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Two Dimensional (2-d) Convection and Diffusion Transport Equation of Galactic Cosmic Rays by Linearization with Cole – Hopf Transformation and Conservative Form

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Abstract: Galactic Cosmic Rays originating from space and shows different characteristics when they enter into atmosphere. We are considering Convection and diffusion Equations in 2 Dimensions, when they are orienting in atmosphere. By using Cole – Hopf transformation Linearization 2-D Convection – diffusion Equation we find the Diffusion and Convection Equation of Galactic Cosmic Rays in atmosphere. Convection – Diffusion is phenomenon of fluid convection and diffusion and traffic flow. It is a nonlinear PDE. With the help of discretization method like finite element method to solve the numerical solutions to nonlinear PDE. Present study says that, it is a reference in solving nonlinear PDE with the above said method. By using Conservative Form, we solve nonlinear convection and diffusion Equation.
Key words: Convection, Diffusion, Finite Element Method, Conservative Form, Cole – Hopf transformation.

INTRODUCTION

By using Parker Transport Equation, we derive Convection Diffusion 2 –D Equation for Galactic Cosmic Rays (GCR) by Linearization method. By applying Cole-Hopf Transformation we convert nonlinear partial differential equation (PDE) in to linear partial differential equation, as well we are applying Conservative Form to solve the 2-D Convection Diffusion Equation. With Cole Hopf Transformation and Conservative Form, we derived PDE of Higher Order Numerical Solutions to Diffusion Convection Equations (1), (2) & (9). High order Lagrangian interpolants (33) with the nodal spectral element method (5), (31), we can evaluate Φ and obtain the numerical solutions. We consider the different values of viscosity (μ), initial and boundary conditions to get the solutions. Cole-Hopf transformation is applicable to nonlinear Partial Differential Equations such as hyperbolic Partial Differential Equations with 2nd order derivatives and the Korteweg-de Vries Equation. We can solve numerically the nonlinear terms without linearization or with linearization by using conservative forms.

COLE – HOPF TRANSFORMATION LINEARIZATION 2-D CONVECTION DIFFUSION EQUATION

The known convection Diffusion Equations in one or two dimensions

$$\frac{\partial \alpha}{\partial t} + (\boldsymbol{\alpha}.\nabla)\boldsymbol{\alpha} = \mu\Delta\boldsymbol{\alpha} \qquad \text{in } \Omega \text{ and } t \geq 0 \qquad (1)$$

The Cole-Hopf transformation linearization 2-D Convection Diffusion equation is

$$\alpha(p,q,t) = -2\mu\frac{\Phi_a(p,q,t)}{\Phi(p,q,t)} \qquad (2)$$

$$\beta(p,q,t) = -2\mu\frac{\Phi_b(p,q,t)}{\Phi(p,q,t)} \qquad (3)$$

The convection diffusion equation in 2-d is given by

$$\frac{\partial \alpha}{\partial t} + (\alpha.\nabla)\ \alpha = \mu \Delta \alpha \qquad (4)$$

The equation 3 is to be written in a scalar equation form as

$$\frac{\partial \Phi}{\partial t} = \mu \Delta \Phi \qquad \text{in } \Omega \text{ and } t \geq 0 \qquad (5)$$

With the help of high order Lagrangian interpolants with the nodal spectral element method, we can evaluate Φ in (5) and obtain the numerical solutions. After that we can evaluate the values of Φ_p and Φ_q from Φ by using high order finite difference scheme. From the Equations (2) and (3), we can obtain the values of α and β respectively from Φ. With the help of basic function of K^{th} order Lagrangian interpolants , in order to get the exponential convergence, we are using $(K + 1)^{st}$ order difference scheme to calculate Φ_a and Φ_b.

