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πgN*CLOSED SETS AND QUASI N*- NORMAL SPACES

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Abstract. In this paper, we introduce a new class of sets called gN*-closed, πgN*closed sets and its properties are studied and we introduce a new concept of quasi-normal spaces called quasi N*-normal spaces by using N*-open sets due to G. Navalagi [**6**] in topological spaces and obtained several properties of such a space. Further we obtain a characterization and preservation theorems for quasi N*-normal spaces and by using N*-open sets.

1. Introduction

The notion of quasi normal space was introduced by Zaitsev [**11**]. Dontchev and Noiri [**2**] introduce the notion of πg-closed sets as a weak form of g-closed sets due to Levine [**4**]. By using πg-closed sets, Dontchev and Noiri [**2**] obtained a new characterization of quasi normal spaces. G.Navalagi [**6**] introduced the concept of N* and *N-closed sets and discuss some of their basic properties. Recently, Jeyanthi and Janaki [**3**] introduced the concepts of quasi r-normal spaces in topological spaces by using regular open sets in topological spaces and obtained some characterizations and preservation theorems of such spaces.We introduce the notion of N*g-closed , N*αg-open, πgN*-closed,πgN*-open sets,πgN*-closed, almost πgN*-closed, πgN*-continuous and almost πgN*-continuous functions and its properties are studied. Further we obtain characterization and preservation theorems for quasi N*-normal spaces.

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2. Preliminaries.

2.1.Definition. A subset A of a topological space X is called

1. **regular closed**[**11**]) if A = cl(int(A)).
2. **generalized closed** [**4**](briefly, **g-closed**) if cl(A) ⊂ U whenever A ⊂ U and U is open in X .
3. **πg-closed** [**2**] if cl(A) ⊂ U whenever A ⊂ U and U is π-open in X.
4. **α-closed**[**8**] if cl(int(cl(A))) ⊆ A .
5. **αg-closed** [5] if α-cl(A) ⊆ U, whenever A ⊆ U and U is in X.
6. **πgα-closed** [**1**] if α-cl(A) ⊂ U whenever A ⊂ U and U is π-open in X.

The finite union of regular open sets is said to be **π-open**. The complement of π-open set is said to be **π-closed** set. The complement of regular closed (resp. g-closed, πg-closed, α-closed, αg-closed, πgα-closed) set is said to be **regular open** (resp. **g-open**, **πg-open**, **α-open**, **αg-open** , **πgα-open**) sets.

2.2.Definition. A subset A of a topological space X is called

1. **N*-closed** [**6**] if αg-cl(A) ⊆ U, whenever A ⊆ U, and U is g-open in X. 2. **N*g-closed** if N*-cl(A) ⊆ U, whenever A ⊆ U, and U is open in X.

3. **πgN*-closed** if N*-cl(A) ⊂ U, whenever A ⊂ U and U is π-open in X.

The complement of N*- closed (resp. N*g-closed, πgN*-closed) sets is said to be **N*-open** (resp. **N*g-open**, **πgN*-open**).The intersection of all N*-closed subsets of X containing A (i.e. super sets of A) is called the **N*-closure of A** and is denoted by **N***-**cl (A)**. The union of all N*-open sets contained in A is called **N*-interior of A** and is denoted by **N***-int(**A**).The family of all N*-open (resp. N*-closed) sets of a space X is denoted by **N*O(X)** (resp. **N*C(X)**).

2.3. lemma.Let X be a topological space. Then

1.Every α-closed subset of X is N*-closed

2.Every α-open subset of X is N*-open.

We have the following implications for the properties of subsets.

closed	$\Rightarrow$	g-closed	$\Rightarrow$	πg-closed
$\Downarrow$		$\Downarrow$		$\Downarrow$
α-closed	$\Rightarrow$	αg-closed	$\Rightarrow$	πgα-closed
$\Downarrow$		$\Downarrow$		$\Downarrow$
N*-closed	$\Rightarrow$	N*g-closed	$\Rightarrow$	πgN*-closed

Where none of the implications is reversible as can be seen from the following examples

2.4.Example. Let X = {a, b, c , d } and τ = { ϕ, {a}, {c, d}, {a, c, d }, {d}, {a, d}, X }. Then the set A = {a} is πgα-closed set as well πgN*-closed set but not g-closed set in X .

