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ON THE NEGATIVE PELL EQUATION

$$y^2 = 13x^2 - 12$$

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Abstract: This paper concerns with the problem of obtaining infinitely many non-zero distinct integer solutions of the binary quadratic Diophantine equation $y^2 = 13x^2 - 12$. Employing the lemma of Brahmagupta, we find infinitely many integral solutions of the above equation. The recurrence relations on the solutions are presented. An interesting relations among the solutions are presented.

Key Words: *Binary quadratic, Pell equation, integral solutions.*
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I. Introduction

Equations with integer co-efficient which are to be solved integer are called Diophanine equations. Consider the linear Diophantine equation $2x+3y=4$ has $(-1,2)$ as a solution. In fact it has finitely many solutions $(2+3t,-2t)$, where t is an arbitrary integer. A Pell equation is a type of non-liear Diophantine equation in the form $y^2 = Dx^2 \pm 1$, where D is non-square positive integer. when D takes different integral values [1-4]. For an extensive review of various problems, one may refer [5-10]. The above equation is also called the Pell Fermat equation.

In this communication, we obtain infinitely many integer solutions to the negative Pell equation $y^2 = 13x^2 - 12$. We also obtained different relations among the solutions, different choices of hyperbolas, parabolas and Pythagorean triangle together with their solutions

II. METHOD OF ANALYSIS:

The Negative Pell equation under consideration is

$$y^2 = 13x^2 - 12 \tag{1.1}$$

whose smallest positive integer solution is

$$x_0 = 1,\ y_0 = 1$$

To obtain the other solutions of (1.1), consider the Pell equation

$$y^2 = 13x^2 + 1 \tag{1.2}$$

whose general solution is given by

$$\tilde{x}_n = \frac{1}{2\sqrt{13}} g_n$$

$$\tilde{y}_n = \frac{1}{2} f_n$$

where

$$f_n = \left(649+180\sqrt{13}\right)^{n+1} + \left(649-180\sqrt{13}\right)^{n+1},$$

$$g_n = \left(649+180\sqrt{13}\right)^{n+1} - \left(649-180\sqrt{13}\right)^{n+1},\ n=-1,0,1...$$

Applying Brahmagupta lemma between (x_0, y_0) and $(\tilde{x}_n, \tilde{y}_n)$, the other integer solutions of (1.1) are given by

$$x_{n+1} = \frac{\sqrt{13}\, f_n + g_n}{2}$$

$$y_{n+1} = \frac{f_n + \sqrt{13}\, g_n}{2}$$

$$\Rightarrow\ 2\sqrt{13}x_{n+1} = \sqrt{13} f_n + g_n \qquad (1.3)$$

$$2\sqrt{13} y_{n+1} = \sqrt{13} f_n + 13 g_n \qquad (1.4)$$

A few numerical examples are given in the following Table: 1.1

Table: 1.1 Numerical Examples

n	x_{n+1}	y_{n+1}
-1	1	1
0	829	2989
1	1076041	3879721
2	1396700389	5035874869
3	1812916028881	6536561700241
4	2353163608787149	6824452051037949
5	3054404551289690521	11012812225685557561

The recurrence relations satisfied by the values of x_{n+1} and y_{n+1} are respectively,

$$x_{n+3} - 1298x_{n+2} + x_{n+1} = 0\ ,\ n = -1,0,1....$$

$$y_{n+3} - 1298y_{n+2} + y_{n+1} = 0,\ n = -1,0,1....$$

1. A few relations among the solutions are given below:

(i). x_{n+1} and y_{n+1} are always odd.

