (x_{m+1}, x_{m+2}, \dots \dots \dots \dots x_n)$$

$f_1(x) = \theta_1\, \beta_1\, x^{-(\beta_1+1)}\left(e^{-\theta_1 x}\right)^{-\beta_1}$ (1.5.1.2)

Where $x_1, x_2, \dots \dots \dots .. x_m, \theta_1, \beta_1 > o$

$f_2(x) = \theta_2\, \beta_2\, x^{-(\beta_2+1)}\left(e^{-\theta_2 x}\right)^{-\beta_2}$ $x_{m+1}, \dots \dots x_n, \theta_2, \beta_2 > 0$, (1.5.1.3)

The likelihood functions of probability density function of the sequence are

$L_1(x, \theta_{1,}\beta_1) = \prod_{j=1}^{m} f(x_{j\ ,}\, \theta_{1,}\beta_1)$

$L_1(x, \theta_{1,}\beta_1) = (\theta_1\beta_1)^m U_1 e^{-\theta_1 T_{2m}}$ (1.5.1.5)

Where $U_1 = \prod_{j=1}^{m} x_j^{-(\beta_1+1)}$

$T_{2m} = \sum_{j=1}^{m} x_j^{-\beta_1}$

$L_2(x, \theta_{2,}\beta_2) = \prod_{j=m+1}^{n} f(x_{j\ ,}\ \theta_{2,}\beta_2)$

$L_2(x, \theta_{2,}\beta_2) = \theta_2^{\ n-m}\beta_2^{\ n-m} \prod_{j=m+1}^{n} x_j^{-(\beta_2+1)} e^{-\theta_2 \sum_{j=1}^{m} x_j^{-\beta_2}}$

$L_2(x, \theta_{2,}\beta_2) = (\theta_2\beta_2)^{n-m} U_2 e^{-\theta_2 (T_{2n}-T_{2m})}$ (1.5.1.5)

Where $U_2 = \prod_{j=m+1}^{n} x_j^{-(\beta_2+1)}$

$T_{2n} - T_{2m} = \sum_{j=m+1}^{n} x_j^{-\beta_2}$

And the joint Likelihood function is given by

$\mathrm{L}(\theta_1, \theta_2 | \underline{x}) \propto (\theta_1\beta_1)^m U_1 e^{-\theta_1 T_{2m}} (\theta_2\beta_2)^{n-m} U_2 e^{-\theta_2 (T_{2n}-T_{2m})}$ (1.5.1.6)

Suppose the marginal prior distributions 0f θ_1, θ_2 are natural conjugate prior

$\pi_1(\theta_1, \underline{\mathrm{x}}) = \frac{b_1^{a_1}}{\Gamma a_1} \theta_1^{(a_1-1)} e^{-b_1\theta_1};$ $a_{1,}b_1 > 0, \theta_1 > 0$ (1.5.1.7)

$\pi_2(\theta_2, \underline{\mathrm{x}}) = \frac{b_2^{a_2}}{\Gamma a_2} \theta_2^{(a_2-1)} e^{-b_2\theta_2}$; $a_2, b_2 > 0, \theta_2 > 0$ (1.5.1.8)

The joint prior distribution of θ_1,θ_2 and change point ‘m’ is

$\pi(\theta_1, \theta_2, m) \propto \frac{b_1^{a_1}}{\Gamma a_1} \frac{b_2^{a_2}}{\Gamma a_2} \theta_1^{(a_1-1)} e^{-b_1\theta_1} \theta_2^{(a_2-1)} e^{-b_2\theta_2}$ (1.5.1.9)

where $\theta_1, \theta_2 > 0$ & m = 1,2.........(n-1)