2.5.Example. Let X = {a, b, c , d } and τ = { ϕ, {a}, {c}, {a, b}, {a, c}, {a, d}, {a, d, c }, {a, b, d}, {a, b, c}, X}. Then the set A = {c} is πgα-closed set as well as πgN*-closed set but not αg-closed and not N*g-closed set in X.

2.6.Example. Let X = {a, b, c , d } and τ = { ϕ, {a}, {c, d}, {a, c, d}, {d}, {a, d}, X}. Then the set A = {c} is πgα-closed set as well as πgN*-closed set but not πg-closed set in X.

2.7.Theorem.

(a) Finite union of πgN*-closed sets are πgN*-closed.

(b) Finite intersection of πgN*-closed need not be a πgN*-closed.

(c) A countable union of πgN*-closed sets need not be a πgN*-closed.

Proof. (a) Let A and B be πgN*-closed sets. Therefore N*-cl (A)$\subset$ U and N*-cl(B) $\subset$ U whenever A $\subset$ U, B $\subset$ U and U is π-open. Let A $\cup$ B $\subset$ U where U is π-open. Since N*-cl(A $\cup$ B) $\subset$ N*-cl(A) $\cup$ N*-cl(B) $\subset$ U, we have A $\cup$ B is πgN*-closed.

(b) Let X = {a, b, c, d} and τ = { ϕ, {a}, {b}, {a, b}, {a, b, c}, {a, b, d}, X}. Let A = {a, b, c}, B = {a, b, d}. A and B are πgN*-closed sets. But A $\cap$ B = {a, b}$\subset${a, b} which is π-open. N*-cl(A $\cap$ B) $\not\subset$ {a, b}. Hence A $\cap$ B is not πgN*-closed.

(c) Let R be the real line with the usual topology. Every singleton is πgN*-closed. But, A = {1/i : i = 2, 3, 4 ……………} is not πgN*-closed. Since A $\subset$ (0, 1) which is π-open but N*-cl(A) $\not\subset$ (0, 1).

2.8.Theorem. If A is πgN*-closed and A ⊂ B ⊂ N*-cl(A) then B is πgN*-closed.

Proof. Since A is πgN*-closed, N*(A) ⊂ U whenever A ⊂ U and U is π-open. Let B ⊂ U and U be π-open. Since B ⊂ N*-cl(A), N*-cl(B) ⊂ N*-cl(A) ⊂ U. Hence B is πgN*-closed.

2.9.Theorem. Let A be a πgN*-closed set in X. Then N*-cl(A) – A does not contain any non empty π-closed set.

Proof. Let F be a non empty π-closed set such that F ⊂ N*-cl(A) – A. Then F ⊂ N*-cl(A) ∩ (X – A) ⊂ X – A implies A ⊂ X – F where X – F is π-open. Therefore N*-cl(A)⊂ X – F implies F ⊂ $(N^*\text{-cl}(A))^C$. Now F ⊂ N*-cl(A) (A) ∩ $(N^*\text{-cl}(A))^C$ implies F is empty.

Reverse implication does not hold.

2.10.Corollary. Let A be πgN*closed. A is N*-closed iff N*-cl(A) – A is π-closed.

Proof. Let A be N*-closed set then A = N*-cl(A) implies N*-cl(A) – A = ϕ which is π-closed.
Conversely if N*-cl(A) – A is π-closed then A is N*-closed.

2.11. Theorem. If A is π-open and πgN*-closed. Then A is N*-closed hence clopen.

Proof. Let A be regular open. Since A is πgN*-closed, N*-cl(A) ⊂ A implies A is N*-closed. Hence A is closed (Since every π-open, N*-closed set is closed). Therefore A is clopen.

2.12. Theorem :- For a topological space X, the following are equivalent :

(a) X is extremally disconnected.

(b) Every subset of X is πgN*-closed.