(ii). $x_{n+1} - y_{n+1} \equiv 0 \pmod 2$

2. Relations among the solutions:

- $649x_{n+2} - x_{n+1} - 180y_{n+2} = 0$
- $842401x_{n+2} - 649x_{n+1} - 180y_{n+3} = 0$
- $x_{n+3} - 842401x_{n+1} - 233640y_{n+1} = 0$
- $x_{n+3} - x_{n+1} - 360y_{n+2} = 0$
- $2340x_{n+1} + 649y_{n+1} - y_{n+2} = 0$
- $3037320x_{n+1} + 842401y_{n+1} - y_{n+3} = 0$
- $2340x_{n+1} + 842401y_{n+2} - 649y_{n+3} = 0$
- $649x_{n+3} - 842401x_{n+2} - 180y_{n+1} = 0$
- $x_{n+3} - 649x_{n+2} - 180y_{n+2} = 0$
- $649x_{n+3} - 180y_{n+3} - x_{n+2} = 0$
- $2340x_{n+2} + y_{n+1} - 649y_{n+2} = 0$
- $4680x_{n+2} + y_{n+1} - y_{n+3} = 0$
- $2340x_{n+2} + 649y_{n+2} - y_{n+3} = 0$
- $649x_{n+3} - 180y_{n+1} - 842401x_{n+2} = 0$
- $649y_{n+1} + 2340x_{n+3} - 842401y_{n+2} = 0$
- $y_{n+1} + 3037320x_{n+3} - 842401y_{n+3} = 0$
- $y_{n+2} + 2340x_{n+3} - 649y_{n+3} = 0$
- $842401y_{n+2} - 649x_{n+1} - 2340x_{n+3} = 0$
- $649x_{n+3} - 180y_{n+3} - x_{n+2} = 0$

3. Each of the following expressions represents a Nasty number:

- $\frac{1}{1401840}\left(23278326x_{2n+2}-6x_{2n+4}+16822080\right)$
- $\frac{1}{6}\left(78x_{2n+2}-6y_{2n+2}+72\right)$
- $\frac{1}{3894}\left(64662x_{2n+2}-6y_{2n+3}+46728\right)$
- $\frac{1}{5054406}\left(83931198x_{2n+2}-6y_{2n+4}+60652872\right)$
- $\frac{1}{1080}\left(23278326x_{2n+3}-17934x_{2n+4}+12960\right)$
- $\frac{1}{3894}\left(78x_{2n+3}-17934y_{2n+2}+46728\right)$
- $\frac{1}{6}\left(64662x_{2n+3}-17934y_{2n+3}+72\right)$
- $\frac{1}{3894}\left(83931198x_{2n+3}-17934y_{2n+4}+46728\right)$
- $\frac{1}{5054406}\left(78x_{2n+4}-23278326y_{2n+2}+60652872\right)$
- $\frac{1}{3894}\left(10777x_{2n+4}-3879721y_{2n+3}+46728\right)$
- $\frac{1}{6}\left(83931198x_{2n+4}-232783267y_{2n+4}+72\right)$
- $\frac{1}{14040}\left(78y_{2n+3}-64662y_{2n+2}+168480\right)$
- $\frac{1}{18223920}\left(78y_{2n+4}-83931198y_{2n+2}+218687040\right)$
- $\frac{1}{14040}\left(64662x_{2n+4}-83931198y_{2n+3}+168480\right)$
- $\frac{1}{1080}\left(17934x_{2n+2}-6x_{2n+3}+12960\right)$

4. Each of the following expressions represents a Cubical Integer:

- $\frac{1}{1401840}\left[3879721x_{3n+3}-x_{3n+5}+11639163x_{n+1}-3x_{n+3}\right]$
- $\frac{1}{6}\left[13x_{3n+3}-y_{3n+3}+39x_{n+1}-3y_{n+1}\right]$
- $\frac{1}{3894}\left[10777x_{3n+3}-y_{3n+4}+32331x_{n+1}-3y_{n+2}\right]$
- $\frac{1}{5054406}\left[13988533x_{3n+3}-y_{3n+5}+41965599x_{n+1}-3y_{n+3}\right]$
- $\frac{1}{1080}\left[3879721x_{3n+4}-2989x_{3n+5}+11639163x_{n+2}-8967x_{n+3}\right]$