The joint posterior density of θ_1, θ_2 and m say $\rho(\theta_1, \theta_2, m/\underline{x})$ is obtaited by using equations (1.5.1.6) & (1.5.1.9)

$$\rho(\theta_1,\theta_2\,,m|\underline{x}) = \frac{L(\theta_1\,,\theta_2/\underline{x})\pi(\theta_1\,,\theta_2,m)}{\sum_m \iint_{\theta_1\theta_2} L(\theta_1\,,\theta_2/\underline{x})\pi(\theta_1,\theta_2,m)d\theta_1 d\theta_2} \qquad (1.5.1.10)$$

$$\rho(\theta_1,\theta_2\,,m|\underline{x}) = \frac{{\theta_1}^{(m+a_1-1)}\, e^{-\theta_1(T_{2m}+b_1)}\, {\theta_2}^{(n-m+a_2-1)}\, e^{-\theta_2(T_{2n}-T_{2m}+b_2)}}{\sum_m \int_0^\infty e^{-\theta_1(T_{2m}+b_1)}\, {\theta_1}^{(m+a_1-1)}\, d\theta_1 \int_0^\infty {\theta_2}^{(n-m+a_2-1)}\, e^{-\theta_2(T_{2n}-T_{2m}+b_2)}\, d\theta_2}$$

Assuming $\theta_1(T_{2m}+b_1) = x$ & $\theta_2(T_{2n}-T_{2m}+b_2) = y$

$\theta_1 = \frac{x}{(T_{2m}+b_1)}$ & $\theta_2 = \frac{y}{T_{2n}-T_{2m}+b_2}$

$d\theta_1 = \frac{dx}{(T_{2m}+b_1)}$ & $d\theta_2 = \frac{dy}{T_{2n}-T_{2m}+b_2}$

$$\rho(\theta_1,\theta_2\,,m|\underline{x}) = \frac{{\theta_1}^{(m+a_1-1)}\, e^{-\theta_1(T_{2m}+b_1)}\, {\theta_2}^{(n-m+a_2-1)}\, e^{-\theta_2(T_{2n}-T_{2m}+b_2)}}{\sum_m \int_0^\infty e^{-x} \frac{x^{(m+a_1-1)}}{(T_{2m}+b_1)^{(m+a_1-1)}} \frac{dx}{(T_{2m}+b_1)} \int_0^\infty e^{-y} \frac{y^{(n-m+a_2-1)}}{(T_{2n}-T_{2m}+b_2)^{(n-m+a_2-1)}} \frac{dy}{(T_{2n}-T_{2m}+b_2)}}$$

$$\rho(\theta_1,\theta_2\,,m|\underline{x}) = \frac{e^{-\theta_1(T_{2m}+b_1)}\, {\theta_1}^{(m+a_1-1)}\, e^{-\theta_2(T_{2n}-T_{2m}+b_2)}\, {\theta_2}^{(n-m+a_2-1)}}{\sum_m \frac{\Gamma(m+a_1)}{(T_{2m}+b_1)^{(m+a_1)}} \frac{\Gamma(n-m+a_2)}{(T_{2n}-T_{2m}+b_2)^{(n-m+a_2)}}}$$

$$\rho(\theta_1,\theta_2\,,m|\underline{x}) = \frac{e^{-\theta_1(T_{2m}+b_1)}\, {\theta_1}^{(m+a_1-1)}\, e^{-\theta_2(T_{2n}-T_{2m}+b_2)}\, {\theta_2}^{(n-m+a_2-1)}}{\xi(a_1,a_2,b_1,b_2,m,n)} \qquad (1.5.1.11)$$

Where $\xi(a_1,a_2,b_1,b_2,m,n) = \sum_{m=1}^{n-1}\left[\frac{\Gamma(m+a_1)}{(T_{2m}+b_1)^{m+a_1}} \frac{\Gamma(n-m+a_2)}{(T_{2n}-T_{2m}+b_2)^{(n-m+a_2)}}\right]$

The Marginal posterior distribution of change point 'm' using the equations (1.5.1.6), (1.5.1.7) & (1.5.1.8)

$$\rho(m|\underline{x}) = \frac{L(\theta_1\,,\theta_2/\underline{x})\,\pi(\theta_1)\,\pi(\theta_2)}{\sum_m L(\theta_1\,,\theta_2/\underline{x})\,\pi(\theta_1)\,\pi(\theta_2)}$$