(c) The topology on X generated by πgN*-closed sets.

Proof. (a) ⇒ (b). Assume X is extremally disconnected. Let A ⊂ U, where U is π-open in X. Since U is π-open , it is the finite union of regular open sets and X is extremally disconnected, U is finite union of clopen sets and hence U is clopen. Therefore N*-cl(A) ⊂ cl(A) ⊂ cl(U) ⊂ U implies A is πgN*-closed.

(b ⇒ (a). Let A be reguler open set of X. Since A is πgN*-closed by **Theorem 2.11** A is clopen. Hence X is extremally disconnected.

(b) ⇔ (c) is obvious.

2.13. Lemma[11]**.** If A is a subset of X, then

1. X – N*-cl(X – A) = N*-int(A).

2. X – N*-int(X – A) = N*-cl(A).

2.14. Theorem. A subset A of a topological space X is πgN*-open iff F ⊂ N*-int(A) whenever F is π-closed and F ⊂ A.

Proof. Let F be π-closed set such that F ⊂ A. Since X – A is πgN*-closed and X – A ⊂ X – F we have F ⊂ N*-int(A) .

Conversely. Let F ⊂ N*-int(A) where F is π-closed and F ⊂ A. Since F ⊂ A and X – F is π-open, N*-cl(X – A) = X – N*-int(A) ⊂ X – F. Therefore A is πgN*-open.

2.15. Theorem. If N*-int(A) ⊂ B ⊂ A and A is πgN*-open then B is πgN*-open.

Proof. Since N*-int(A) ⊂ B ⊂ A using **Theorem 2.8**, N*-cl(X – A) ⊃ (X – B) implies B is πgN*-open.

2.16. Remark. For any A ⊂ X, N*-int(A) (N*-cl(A)) – A)) = ϕ.

2.17. Theorem. If A ⊂ X is πgN*closed then N*cl(A) – A is πgN*-open.

Proof. Let A be πgN*-closed and F be a π-closed set such that F ⊂ N*cl(A) – A. By **Theorem 2.9**, F = ϕ implies F ⊂ N*-int(A) (N*cl(A) – A)). By **Theorem 2.14**, N*-cl(A) – A is πgN*-open.

Converse of the above theorem is not true.

2.18.Example. Let X = {a, b, c } and τ = { ϕ, {a}, {b}, {a, b}, X}. Let A = {b }.Then A is not πgN*-closed but N*-cl(A) – A = {a, b} πgN*-open.

3. Quasi N*-normal spaces

3.1.Definition. A topological space X is said to be **N*-normal** (resp. **quasi N*-normal** , **mildly N*-normal**) if for every pair of disjoint closed (resp. π-closed, regularly closed) subsets H, K of X, there exist disjoint N*-open sets U, V of X such that H ⊂ U and K ⊂ V.

3.2.Example. Let X = {a, b, c, d} and τ = {ϕ, {a }, {b},{a, b}, {a, b, c }, X}. The pair of disjoint closed subsets of X are A = ϕ and B = {d}.Then N*-closed sets in X are X, ϕ , {a}, {b}, {c}, {d}, {c, d}, {a, d}, {b, c},{a, c}, {b, d}, {a, c, d}, {b, c, d}. Also U = { b} and V = {c, d} are N*-open sets such that $A \subset U$ and $B \subset V$. Hence X is N*-normal but it is not normal.

3.3.Example. Let X = { a, b, c, d } and τ = { ϕ, {a}, {c}, {a, c},{ b, d}, {a, b, d}, {b, c, d}, X }. The pair of disjoint π-closed subsets of X are A = {a} and B = {c}. Also U = {a} and V = {b, c, d} are open sets such that $A \subset U$ and $B \subset V$. Hence X is quasi-normal as well as quasi N*-normal because every open set is N*-open set .

By the definitions and examples stated above, we have the following diagram:

normality	$\Rightarrow$	quasi-normality	$\Rightarrow$	mild-normality
$\Downarrow$		$\Downarrow$		$\Downarrow$
N*-normality	$\Rightarrow$	quasi N*-normality	$\Rightarrow$	mild N*-normality

3.4. Theorem. For topological space X , the following are equivalent:

(a) X is quasi N*-normal.