NUMERICAL RESULTS

The First Set of 2-d Example

If we consider the value of viscosity σ = 0.003 and boundary and initial condition for Equation (4) as follows:

$$\alpha(p, q, 0) = \sin(\pi p).\sin(\pi q);\ \beta(p, q, 0) = q; \qquad \Omega: 0 \leq p, q \leq 1;\ t \geq 0 \quad (6)$$

$$\alpha(p, q, t) = 0, \qquad \beta(p, q, t) = q, \qquad p, q \in \partial\,\Omega \qquad (7)$$

The vector α(p, q, t) can reduced into a scalar in Φ (p, q, t) by using Cole – Hopf transformation. We used Lagrangian interpolants over Gauss-Lobatto-Legendre point s and fifth order basis function of Nodal Spectral Element Method (SEM) were used to solve 2-D Cosmic Ray Solar heat equation. To solve Φ_p and Φ_q we still use the Difference method, Lagrangian interpolation Method. From the Equations (2) and (3), we calculate the solutions to α(p, q). From the figures 2 (a) and (b) gives an idea about the contour lines at t = 0.2 for the velocity components α and β. The Quadrature points and the Elements represented by the figure 2 (c). The vector field of $\bar{\alpha}$ = (α, β) is represented by figure 2 (d). The field vector explains in both y and x directions due to velocity components , β were asymmetric and coupled together.

The Second Set of 2-d Example

In the 2nd set of two dimensional example we can change the domain into Ω = {(p, q) : 0 ≤ p, q ≤ 2π}, we assume μ = 0.004, and with the help of basic conditions

$$\alpha(p, q, 0) = e^{[1-(p-\pi)^2-(q-\pi)^2]} \qquad (8)$$

$$\beta(p, q, 0) = \sin(p).\ \sin(q) \qquad (9)$$

and on the limit 3 Ω the new boundary conditions

$$\alpha(p, q, t) = 0,\ \beta(p, q, 0) = 0,\ p, q \in \partial\Omega \qquad (10)$$

The above set of domain was divided into 20 elements. We can use fifth order basis function of Nodal Spectral Element Method (SEM) were used to solve 2-D Cosmic Ray Solar heat equation and to solve the first derivatives used Lagrange interpolation and compact scheme. At last we get the solutions of Equations (2) and (3). At time t = 1.0 we get the numerical solutions. The Contour lines were shown in figure 3 (a) and (b) for α and β respectively. 20 elements and quadrature points were shown in figure 3 (c). The vector field $\bar{\alpha}$ = (α, β) shown in the figure 3 (d).
The essential blessings of Cole-Hopf transformation are permit an implicit temporal remedy wherein the temporal pitch is unrestricted via way of means of the circumstance of balance and temporal integration is unconditional**.** But Hopf-Cole transformation is applicable to nonlinear Partial Differential Equations such as hyperbolic Partial Differential Equations with 2nd order derivatives and the Korteweg-de Vries Equation.

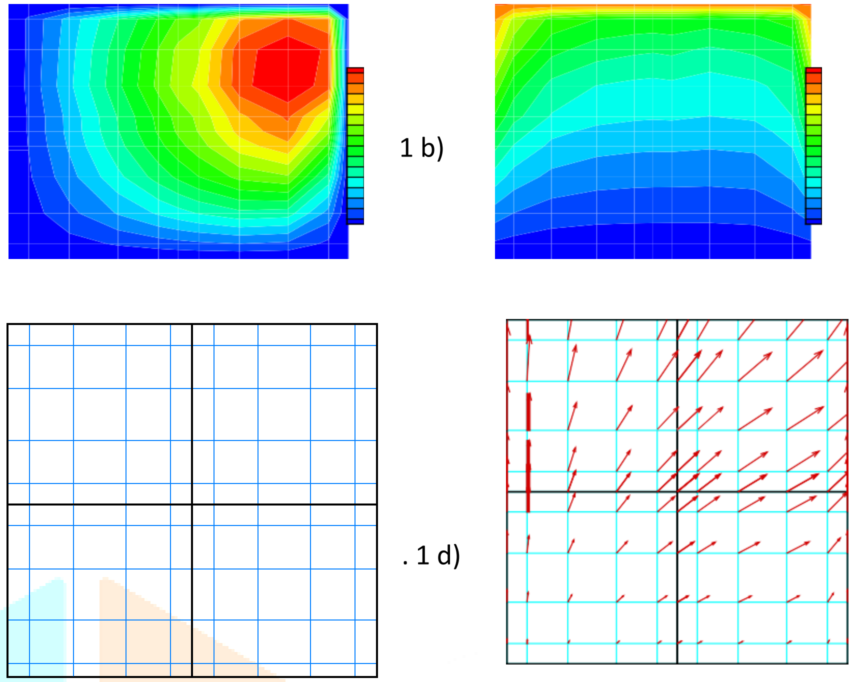


Figure 1: The 2-D Convection Diffusion Equation (First set of example) linearized with Velocity Components α and β the computational elements and quadrant points.