On solving which gives

$$\rho(m|\underline{x}) = \frac{(\theta_1\beta_1)^m\, U_1\, e^{-\theta_1 T_{2m}}\, (\theta_2\beta_2)^{n-m}\, U_2\, e^{-\theta_2(T_{2n}-T_{2m})} \frac{b_2^{a_2}}{\Gamma a_2}\theta_2^{(a_2-1)} e^{-b_2\theta_2} \frac{b_1^{a_1}}{\Gamma a_1}\theta_1^{(a_1-1)} e^{-b_1\theta_1}}{\sum_m (\theta_1\beta_1)^m\, U_1\, e^{-\theta_1 T_{2m}}\, (\theta_2\beta_2)^{n-m}\, U_2\, e^{-\theta_2(T_{2n}-T_{2m})} \frac{b_2^{a_2}}{\Gamma a_2}\theta_2^{(a_2-1)} e^{-b_2\theta_2} \frac{b_1^{a_1}}{\Gamma a_1}\theta_1^{(a_1-1)} e^{-b_1\theta_1}}$$

$$\rho(m|\underline{x}) = \frac{{\theta_1}^{(m+a_1-1)}\, e^{-\theta_1(T_{2m}+b_1)}\, {\theta_2}^{(n-m+a_2-1)}\, e^{-\theta_2(T_{2n}-T_{2m}+b_2)}}{\sum_m {\theta_1}^{(m+a_1-1)}\, e^{-\theta_1(T_{2m}+b_1)}\, {\theta_2}^{(n-m+a_2-1)}\, e^{-\theta_2(T_{2n}-T_{2m}+b_2)}}$$

$$\rho(m|\underline{x}) = \frac{\int_0^\infty e^{-\theta_1(T_{2m}+b_1)}\, {\theta_1}^{(m+a_1-1)}\, d\theta_1 \int_0^\infty e^{-\theta_2(T_{2n}-T_{2m}+b_2)}\, {\theta_2}^{(n-m+a_2-1)}\, d\theta_2}{\sum_m \int_0^\infty e^{-\theta_1(T_{2m}+b_1)}\, {\theta_1}^{(m+a_1-1)}\, d\theta_1 \int_0^\infty e^{-\theta_2(T_{2n}-T_{2m}+b_2)}\, {\theta_2}^{(n-m+a_2-1)}\, d\theta_2}$$

Assuming $\theta_1(T_{2m}+b_1) = y$ & $\theta_2(T_{2n}-T_{2m}+b_2) = z$

$\theta_1 = \frac{y}{(T_{2m}+b_1)}$ & $\theta_2 = \frac{z}{T_{2n}-T_{2m}+b_2}$

$d\theta_1 = \frac{dy}{(T_{2m}+b_1)}$ & $d\theta_2 = \frac{z}{T_{2n}-T_{2m}+b_2}$

$$\rho(m|\underline{x}) = \frac{\int_0^\infty e^{-y} \frac{y^{(m+a_1-1)}}{(T_{2m}+b_1)^{(m+a_1-1)}} \frac{dy}{(T_{2m}+b_1)} \int_0^\infty e^{-z} \frac{z^{(n-m+a_2-1)}}{(T_{2n}-T_{2m}+b_2)^{(n-m+a_2-1)}} \frac{dz}{(T_{2n}-T_{2m}+b_2)}}{\sum_m \int_0^\infty e^{-y} \frac{y^{(m+a_1-1)}}{(T_{2m}+b_1)^{(m+a_1-1)}} \frac{dy}{(T_{2m}+b_1)} \int_0^\infty e^{-z} \frac{z^{(n-m+a_2-1)}}{(T_{2n}-T_{2m}+b_2)^{(n-m+a_2-1)}} \frac{dz}{(T_{2n}-T_{2m}+b_2)}}$$

$$\rho(m|\underline{x}) = \frac{\frac{\Gamma(m+a_1)}{(T_{2m}+b_1)^{(m+a_1)}} \frac{\Gamma(n-m+a_2)}{(T_{2n}-T_{2m}+b_2)^{(n-m+a_2)}}}{\sum_m \frac{\Gamma(m+a_1)}{(T_{2m}+b_1)^{(m+a_1)}} \frac{\Gamma(n-m+a_2)}{(T_{2n}-T_{2m}+b_2)^{(n-m+a_2)}}}$$

$$\rho(m|\underline{x}) = \frac{\frac{\Gamma(m+a_1)}{(T_{2m}+b_1)^{(m+a_1)}} \frac{\Gamma(n-m+a_2)}{(T_{2n}-T_{2m}+b_2)^{(n-m+a_2)}}}{\xi(a_1, a_2, b_1, b_2, m, n)} \qquad (1.5.1.12)$$