- $\frac{1}{3894}\left[13x_{3n+4}-2989y_{3n+3}+39x_{n+2}-8967y_{n+1}\right]$
- $\frac{1}{6}\left[10777x_{3n+4}-2989y_{3n+4}+32331x_{n+2}-8967y_{n+2}\right]$
- $\frac{1}{3894}\left[13988533x_{3n+4}-2989y_{3n+5}+41965599x_{n+2}-8967y_{n+3}\right]$
- $\frac{1}{5054406}\left[13x_{3n+5}-3879721y_{3n+3}+39x_{n+3}-11639163y_{n+1}\right]$
- $\frac{1}{3894}\left[10777x_{3n+5}-3879721y_{3n+4}+32331x_{n+3}-11639163y_{n+2}\right]$
- $\frac{1}{6}\left[13988533x_{3n+5}-3879721y_{3n+5}+41965599x_{n+3}-11639163y_{n+3}\right]$
- $\frac{1}{14040}\left[10777y_{3n+3}-13y_{3n+4}+32331y_{n+1}-39y_{n+2}\right]$
- $\frac{1}{18223920}\left[13y_{3n+5}-13988533y_{3n+3}+39y_{n+3}-41965599y_{n+1}\right]$
- $\frac{1}{14040}\left[10777y_{3n+5}-13988533y_{3n+4}+32331y_{n+3}-41965599y_{n+2}\right]$
- $\frac{1}{1080}\left[2989x_{3n+3}-x_{3n+4}+8967x_{n+1}-3x_{n+2}\right]$

5. Each of the following expressions represents a Bi-quadratic integer:

- $\frac{1}{1401840}\left[\begin{array}{l}3879721x_{4n+4}-x_{4n+6}+15518884x_{2n+2}-4x_{2n+4}\\+8411040\end{array}\right]$
- $\frac{1}{14040}\left[\begin{array}{l}10777y_{4n+4}-13y_{4n+5}+52y_{2n+3}-43108y_{2n+2}\\+84240\end{array}\right]$
- $\frac{1}{18223920}\left[\begin{array}{l}13y_{4n+6}+13988533y_{4n+4}+48y_{2n+4}-55954132y_{2n+2}\\+109343520\end{array}\right]$
- $\frac{1}{5054406}\left[\begin{array}{l}13988533x_{4n+4}-y_{4n+6}+55954132x_{2n+2}-4y_{2n+4}\\+30326436\end{array}\right]$
- $\frac{1}{1080}\left[\begin{array}{l}3879721x_{4n+5}-2989x_{4n+6}+15518884x_{2n+3}-11956x_{2n+4}\\+6480\end{array}\right]$
- $\frac{1}{3894}\left[13x_{4n+5}-2989y_{4n+4}+52x_{2n+3}-11956y_{2n+2}+23364\right]$
- $\frac{1}{6}\left[10777x_{4n+5}-2989y_{4n+5}+43108x_{2n+3}-11956y_{2n+3}+36\right]$
- $\frac{1}{3894}\left[\begin{array}{l}13988533x_{4n+5}-2989y_{4n+6}+55954132x_{2n+3}-11956y_{2n+4}\\+23364\end{array}\right]$
- $\frac{1}{5054406}\left[\begin{array}{l}13x_{4n+6}-3879721y_{4n+4}+52x_{2n+4}-15518884y_{2n+2}\\+30326436\end{array}\right]$

- $\frac{1}{3894}\begin{bmatrix}10777x_{4n+6}-3879721y_{4n+5}+43108x_{2n+4}-15518884y_{2n+3}\\+23364\end{bmatrix}$
- $\frac{1}{6}[13x_{4n+4}-y_{4n+4}+52x_{2n+2}-4y_{2n+2}+36]$
- $\frac{1}{3894}[10777x_{4n+4}-y_{4n+5}+43108x_{2n+2}-4y_{2n+3}+23364]$
- $\frac{1}{1080}[2989x_{4n+4}-x_{4n+5}+11956x_{2n+2}-4x_{2n+3}+6480]$