(b) For any disjoint π-closed sets H and K, there exist disjoint N*g-open sets U and V such that $H \subset U$ and $K \subset V$.

(c) For any disjoint π-closed sets H and K, there exist disjoint πgN*-open sets U and V such that $H \subset U$ and $K \subset V$.

(d) For any π -closed set H and any π-open set V containing H, there exist a N*g-open set U of X such that $H \subset U \subset$ N*-cl(U) $\subset V$.

(e) For any π-closed set H and any π-open set V containing H, there exist a πgN*-open set U of X such that $H \subset U \subset$ N*-cl(U) $\subset V$.

Proof. (a) $\Rightarrow$ (b), (b) $\Rightarrow$ (c), (d) $\Rightarrow$ (e), (c) $\Rightarrow$ (d) and (e) $\Rightarrow$ (a).

(a) $\Rightarrow$ (b). Let X be quasi N*-normal. Let H, K be disjoint π-closed sets of X. By assumption, there exist disjoint N*-open sets U, V such that $H \subset U$ and $K \subset V$. Since every N*-open set is N*g-open, U and,V are N*g-open sets such that $H \subset U$ and $K \subset V$.

(b) $\Rightarrow$ (c). Let H, K be two disjoint π -closed sets. By assumption, there exists N*g-open sets U and V such that $H \subset U$ and $K \subset V$. Since N*g-open set is πgN*-open, U and V are πgN*-open sets such that $H \subset U$ and $K \subset V$.

(d) $\Rightarrow$ (e). Let H be any π -closed set and V be any π -open set containing H. By assumption, there exist N*g-open set U of X such that $H \subset U \subset$ N*-cl(U) $\subset V$. Since, every N*g-open set is πgN*-open, there exist πgN*-open sets U of X such that $H \subset U \subset$ N*-cl(U) $\subset V$.

(c) $\Rightarrow$ (d). Let H be any π-closed set and V be any π-open set containing H. By assumption, there exist πg N*-open sets U and W such that $H \subset U$ and $X - V \subset W$. By **Theorem 2.14**, we get $X - V \subset$ N*-int(W) and N*-cl (U) $\cap$ N*-int(W)= ϕ. Hence $H \subset U \subset$ N*-cl (U) $\subset X -$ N*-int (W) $\subset V$.

(e) $\Rightarrow$ (a). Let H, K be any two disjoint π-closed set of X. Then $H \subset X - K$ and X–K is π -open. By assumption, there exist πg N*-open set G of X such that $H \subset G \subset$ N*-cl(G) $\subset X - K$. Put U = N*-int(G), V = X – N*-cl(G). Then U and V are disjoint N*-open sets of X such that $H \subset U$ and $K \subset V$.

4 . Some Functions

4.1. Definition. A function $f : X \to Y$ is said to be

1. **almost closed** [9](resp. **almost N*-closed** , **almost N*-g-closed**) if f (F) is closed (resp. N*-closed , N*g-closed) in Y for every $F \in RC(X)$.

2. **πgN*-closed** (resp. **almost πgN*-closed**) if for every closed set (resp. regularly closed) F of X , f(F) is πgN*-closed in Y.

3. **π-continuous** [2] (resp. **πgα-continuous**[1] , **πgN*-continuous**) if $f^{-1}(F)$ is π-closed (resp.πgα-closed, πgN*-closed) in X for every closed set F of Y.

4. **almost continuous** [9] (resp. **almost π-continuous** [2], **almost πgα-continuous**[1] , **almost πgN*-continuous**) if $f^{-1}(F)$ is closed (resp. π-closed, πgα-closed, πgN*-closed) in X for every regularly closed set F of Y.

5. **rc-preserving [7]** if f(F) is regularly closed in Y for every $F \in RC(X)$.