The marginal posterior distribution of θ_1, using equations (1.5.1.6)& (1.5.1.7)

$$\rho(\theta_1|\underline{x}) = \frac{L(\theta_1, \theta_2/\underline{x})\, \pi(\theta_1)}{\int_0^\infty L(\theta_1, \theta_2/\underline{x})\, \pi(\theta_1) d\theta_1}$$

$$\rho(\theta_1|\underline{x}) == \frac{\sum_m \int_0^\infty L(\theta_1, \theta_2/\underline{x})\, \pi(\theta_1)\, \pi(\theta_2)\, d\theta_2}{\sum_m \iint_0^\infty L(\theta_1, \theta_2/\underline{x})\, \pi(\theta_1)\, \pi(\theta_2)\, d\theta_1\, d\theta_2}$$

On solving which gives

$$\rho(\theta_1|\underline{x}) == \frac{\sum_m e^{-\theta_1(T_{2m}+b_1)}\, {\theta_1}^{(m+a_1-1)} \int_0^\infty e^{-\theta_2(T_{2n}-T_{2m}+b_2)}\, {\theta_2}^{(n-m+a_2-1)}\, d\theta_2}{\sum_m \int_0^\infty e^{-\theta_1(T_{2m}+b_1)}\, {\theta_1}^{(m+a_1-1)}\, d\theta_1 \int_0^\infty e^{-\theta_2(T_{2n}-T_{2m}+b_2)}\, {\theta_2}^{(n-m+a_2-1)}\, d\theta_2}$$

Assuming $\theta_1(T_{2m} + b_1) = y$ & $\theta_2(T_{2n} - T_{2m} + b_2) = z$

$\theta_1 = \frac{y}{(T_{2m}+b_1)}$ & $\theta_2 = \frac{z}{T_{2n}-T_{2m}+b_2}$

$d\theta_1 = \frac{dy}{(T_{2m}+b_1)}$ & $d\theta_2 = \frac{dz}{T_{2n}-T_{2m}+b_2}$

$$\rho(\theta_1|\underline{x}) == \frac{\sum_m e^{-\theta_1(T_{2m}+b_1)}\, {\theta_1}^{(m+a_1-1)} \int_0^\infty e^{-z} \frac{z^{(n-m+a_2-1)}}{(T_{2n}-T_{2m}+b_2)^{(n-m+a_2-1)}} \frac{dz}{(T_{2n}-T_{2m}+b_2)}}{\sum_m \int_0^\infty e^{-y} \frac{y^{(m+a_1-1)}}{(T_{2m}+b_1)^{(m+a_1-1)}} \frac{dy}{(T_{2m}+b_1)} \int_0^\infty e^{-z} \frac{z^{(n-m+a_2-1)}}{(T_{2n}-T_{2m}+b_2)^{(n-m+a_2-1)}} \frac{dz}{(T_{2n}-T_{2m}+b_2)}}$$

$$\rho(\theta_1|\underline{x}) == \frac{\sum_m e^{-\theta_1(T_{2m}+b_1)}\, {\theta_1}^{(m+a_1-1)} \frac{\Gamma(n-m+a_2)}{(T_{2n}-T_{2m}+b_2)^{(n-m+a_2)}}}{\sum_m \frac{\Gamma(m+a_1)}{(T_{2m}+b_1)^{(m+a_1)}} \frac{\Gamma(n-m+a_2)}{(T_{2n}-T_{2m}+b_2)^{(n-m+a_2)}}}$$

$$\rho(\theta_1|\underline{x}) == \frac{\sum_m e^{-\theta_1(T_{2m}+b_1)}\, {\theta_1}^{(m+a_1-1)} \frac{\Gamma(n-m+a_2)}{(T_{2n}-T_{2m}+b_2)^{(n-m+a_2)}}}{\xi(a_1, a_2, b_1, b_2, m, n)} \qquad (1.5.1.13)$$