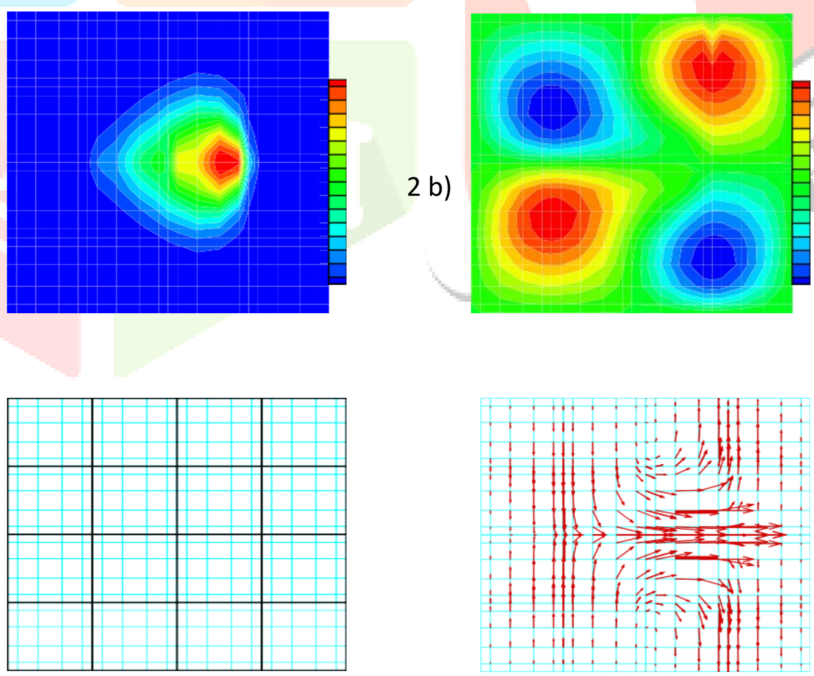


Figure 2: The 2-D Convection Diffusion Equation (Second set of example) linearized with Velocity Components α and β the mesh with quadrature points 15 elements.

2-D DIFFUSION CONVECTION EQUATIONS USING COSERVATIVE FORM

If we consider velocity components α and β along x and y directions, the equation (1) is subject to basic and final conditions and the component form of equation (1) is

$$\frac{\partial\alpha}{\partial t} + \alpha\frac{\partial\alpha}{\partial p} + \beta\frac{\partial\alpha}{\partial q} = \mu\left(\frac{\partial^2\alpha}{\partial p^2} + \frac{\partial^2\alpha}{\partial q^2}\right) \qquad (11)$$

$$\frac{\partial\beta}{\partial t} + \alpha\frac{\partial\beta}{\partial p} + \beta\frac{\partial\beta}{\partial q} = \mu\left(\frac{\partial^2\beta}{\partial p^2} + \frac{\partial^2\beta}{\partial q^2}\right) \qquad (12)$$

The above Equations are conditionally conservative. We convert the equations (11) and (12) into Partial Conservative Form as

$$\frac{\partial\alpha}{\partial t} + \left(\frac{\alpha^2}{2}\right)_p + \beta\frac{\partial\alpha}{\partial q} = \mu\left(\frac{\partial^2\alpha}{\partial p^2} + \frac{\partial^2\alpha}{\partial q^2}\right) \qquad (13)$$

$$\frac{\partial\beta}{\partial t} + \alpha\frac{\partial\beta}{\partial p} + \left(\frac{\beta^2}{2}\right)_q = \mu\left(\frac{\partial^2\beta}{\partial p^2} + \frac{\partial^2\beta}{\partial q^2}\right) \qquad (14)$$