6. Each of the following expressions represents a Quintic integer:

- $\frac{1}{14040}\begin{bmatrix}107770y_{n+3}-139885330y_{n+2}+53885y_{3n+5}-69942665y_{3n+4}\\+10777y_{5n+7}-13988533y_{5n+6}\end{bmatrix}$
- $\frac{1}{1401840}\begin{bmatrix}38797210x_{n+1}-10x_{n+3}+19398605x_{3n+3}-5x_{3n+5}\\+3879721x_{5n+5}-x_{5n+7}\end{bmatrix}$
- $\frac{1}{14040}\begin{bmatrix}130y_{n+2}-107770y_{n+1}+65y_{3n+4}-53885y_{3n+3}+13y_{5n+6}\\-10777y_{5n+5}\end{bmatrix}$
- $\frac{1}{18223920}\begin{bmatrix}130y_{n+3}-139885330y_{n+1}+165y_{3n+5}-69942665y_{3n+3}\\+13y_{5n+7}-13988533y_{5n+5}\end{bmatrix}$
- $\frac{1}{3894}\begin{bmatrix}107770x_{n+1}-10y_{n+2}+53885x_{3n+3}-5y_{3n+4}\\+10777x_{5n+5}-y_{5n+6}\end{bmatrix}$
- $\frac{1}{5054406}\begin{bmatrix}139885330x_{n+1}-10y_{n+3}+69942665x_{3n+3}-5y_{3n+5}\\+13988533x_{5n+5}-y_{5n+7}\end{bmatrix}$
- $\frac{1}{1080}\begin{bmatrix}38797210x_{n+2}-29890x_{n+3}+19398605x_{3n+4}-14945x_{3n+5}\\+3879721x_{5n+6}-2989x_{5n+7}\end{bmatrix}$
- $\frac{1}{3894}\begin{bmatrix}130x_{n+2}-29890y_{n+1}+65x_{3n+4}-14945y_{3n+3}+13x_{5n+6}\\-2989y_{5n+5}\end{bmatrix}$
- $\frac{1}{6}[130x_{n+1}-10y_{n+1}+65x_{3n+3}-5y_{3n+3}+13x_{5n+5}-y_{5n+5}]$
- $\frac{1}{6}\begin{bmatrix}107770x_{n+2}-29890y_{n+2}+53885x_{3n+4}-14945y_{3n+4}\\+10777x_{5n+6}-2989y_{5n+6}\end{bmatrix}$
- $\frac{1}{3894}\begin{bmatrix}139885330x_{n+2}-29890y_{n+3}+69942665x_{3n+4}-14945x_{3n+5}\\13988533x_{5n+6}-2989y_{5n+7}\end{bmatrix}$
- $\frac{1}{5054406}\begin{bmatrix}130x_{n+3}-38797210y_{n+1}+65x_{3n+5}-19398605y_{3n+3}\\+13x_{5n+7}-3879721y_{5n+5}\end{bmatrix}$
- $\frac{1}{3894}\begin{bmatrix}107770x_{n+3}-38797210y_{n+2}+53885x_{3n+5}-19398605y_{3n+4}\\+10777x_{5n+7}-3879721y_{5n+6}\end{bmatrix}$
- $\frac{1}{1080}[2989x_{5n+5}-x_{5n+6}+14945x_{3n+3}-5x_{3n+4}+29890x_{n+1}-10x_{n+2}]$

➢ $$\frac{1}{6}\begin{bmatrix} 139885330x_{n+3} - 38797210y_{n+3} + 69942665x_{3n+5} - 19398605y_{3n+5} \\ +13988533x_{5n+7} - 3879721y_{5n+7} \end{bmatrix}$$

III. REMARKABLE OBSERVATIONS:

1. Employing linear combinations among the solutions of (1.1), one may generate integer solutions for other choices of hyperbola which are presented below:

Table: 1.2 Hyperbolas

S. No	Hyperbolas	(X, Y)
1	$X^2 - 13Y^2 = 7860621542400$	$\begin{pmatrix} 3879721x_{n+1} - x_{n+3}, \\ x_{n+3} - 1076041x_{n+1} \end{pmatrix}$
2	$X^2 - 13Y^2 = 144$	$\begin{pmatrix} 13x_{n+1} - y_{n+1}, \\ y_{n+1} - x_{n+1} \end{pmatrix}$
3	$X^2 - 13Y^2 = 60652944$	$\begin{pmatrix} 10777x_{n+1} - y_{n+2}, \\ y_{n+2} - 2989x_{n+1} \end{pmatrix}$
4	$X^2 - 13Y^2 = 102188080051344$	$\begin{pmatrix} 13988533x_{n+1} - y_{n+3}, \\ y_{n+3} - 3879721x_{n+1} \end{pmatrix}$
5	$X^2 - 13Y^2 = 4665600$	$\begin{pmatrix} 3879721x_{n+2} - 2989x_{n+3}, \\ 829x_{n+3} - 1076041x_{n+2} \end{pmatrix}$
6	$X^2 - 13Y^2 = 60652944$	$\begin{pmatrix} 13x_{n+2} - 2989y_{n+1}, \\ 829y_{n+1} - x_{n+2} \end{pmatrix}$
7	$X^2 - 13Y^2 = 144$	$\begin{pmatrix} 10777x_{n+2} - 2989y_{n+2}, \\ 829y_{n+2} - 2989x_{n+2} \end{pmatrix}$
8	$X^2 - 13Y^2 = 60652944$	$\begin{pmatrix} 13988533x_{n+2} - 2989y_{n+3}, \\ 829y_{n+3} - 3879721x_{n+2} \end{pmatrix}$
9	$X^2 - 13Y^2 = 102188080051344$	$\begin{pmatrix} 13x_{n+3} - 3879721y_{n+1}, \\ 1076041y_{n+1} - x_{n+3} \end{pmatrix}$
10	$X^2 - 13Y^2 = 60652944$	$\begin{pmatrix} 10777x_{n+3} - 3879721y_{n+2}, \\ 1076041y_{n+2} - 2989x_{n+3} \end{pmatrix}$
11	$X^2 - 13Y^2 = 144$	$\begin{pmatrix} 13988533x_{n+3} - 3879721y_{n+3}, \\ 1076041y_{n+3} - 3879721x_{n+3} \end{pmatrix}$

12	$X^2 - 13Y^2 = 788486400$	$\left(13y_{n+2} - 10777y_{n+1},\ 2989y_{n+1} - y_{n+2}\right)$
13	$X^2 - 13Y^2 = 1328445040665600$	$\left(13y_{n+3} - 13988533y_{n+1},\ 3879721y_{n+1} - y_{n+3}\right)$
14	$X^2 - 13Y^2 = 788486400$	$\left(10777y_{n+3} - 13988533y_{n+2},\ 3879721y_{n+2} - 2989y_{n+3}\right)$

2. Employing linear combinations among the solutions of (1.1), one may generate integer solutions for other choices of parabolas which are presented below:

Table: 1.3 Parabolas

S. No	Parabolas	(X, Y)
1	$1401840X - 13Y^2 = 7860621542400$	$\left(3879721x_{n+1} - x_{n+3},\ x_{n+3} - 1076041x_{n+1}\right)$
2	$6X - 13Y^2 = 144$	$\left(13x_{2n+2} - y_{2n+2},\ y_{n+1} - x_{n+1}\right)$
3	$3894X - 13Y^2 = 60652944$	$\left(10777x_{2n+2} - y_{2n+3},\ y_{n+2} - 2989x_{n+1}\right)$
4	$5054406X - 13Y^2 = 1021880800051344$	$\left(13988533x_{2n+2} - y_{2n+4},\ y_{n+3} - 3879721x_{n+1}\right)$
5	$1080X - 13Y^2 = 4665600$	$\left(3879721x_{2n+3} - 2989x_{2n+4},\ 829x_{n+3} - 1076041x_{n+2}\right)$
6	$3894X - 13Y^2 = 60652944$	$\left(13x_{2n+3} - 2989y_{2n+2},\ 829y_{n+1} - x_{n+2}\right)$
7	$6X - 13Y^2 = 144$	$\left(10777x_{2n+3} - 2989y_{2n+3},\ 829y_{n+2} - 2989x_{n+2}\right)$