From the definitions stated above, we obtain the following diagram:

closed $\Rightarrow$ α-closed $\Rightarrow$ αg-closed $\Rightarrow$ πgα-closed

$\Downarrow$ $\Downarrow$ $\Downarrow$ $\Downarrow$

al.-closed $\Rightarrow$ al.N*-closed $\Rightarrow$ al. N*g-closed $\Rightarrow$ al. πgN*-closed

where al. = almost

Moreover, by the following examples, we realize that none of the implications is reversible.

4.2. Example. $X = \{a, b, c, d\}$, $\tau = \{\phi, X, \{c\}, \{a, b, d\}$ and $\sigma = \{\phi, \{a\}, \{c, d\}, \{a, c, d\} \{d\}, \{a, d\}, X\}$. Let $f : (X, \tau) \to (X, \sigma)$ be the identity function, then f is πgα-closed as well as πgN*-closed but not πg-closed. Since A= {c} is not πg-closed in (X, σ).

4.3. Example. Let $X = \{a, b, c, d\}$, $\tau = \{\phi, X, \{c\},\{a, b, d\}, \{b, d\}, \{b, c, d\}, X\}$ and $\sigma = \{\phi, X, \{a\},\{c, d\}, \{a, c, d\}, \{d\}, \{a, d\}\}$.Let $f : (X, \tau) \to (X, \sigma)$ be the identity function. Then f is almost πgα-closed as well as almost πgN*-closed but not πgN*-closed. Since A = {a} is not πgN*-closed

4.4.Theorem. If $f : X \to Y$ is an almost π-continuous and πgN*-closed function, then f(A) is πgN*-closed in Y for every πgN*-closed set A of X.

Proof. Let A be any πgN*-closed set A of X and V be any π-open set of Y containing f(A). Since f is almost π-continuous, $f^{-1}(V)$ is π-open in X and $A \subset f^{-1}(V)$. Therefore N*-cl(A) $\subset f^{-1}(V)$ and hence f(N*-cl(A)) $\subset$ V. Since f is πgN*-closed, f(N*-cl(A)) is πgN*-closed in Y. And hence we obtain N*-cl(f(A)) $\subset$ N*-cl(f(N*-cl(A))) $\subset$ V. Hence f(A) is πgN*-closed in Y.

4.5. Theorem. A surjection $f : X \to Y$ is almost πgN*-closed if and only if for each subset S of Y and each $U \in RO(X)$ containing $f^{-1}(S)$ there exists a πgN*-open set V of Y such that $S \subset V$ and $f^{-1}(V) \subset U$.

Proof. Necessity. Suppose that f is almost πgN*-closed. Let S be a subset of Y and $U \in RO(X)$ containing $f^{-1}(S)$. If $V = Y - f(X - U)$, then V is a πgN*-open set of Y such that $S \subset V$ and $f^{-1}(V) \subset U$.

Sufficiency. Let F be any regular closed set of X. Then $f^{-1}(Y - f(F)) \subset X - F$ and $X - F \in RO(X)$. There exists πgN*-open set V of Y such that $Y - f(F) \subset V$ and $f^{-1}(V) \subset X - F$. Therefore, we have $f(F) \supset Y - V$ and $F \subset X - f^{-1}(V) \subset f^{-1}(Y - V)$. Hence we obtain $f(F) = Y - V$ and f(F) is πgN*-closed in Y which shows that f is almost πgN*-closed.

5. Preservation Theorem

5.1.Theorem. If f : X → Y is an almost πgN*-continuous, rc-preserving injection and Y is quasi N*-normal then X is quasi N*-normal.

Proof. Let A and B be any disjoint π-closed sets of X. Since f is an rc-preserving injection, f(A) and f(B) are disjoint π-closed sets of Y. Since Y is quasi N*-normal, there exist disjoint N*-open sets U and V of Y such that f(A) ⊂ U and f(B) ⊂ V. Now if G = int(cl(U)) and H = int(cl(V)). Then G and H are regularly open sets such that f(A) ⊂ G and f(B) ⊂ H. Since f is almost πgN*-continuous, f^{-1} (G) and f^{-1}(H) are disjoint πgN*-open sets containing A and B which shows that X is quasi N*-normal.

5.2.Theorem. If f : X →Y is π-continuous, almost N*-closed surjection and X is quasi N*-normal space then Y is N*-normal .