The marginal posterior distribution of θ_2, using the equation (1.5.1.6) & (1.5.1.8) is

$$\rho(\theta_2|\underline{x}) = \frac{L(\theta_1, \theta_2/\underline{x})\, \pi(\theta_2)}{\int_0^\infty L(\theta_1, \theta_2/\underline{x})\, \pi(\theta_2)\, d\theta_2}$$

$$\rho(\theta_2|\underline{x}) == \frac{\sum_m \int_0^\infty L(\theta_1, \theta_2/\underline{x})\, \pi(\theta_1)\, \pi(\theta_2)\, d\theta_1}{\sum_m \iint_0^\infty L(\theta_1, \theta_2/\underline{x})\, \pi(\theta_1)\, \pi(\theta_2)\, d\theta_1\, d\theta_2}$$

$$\rho(\theta_2|\underline{x}) = = \frac{\sum_m e^{-\theta_2(T_{2n}-T_{2m}+b_2)}\, \theta_2{}^{(n-m+a_2-1)} \int_0^\infty e^{-\theta_1(T_{2m}+b_1)}\, \theta_1{}^{(m+a_1-1)}\, d\theta_1}{\sum_m \int_0^\infty e^{-\theta_1(T_{2m}+b_1)}\, \theta_1{}^{(m+a_1-1)}\, d\theta_1 \int_0^\infty e^{-\theta_2(T_{2n}-T_{2m}+b_2)}\, \theta_2{}^{(n-m+a_2-1)}\, d\theta_2}$$

Assuming $\theta_1(T_{2m}+b_1) = y$ & $\theta_1 = \frac{y}{(T_{2m}+b_1)}$

$$\rho(\theta_2|\underline{x}) = = \frac{\sum_m e^{-\theta_2(T_{2n}-T_{2m}+b_2)}\, \theta_2{}^{(n-m+a_2-1)} \int_0^\infty e^{-y} \frac{y^{(m+a_1-1)}}{(T_{2m}+b_1)^{(m+a_1-1)}} \frac{dy}{(T_{2m}+b_1)}}{\sum_m \int_0^\infty e^{-y} \frac{y^{(m+a_1-1)}}{(T_{2m}+b_1)^{(m+a_1-1)}} \frac{dy}{(T_{2m}+b_1)} \int_0^\infty e^{-z} \frac{z^{(n-m+a_2-1)}}{(T_{2n}-T_{2m}+b_2)^{(n-m+a_2-1)}} \frac{dz}{(T_{2n}-T_{2m}+b_2)}}$$

$$\rho(\theta_2|\underline{x}) = = \frac{\sum_m \frac{\Gamma(m+a_1)}{(T_{2m}+b_1)^{(m+a_1)}} e^{-\theta_2(T_{2n}-T_{2m}+b_2)}\, \theta_2{}^{(n-m+a_2-1)}}{\sum_m \frac{\Gamma(m+a_1)}{(T_{2m}+b_1)^{(m+a_1)}} \frac{\Gamma(n-m+a_2)}{(T_{2n}-T_{2m}+b_2)^{(n-m+a_2)}}}$$

$$\rho(\theta_2|\underline{x}) = \frac{\sum_m \frac{\Gamma(m+a_1)}{(T_{2m}+b_1)^{(m+a_1)}} e^{-\theta_2(T_{2n}-T_{2m}+b_2)}\, \theta_2{}^{(n-m+a_2-1)}}{\xi(a_1,a_2,b_1,b_2,m,n)} \qquad (1.5.1.14)$$

1. 5. 2 Bayes Estimators under General Entropy Loss Function (GELF)