8	$3894X - 13Y^2 = 60652944$	$\left(13988533x_{2n+3} - 2989y_{2n+4}, 829y_{n+3} - 3879721x_{n+2}\right)$
9	$5054406X - 13Y^2 = 144$	$\left(13x_{2n+4} - 3879721y_{2n+2}, 1076041y_{n+1} - x_{n+3}\right)$
10	$3894X - 13Y^2 = 60652944$	$\left(10777x_{2n+4} - 3879721y_{2n+3}, 1076041y_{n+2} - 2989x_{n+3}\right)$
11	$6X - 13Y^2 = 144$	$\left(13988533x_{2n+4} - 3879721y_{2n+4}, 1076041y_{n+3} - 3879721x_{n+3}\right)$
12	$14040X - 13Y^2 = 788486400$	$\left(13y_{2n+3} - 10777y_{2n+2}, 2989y_{n+1} - y_{n+2}\right)$
13	$18223920X - 13Y^2 = 1328445040665600$	$\left(13y_{2n+4} - 13988533y_{2n+2}, 3879721y_{n+1} - y_{n+3}\right)$
14	$14040X - 13Y^2 = 788486400$	$\left(10777y_{2n+4} - 13988533y_{2n+3}, 3879721y_{n+2} - 2989y_{n+3}\right)$

3. Employing linear combinations among the solutions of (1.1), one may generate integer solutions for other choices of straight lines which are presented below:

Table: 1.4 Straight lines

S.No	**Straight lines**	(X_n, Y_n)
1	$Y_n = 180X_n$	$\left(2989x_{n+1} - x_{n+2}, 13x_{n+1} - y_{n+1}\right)$
2	$Y_n = \frac{180}{649}X_n$	$\left(2989x_{n+1} - x_{n+2}, 10777x_{n+1} - y_{n+2}\right)$
3	$Y_n = \frac{180}{842401}X_n$	$\left(2989x_{n+1} - x_{n+2}, 13988533x_{n+1} - y_{n+3}\right)$
4	$Y_n = X_n$	$\left(2989x_{n+1} - x_{n+2}, 3879721x_{n+2} - 2989x_{n+3}\right)$
5	$Y_n = \frac{1}{13}X_n$	$\left(2989x_{n+1} - x_{n+2}, 13y_{n+2} - 10777y_{n+1}\right)$

6	$Y_n = \frac{1}{16874} X_n$	$\begin{pmatrix} 2989x_{n+1} - x_{n+2}, \\ 13y_{n+3} - 13988533y_{n+1} \end{pmatrix}$
7	$Y_n = 233640X_n$	$\begin{pmatrix} x_{n+3} - 1076041x_{n+1}, \\ 13x_{n+1} - y_{n+1} \end{pmatrix}$
8	$Y_n = \frac{233640}{649} X_n$	$\begin{pmatrix} x_{n+3} - 1076041x_{n+1}, \\ 10777x_{n+1} - y_{n+2} \end{pmatrix}$
9	$Y_n = \frac{233640}{842401} X_n$	$\begin{pmatrix} x_{n+3} - 1076041x_{n+1}, \\ 13988533x_{n+1} - y_{n+3} \end{pmatrix}$
10	$Y_n = 1298X_n$	$\begin{pmatrix} x_{n+3} - 1076041x_{n+1}, \\ 3879721x_{n+2} - 2989x_{n+3} \end{pmatrix}$
11	$Y_n = 360X_n$	$\begin{pmatrix} x_{n+3} - 1076041x_{n+1}, \\ 13x_{n+2} - 2989y_{n+1} \end{pmatrix}$
12	$Y_n = \frac{1298}{13} X_n$	$\begin{pmatrix} x_{n+3} - 1076041x_{n+1}, \\ 13y_{n+2} - 10777y_{n+1} \end{pmatrix}$
13	$Y_n = \frac{1}{649} X_n$	$\begin{pmatrix} 13x_{n+1} - y_{n+1}, \\ 10777x_{n+1} - y_{n+2} \end{pmatrix}$
14	$Y_n = \frac{1}{842401} X_n$	$\begin{pmatrix} 13x_{n+1} - y_{n+1}, \\ 13988533x_{n+1} - y_{n+3} \end{pmatrix}$
15	$Y_n = \frac{1}{180} X_n$	$\begin{pmatrix} 13x_{n+1} - y_{n+1}, \\ 3879721x_{n+2} - 2989x_{n+3} \end{pmatrix}$
16	$Y_n = \frac{1}{2340} X_n$	$\begin{pmatrix} 13x_{n+1} - y_{n+1}, \\ 13y_{n+2} - 10777y_{n+1} \end{pmatrix}$
17	$Y_n = \frac{1}{3037320} X_n$	$\begin{pmatrix} 13x_{n+1} - y_{n+1}, \\ 13y_{n+3} - 13988533y_{n+1} \end{pmatrix}$
18	$Y_n = \frac{649}{842401} X_n$	$\begin{pmatrix} 10777x_{n+1} - y_{n+2}, \\ 13988533x_{n+1} - y_{n+3} \end{pmatrix}$
19	$Y_n = \frac{649}{180} X_n$	$\begin{pmatrix} 10777x_{n+1} - y_{n+2}, \\ 3879721x_{n+2} - 2989x_{n+3} \end{pmatrix}$