Proof. Let A and B be any two disjoint closed sets of Y. Then f^{-1} (A) and f^{-1}(B) are disjoint π-closed sets of X. Since X is quasi N*-normal, there exist disjoint N*-open sets of U and V such that f^{-1}(A) ⊂ U and f^{-1}(B) ⊂ V. Let G = int(cl(V)) and H = int(cl(V)). Then G and H are disjoint regularly open sets of X such that f^{-1}(A) ⊂ G and f^{-1}(B) ⊂ H. Set K = Y – f(X – G) and L = Y – f (X – H).Then K and L are N*-open sets of Y such that A ⊂ K, B ⊂ L, f^{-1}(K) ⊂ G , f^{-1}(L) ⊂ H. Since G and H are disjoint, K and L are disjoint. Since K and L are N*-open and we obtain A ⊂ N*int(K), B ⊂ N*-int(L) and N*-int(K) ∩N*-int(L) = ϕ. Therefore Y is N*-normal.

5.3.Theorem. Let f : X → Y be an almost π-continuous and almost πgN*-closed surjection. If X is quasi N*-normal space then Y is quasi N*-normal.

Proof. Let A and B be any disjoint π-closed sets of Y. Since f is almost π-continuous, f^{-1}(A) , f^{-1}(B) are disjoint closed subsets of X. Since X is quasi N*-normal, there exist disjoint N*-open sets U and V of X such that f^{-1}(A) ⊂ U and f^{-1}(B) ⊂ V. Put G = int(cl(U)) and H = int(cl(V)). Then G and H are disjoint regularly open sets of X such that f^{-1}(A) ⊂ G and f^{-1}(B) ⊂ H. By **Theorem 4.5**, there exist πgN*-open sets K and L of Y such that A ⊂ K, B ⊂ L, f^{-1}(K) ⊂ G and f^{-1}(L) ⊂ H. Since G and H are disjoint, so are K and L by **Theorem 2.14,** A ⊂ N*int(K), B ⊂ N*-int(L) and N*-int(K) ∩N*-int(L) = ϕ. Therefore, Y is quasi N*-normal.

5.3.Corollary. If f : X → Y is almost continuous and almost closed surjection and X is a normal space, then Y is quasi N*-normal.

Proof. Since every almost closed function is almost πgN*-closed so Y is quasi N*-normal.

REFERENCES

1. Arockiarani and C. Janaki, πgα-closed sets and quasi α-normal spaces, *Acta Ciencia Indica*, Vol. XXXIII M. No. 2, (2007), 657-666.
2. J. Dontchev and T. Noiri, Quasi-normal spaces and πg-closed sets, *Acta Math. Hungar*. **89**(3)(2000), 211-219.
3. V. Jeyanthi and C. Janaki, Quasi r-normal spaces, *Scholar J. of Physics, Math. and Stat*. 1(2) (2014),108-110.
4. N. Levine, Generalized closed sets in topology, *Rend. Circ. Mat. Palermo* **19**(1970),89-96.
5. H. Maki, R. Devi and Balachandran K. , Generalized α-closed sets in topology, *Bull. Fukuoka Univ. ed. Part* III **42** (1993), 13-21.
6. G.Navalagi,Properties of N*-and *N-closed sets in topology, Internat. J.of Innovative Res.in Sci.,Engg.and Tech.(2020),6485-6496.
7. T. Noiri, Mildly - normal spaces and some functions. *Kyungpook Math. J.* **36**(1996),183 - 190.
8. Njastad, O.,On some class of nearly open sets, *Pacific. J. Math*., **15**(1965), 961-970.
9. M. K. Singal and A. R. Singal, Almost continuous mappings, *Yokohama Math. J*. **16**(1968), 63-73.
10. M.H. Stone, Applications of the theory of Boolean rings to general topology, *Trans. Amer. Math. Soc.* **41** (1937), 375-381.
11. Zaitsev V., On certain classes of topological spaces and their biocompactifications, *Dokl Akad Nauk SSSR* **178**(1968), 778-779.