Occasionally, the use of symmetric loss function, namely SELF, was found to inappropriate, since for example, an overestimation of the reliability function usually much more serious than an underestimation. Here was consider asymmetric loss function namely GELF proposed by Calabria and Pulcini (1994), is given by

$$L_5(\theta, d) = \left(\frac{d}{\theta}\right)^{\alpha_2} - \alpha_2 \ln\left(\frac{d}{\theta}\right) - 1 \;;\; (\alpha_2 \neq 0) \qquad (1.5.2.1)$$

Where as for the change point m, the loss function is defined as

$$L_5(m, \widehat{m}_{BE}) = \left(\frac{\widehat{m}_{BE}}{m}\right)^{\alpha_2} - \alpha_2 \ln\left(\frac{\widehat{m}_{BE}}{m}\right) - 1 \;;\; (\alpha_2 \neq 0) \qquad (1.5.2.2)$$

where, $\alpha_2 \neq 0$, m = 1,2,….(n-1), and $\widehat{m}_{BE} = 1,2, \ldots\ldots (n-1)$ Here, $\hat{\theta}_{BE}$ is the smallest integer greater than the analytical solution. The sign of the shape parameter $\alpha_2 > 0$, if over estimation is more serious than under estimation, and vice versa, and the magnitude of α_2 reflects the degree of asymmtery. The Bayes estimator of θ under the GELF is given by

$$\hat{\theta}_{BE} = \left[E_\rho(\theta^{-k_2})\right]^{1/k_2} \qquad (1.5.2.3)$$

The Bayes estimate $\widehat{m}_{BE}$ of m under GELF using marginal posterior distribution equation (1.5.1.13), we get as $\widehat{m}_{BE} = \left[\sum_m m^{-k_2}\, \rho(m|\underline{x})\right]^{-1/k_2}$

$$\widehat{m}_{BE} = \left[\frac{\sum_m m^{-k_2} \frac{\Gamma(m+a_1)}{(T_{2m}+b_1)^{(m+a_1)}} \frac{\Gamma(n-m+a_2)}{(T_{2n}-T_{2m}+b_2)^{(n-m+a_2)}}}{\xi(a_1,a_2,b_1,b_2,m,n)}\right]^{-1/k_2} \qquad (1.5.2.4)$$

The Bayes Estimate $\hat{\theta}_{1BE}$ of θ_1 under GELF using marginal posterior distribution equation (1.5.1.14), we get

$$\hat{\theta}_{1BE} = \left[E_\rho\left({\theta_1}^{-k_2}\right)\right]^{-1/k_2}$$

$$\hat{\theta}_{1BE} = \left[\frac{\sum_m {\theta_1}^{-k_2} e^{-\theta_1(T_{2m}+b_1)}\, {\theta_1}^{(m+a_1-1)} \dfrac{\Gamma(n-m+a_2)}{(T_{2n}-T_{2m}+b_2)^{(n-m+a_2)}}}{\xi(a_1,a_2,b_1,b_2,m,n)}\right]^{-1/k_2}$$

$$\hat{\theta}_{1BE} = \left[\frac{\sum_m \frac{\Gamma(n-m+a_2)}{(T_{2n}-T_{2m}+b_2)^{(n-m+a_2)}} \int_0^\infty e^{-\theta_1(T_{2m}+b_1)}\, {\theta_1}^{(m+a_1-k_2-1)} d\theta_1}{\xi(a_1,a_2,b_1,b_2,m,n)}\right]^{-1/k_2}$$

Assuming $\theta_1(T_{2m}+b_1) = y$ & $\theta_1 = \frac{y}{(T_{2m}+b_1)}$

$$\hat{\theta}_{1BE} = \left[\frac{\sum_m \frac{\Gamma(n-m+a_2)}{(T_{2n}-T_{2m}+b_2)^{(n-m+a_2)}} \int_0^\infty e^{-y} \frac{y^{(m+a_1-k_2-1)}}{(T_{2m}+b_1)^{(m+a_1-k_2-1)}} \frac{dy}{(T_{2m}+b_1)}}{\xi(a_1,a_2,b_1,b_2,m,n)}\right]^{-1/k_2}$$