20	$Y_n = 649X_n$	$\begin{pmatrix} 10777x_{n+1} - y_{n+2}, \\ 10777x_{n+2} - 2989y_{n+2} \end{pmatrix}$
21	$Y_n = \frac{649}{2340}X_n$	$\begin{pmatrix} 10777x_{n+1} - y_{n+2}, \\ 13y_{n+2} - 10777y_{n+1} \end{pmatrix}$
22	$Y_n = \frac{649}{3037320}X_n$	$\begin{pmatrix} 10777x_{n+1} - y_{n+2}, \\ 13y_{n+3} - 13988533y_{n+1} \end{pmatrix}$
23	$Y_n = \frac{561601}{120}X_n$	$\begin{pmatrix} 13988533x_{n+1} - y_{n+3}, \\ 3879721x_{n+2} - 2989x_{n+3} \end{pmatrix}$
24	$Y_n = \frac{842401}{649}X_n$	$\begin{pmatrix} 13988533x_{n+1} - y_{n+3}, \\ 13x_{n+2} - 2989y_{n+1} \end{pmatrix}$
25	$Y_n = 842401X_n$	$\begin{pmatrix} 13988533x_{n+1} - y_{n+3}, \\ 10777x_{n+2} - 2989y_{n+2} \end{pmatrix}$
26	$Y_n = \frac{842401}{2340}X_n$	$\begin{pmatrix} 13988533x_{n+1} - y_{n+3}, \\ 13y_{n+2} - 10777y_{n+1} \end{pmatrix}$
27	$Y_n = \frac{842401}{3037320}X_n$	$\begin{pmatrix} 13988533x_{n+1} - y_{n+3}, \\ 13y_{n+3} - 13988533y_{n+1} \end{pmatrix}$

4: Consider $p = x_{n+1} + y_{n+1}$, $q = x_{n+1}$. Note that $p > q > 0$.

Treat p, q as the generators of the Pythagorean triangle $T(X,Y,Z)$

where $X = 2pq, Y = p^2 - q^2, Z = p^2 + q^2$.

Then the following results are obtained:

a) $2X - 13Y + 11Z - 24 = 0.$

b) $\frac{2A}{P} = x_{n+1}y_{n+1}.$

c) $3(Z - Y)$ is a nasty number.

d) $3(X - \frac{4A}{P})$ is a nasty number.

e) $X - \frac{4A}{P} + Y$ is written as the sum of two squares.

IV. CONCLUSION

In this paper, we have presented infinitely many integer solutions for the negative Pell Equations $y^2 = 13x^2 - 12$. As the binary quadratic Diophantine equations are rich in variety, one may search for the other choices of Pell Equations and determine their integer solutions along with suitable properties

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