The Conservative Vector Field ***α* = (**α, β), it means $\frac{\partial\alpha}{\partial q} = \frac{\partial\beta}{\partial p}$ otherwise for incompressible flow field $\frac{\partial\alpha}{\partial p} + \frac{\partial\beta}{\partial q} = 0$, in the Conservative Form we do not consider to change the coupled terms $\beta\frac{\partial\alpha}{\partial q}$ and $\alpha\frac{\partial\beta}{\partial p}$. The term $\beta\frac{\partial\alpha}{\partial q}$ should be nonlinear and kept constant in Equation (13). In the Conservative Form, Linearize the tem $\beta\frac{\partial\alpha}{\partial y}$ by treating β as known coefficient for $\frac{\partial\alpha}{\partial y}$. In the same way the term $\alpha\frac{\partial\beta}{\partial p}$ in equation (14) treating α as the coefficient for $\frac{\partial\beta}{\partial p}$. By considering the K^{th} order of Lagrangian interpolants, Ÿ, we use rectangular elements in terms of spatial discretization, 0n Gauss – Legendra - Lobatto as the basis points function of both y and x directions. We expand $\alpha(p,q)$ as following

$$\alpha(p,q) = \sum_{i=0}^{n}\sum_{j=0}^{n} \mathcal{U}_{ij}\ \ddot{Y}_i(p)\ \ddot{Y}_j(q) \qquad (15)$$

In the same way for β(p, q), with respect to q and p, the derivatives of α are as follows

$$\frac{\partial\alpha(p,q)}{\partial p} = \sum_{i=0}^{n}\sum_{j=0}^{n} \mathcal{U}_{ij}\frac{\partial\ddot{Y}_i(p)}{\partial p}\ddot{Y}_j(q) \qquad (16)$$

$$\frac{\partial\alpha(p,q)}{\partial q} = \sum_{i=0}^{n}\sum_{j=0}^{n} \mathcal{U}_{ij}\frac{\partial\ddot{Y}_j(q)}{\partial q}\ddot{Y}_i(q) \qquad (17)$$

By using Galerkin Projection with the test function Ŧ to get the global variables of linear algebraic function system.

$$Ŧ(a,b) = \ddot{Y}_{lm}(p,q) = \sum_{i=0}^{n}\sum_{j=0}^{n} \ddot{Y}_l(p)\ \ddot{Y}_m(q) \qquad (18)$$

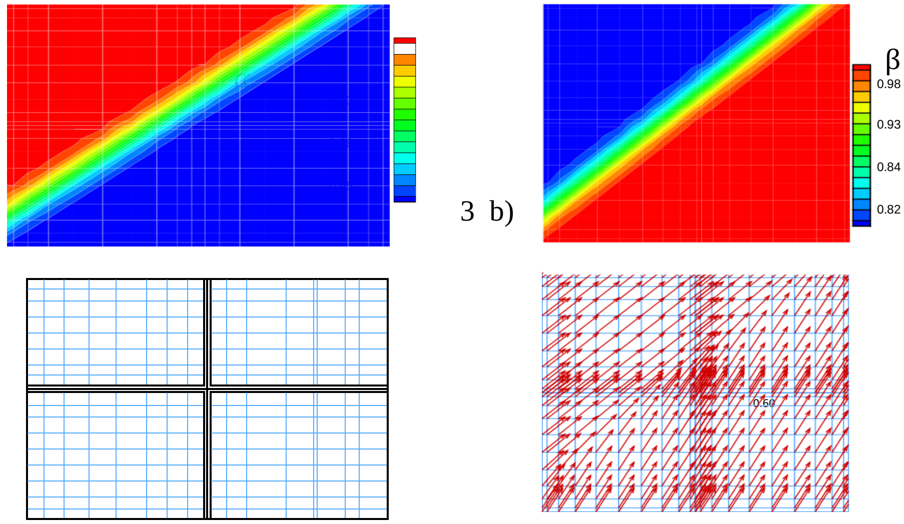
Numerical Example for 2-D Domain

The 2 - D field is set to be **Ω** = {(p,q) : 0 ≤ p, q ≤ 1, t ≥ 0}. Initial conditions and limitations are generated by the exact solution of α and β foe the equations (11) and (12) which are given below:

$$\alpha(p,q,t) = \frac{3}{4} - \frac{1}{4\left(1+e^{200(-t-4p+4q)/32}\right)} \qquad (19)$$

$$\beta(p,q,t) = \frac{3}{4} - \frac{1}{4\left(1+e^{200(-t-4p+4q)/32}\right)} \qquad (20)$$