$$\hat{\theta}_{1BE} = \left[\frac{\sum_m \frac{\Gamma(m+a_1-k_2)}{(T_{2m}+b_1)^{(m+a_1-k_2)}} \frac{\Gamma(n-m+a_2)}{(T_{2n}-T_{2m}+b_2)^{(n-m+a_2)}}}{\xi(a_1,a_2,b_1,b_2,m,n)}\right]^{-1/k_2}$$

$$\hat{\theta}_{1BE} = \left[\frac{\xi[(a_1-k_2),a_2,b_1,b_2,m,n]}{\xi(a_1,a_2,b_1,b_2,m,n)}\right]^{-1/k_2} \qquad (1.5.2.5)$$

The Bayes Estimate $\hat{\theta}_{2BE}$ of θ_2 under GELF using marginal posterior distribution equation (1.5.1.15), we get

$$\hat{\theta}_{2BE} = \left[E_\rho\left({\theta_2}^{-k_2}\right)\right]^{-1/k_2}$$

$$\hat{\theta}_{2BE} = \left[\frac{\sum_m {\theta_2}^{-k_2} \frac{\Gamma(m+a_1)}{(T_{2m}+b_1)^{(m+a_1)}} e^{-\theta_2(T_{2n}-T_{2m}+b_2)}\, {\theta_2}^{(n-m+a_2-1)}}{\xi(a_1,a_2,b_1,b_2,m,n)}\right]^{-1/k_2}$$

$$\hat{\theta}_{2BE} = \left[\frac{\sum_m \frac{\Gamma(m+a_1)}{(T_{2m}+b_1)^{(m+a_1)}} \int_0^\infty e^{-\theta_2(T_{2n}-T_{2m}+b_2)}\, {\theta_2}^{(n-m+a_2-k_2-1)} d\theta_2}{\xi(a_1,a_2,b_1,b_2,m,n)}\right]^{-1/k_2}$$

Assuming $\theta_2(T_{2n}-T_{2m}+b_2) = y$ & $\theta_2 = \frac{y}{(T_{2n}-T_{2m}+b_2)}$

Then $$\hat{\theta}_{2BE} = \left[\frac{\sum_m \frac{\Gamma(m+a_1)}{(T_{2m}+b_1)^{(m+a_1)}} \int_0^\infty \mathrm{e}^{-y} \frac{y^{(n-m+a_2-k_2-1)}}{(T_{2n}-T_{2m}+b_2)^{(n-m+a_2-k_2-1)}} \frac{dy}{(T_{2n}-T_{2m}+b_2)}}{\xi(a_1,a_2,b_1,b_2,m,n)}\right]^{-1/k_2}$$

$$\hat{\theta}_{2BE} = \left[\frac{\sum_m \frac{\Gamma(m+a_1)}{(T_{2m}+b_1)^{(m+a_1)}} \frac{\Gamma(n-m+a_2-k_2)}{(T_{2n}-T_{2m}+b_2)^{(n-m+a_2-k_2)}}}{\xi(a_1,a_2,b_1,b_2,m,n)}\right]^{-1/k_2}$$

$$\hat{\theta}_{2BE} = \left[\frac{\xi[a_1,(a_2-k_2),b_1,b_2,m,n]}{\xi(a_1,a_2,b_1,b_2,m,n)}\right]^{-1/k_2} \qquad (1.5.2.6)$$

Numerical Comparison for Exponentiated Inverted Weibull Distribution

We have generated 20 random observations from Exponentiated Inverted Weibull distribution with parameter θ = 2 and β = 0.5. The observed data mean is μ = 1.5616 and variance σ^2 = 0.6812. Let the change in sequence is at 11th observation, so the means and variances of both sequences ($x_1, x_2, \ldots, x_m$) and ($x_{(m+1)}, x_{(m+2)}, \ldots, x_n$) are μ_1 = 1.5491, μ_2 = 1.5768, σ_1^2 = 1.0197 and σ_2^2 = 0.3427. If the target value of μ_1 is unknown, its estimating ($\hat{\mu}_1$) is given by the mean of first m sample observation given m=11, μ = 1.5491.

Sensitivity Analysis of Bayes Estimates

In this section we have studied the sensitivity of the Bayes estimates with respect to changes in the parameters of prior distribution a_1, b_1, a_2 and b_2. The means and variances of the prior distribution are used as prior information in computing these parameters. Then with these parameter values we have computed the Bayes estimates of m, θ_1 and θ_2 under GELF considering different set of values of (a_1, b_1) and (a_2, b_2).We have also considered the other values like parameter of loss function α_2= - 2 and different sample sizes n=10(10)30. The Bayes estimates of the change point 'm' and the parameters θ_1 and θ_2 are given in table-(1) under GELF. Their respective mean squared errors (M.S.E's) are calculated by repeating this process 1000 times and presented in same table in small parenthesis under the estimated values of parameters. All these values appears to be robust with respect to correct choice of prior parameter values and appropriate sample size. All the estimators perform better with sample size n=20 .Similarly the Bayes estimates of GELF are presented in table (1) appears to be sensitive with wrong choice of prior parameters and sample size. All the calculations are done by R- programming. From the below table we conclude that –

The Bayes estimates of the parameters θ_1 and θ_2 of EIW obtained with loss function GELF have more or less same numerical values. The respective M.S.E's shows that the Bayes estimates are uniformly smaller of $\hat{\theta}_{1BE}$ and $\hat{\theta}_{2BE}$ under GELF except of $\hat{m}_{BE}$.

Table 1.1

Bayes Estimates of m, θ_1 & θ_2 for EIW sequences and their respective M.S.E.'s Under GELF

(a_1, b_1)	(a_2, b_2)	n	$\hat{m}_{BE}$	$\hat{\theta}_{1BE}$	$\hat{\theta}_{2BE}$
(1.25,1.50)	(1.50,1.60)	10	2.3457 **(0.1234)**	0.2427 **(1.6181)**	0.4926 **(1.2458)**
		20	3.0864 **(0.0666)**	0.1283 **(1.6788)**	0.3939 **(1.2641)**
		30	2.3416 **(0.0424)**	0.1096 **(1.7039)**	0.4288 **(1.2167)**
(1.50,1.75)	(1.70,1.80)	10	2.2426 **(0.1263)**	0.2909 **(1.5788)**	0.5732 **(1.2447)**
		20	3.3456 **(0.0153)**	0.1152 **(1.7600)**	0.5044 **(1.0572)**
		30	3.2059 **(0.0069)**	0.0811 **(1.7054)**	0.3582 **(1.0290)**
(1.75,2.0)	(1.90,2.0)	10	2.3890	0.1870	0.5096

			(0.0013)	**(1.7158)**	**(1.2211)**
		20	3.5386 **(0.1222)**	0.1289 **(1.7094)**	0.5053 **(1.4499)**
		30	7.7266 **(0.0017)**	0.1656 **(1.7046)**	0.3501 **(1.0839)**
(2.0,2.25)	(2.10,2.20)	10	2.3171 **(0.0275)**	0.1793 **(1.6589)**	0.3901 **(1.1961)**
		20	2.8099 **(0.0474)**	0.1679 **(1.6708)**	0.5211 **(1.3038)**
		30	3.7478 **(0.3713)**	0.1086 **(1.5872)**	0.3994 **(1.3689)**
(2.25,2.50)	(2.30,2.40)	10	2.5595 **(0.0930)**	0.2216 **(1.6251)**	0.4293 **(1.2699)**
		20	3.0246 **(0.3996)**	0.1462 **(1.7026)**	0.5035 **(1.1441)**
		30	3.6281 **(1.5774)**	0.1026 **(1.80453)**	0.3350 **(1.2933)**
(2.50,2.75)	(2.50,2.60)	10	1.9741 **(0.1213)**	0.1652 **(1.7217)**	0.5663 **(1.1375)**
		20	3.6849 **(1.3200)**	0.1428 **(1.7733)**	0.4229 **(1.3479)**
		30	3.3825 **(2.1143)**	0.1335 **(1.8388)**	0.4328 **(1.3039)**

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