To solve the Rectangular Domain, we divided it into 4 and 16 elements by using Conservative form. The given below figures 3 and 4 show the Numerical results of 4 elements. In both the figures (3a) and (3b) show the level curves of the velocity components α and β individually at t = 0.6. The figure (c) shows the points of quadrature of the elements. Figure (d) says vector field velocity. The Raynolds Number is approximately 225, due to viscous dissipation, the magnitude of solution waves decreases the time. The figure 3 offers records of the convective go with the drift because it has lesser decision than Figure 4, the use of four elements.



3 a) 3 b)

3 c) 3 d)

Figure 3: 2 – D Convention Diffusion Equation, Velocity components α and β with N = 4 the Polynomial Order Po = 10

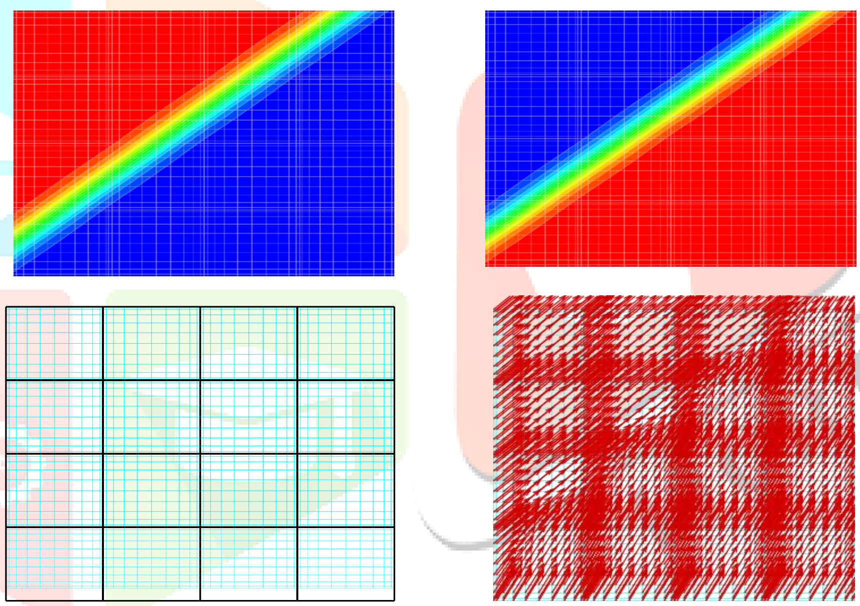


FIGURE 4: 2 – D Convention Diffusion Equation, Velocity components α and β with the Polynomial Order Po = 10, N = 16

The convergence of exponential function shows in the figure 5. Up to 40 we arrange the Polynomial Order. To satisfy the Diffusion and CFL conditions the time step Δt was chosen. If chose in the infinite norm, $\Delta t = 0.002$ and observe the numerical results of α at time t = 0.002 with absolute solution. Using 4 elements gives slower the rate of
Convergence then using 16 elements and at the polynomial order of 24 the error reaches machine zero.

CONCLUSION

The advantage of Cole – Hopf transformation is to convert the nonlinear vector transformation into Partial Differential Equation. The implicit characteristic is higher in time integration, Cole – Hopf transformation is tough to use for better order spatial derivatives if each express and implicit time schemes may be used afterwards. We use and suitable linearization with managed mistakes to open and alternative for treatment of implicit time, which loosen up the time step constraint length and will be useful to long term integration.

The Conservative Form can be the pleasant in nonlinearity and complicated geometry. It is easy for more than one dimensions and is the satisfactory for well-known weighted residual methods; anyhow it does not depend on mesh grids. This method depends on more memory in computations. To capture discontinuity the conservative Form is the better and effective method. The exactness is usually with Conservative Form is usually unaffected by discontinuity in the domain.

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