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GENERALIZED SYMMETRIC RATIONAL CONTRACTION PRINCIPLE IN METRIC SPACE

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Abstract : *Our aim of this paper is to introduced the concept of week symmetric rational contraction principle in metric space and prove some fixed point theorems in metric spaces. Our results are generalization and extended some previous known results.*

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Introduction and Preliminaries

In past few decades study of fixed point theory is one of the most interesting fields to researchers. In this direction Banach contraction mapping principle is one of the most interesting results which states as follows:-

Let (X,d) *be a complete metric space and* $T : X \to X$. *Then* T *is said to be contraction mapping if for all* $x, y \in X$,

$$d(Tx,Ty) \leq kd(x,y) \qquad (1.1)$$

where $0 < k < 1$.

It is easy to see that the contraction mapping principle, any mapping T satisfying (1.1) will have a unique fixed point.

Number of mathematicians generalized the above principle. ***Boyd and Wong***[1] *proved that the constant k in (1.1) can be replaced by the use of upper semi continuous function.* ***Suzuki***[2] *has proved a generalization of the same principle which characterizes metric completeness. The contraction principle has also been extended to probabilistic metric space [5].*

One of the most interesting generalization was presented by ***Khan et al.***[3] *which addressed a new category of fixed point problems by using control function which they called an altering distance function. In fact* ***Khan et al.***[3] *presented following definition of altering distance function.*

Definition 1.1: *A function* $\psi: [0,\infty) \rightarrow [0,\infty)$ *is called an altering distance function if the following properties are satisfied:*

i. $\psi(0) = 0$
ii. ψ *is continuous and monotonically non decreasing.*

In $\boldsymbol{Khan\ et\ al.}$[3] *proved the following fixed point result.*

Theorem 1.2: *Let* (X,d) *is a complete metric space, let* ψ *be an altering distance function, and let* $T: X \rightarrow X$ *be a self-mapping which satisfies the following inequality:*

$$\psi\big(d(Tx,Ty)\big) \leq c\psi\big(d(x,y)\big) \qquad (1.2)$$

for all $x,y \in X$ *and for some* $0 < c < 1$. *Then* T *has a unique fixed point.*

After the publication of $\boldsymbol{Khan\ et\ al.}$[3] *work there is lot of work done by using this concept. Some of works utilizing the concept of altering distance function are noted in* $[\mathbf{4} - \mathbf{14}]$. *In* $[\mathbf{9}]$, *2-variable and in* $[\mathbf{10}]$ *3-variable altering distance functions have been introduced as generalizations of the concept of altering distance function. It has also been extended in the context if multivalued* $[\mathbf{11}]$ *and fuzzy mappings* $[\mathbf{12}]$. *The concept of altering distance function has also been introduced in Menger spaces* $[\mathbf{13}]$.

$\boldsymbol{Alber\ and\ Guerre - Delabriere}$[5] *gave an another generalization of the contraction principle in Hilbert spaces. Rhoades [17] has shown that the result which Alber and Guerre- Delabriere have been proved in* $[\mathbf{5}]$ *is also valid in complete metric spaces.* $\boldsymbol{Rhoades}$[14] *gave the following definition of contraction principle,*

Definition 1.3: *A mapping* $T: X \rightarrow X$, *where* (X,d) *is a metric space, is said to be weakly contractive if*

$$d(Tx,Ty) \leq d(x,y) - \phi(dx,y)) \qquad (1.3)$$

where $x,y \in X$ *and* $\phi: [0,\infty) \rightarrow [0,\infty)$ *is a continuous and nondecreasing function such that* $\phi(t) = 0$ *if and only if* $t = 0$.

It should be noted that if we take $\phi(t) = kt$ *where* $0 < k < 1$, *in (1.3) then we get (1.1).*

Also following theorem is the main results of $\boldsymbol{Rhoades}$[14].

Theorem 1.4: *If* $T : X \rightarrow X$ *is a weakly contractive mapping, where* (X,d) *is a complete metric space, then* T *has a unique fixed point.*

In fact, Alber and Guerre- Delabriere assumed an additional condition on ϕ *which is* $\lim_{t\rightarrow\infty} \phi(t) = \infty$. *But* $\boldsymbol{Rhoades}$[14] *obtained the result noted in* ***Theorem 1.4*** *without using this particular assumption.*

It may be observed that though the function ϕ *has been defined in the same way as the altering distance function, the way it has been used in* ***Theorem 1.4*** *is completely different from the use of altering distance function.*

The purpose of this paper is to introduce a new type contraction principle which is a generalization of Banach contraction principle which includes the generalizations noted in Theorem 1.2 and 1.4 .

Main Results

Our main investigated result of this paper is as follows ,

Theorem 2.1: *Let* (X, d) *be a complete metric space and let* $T: X \to X$ *be a self mapping satisfying the inequality*

$$\psi\big(d(Tx,Ty)\big) \le \psi\left(\big(d(x,y).d(x,Tx).d(y,Ty)\big)^{\frac{1}{3}}\right) - \phi\left(\big(d(x,y).d(x,Tx).d(y,Ty)\big)^{\frac{1}{3}}\right) \qquad (2.1(a))$$

where $\psi, \phi : [0,\infty) \to [0,\infty)$ *are both continuous and monotone nondecreasing functions with* $\psi(t) = 0 = \phi(t)$ *if and only if* $t = 0$. *Then* T *has a unique fixed point.*

Proof: *For any* $x_0 \in X$, *we construct the sequence* $\{x_n\}$ *by* $x_n = Tx_{n-1}$, $n = 1,2, \ldots$

Substituting $x = x_{n-1}$ *and* $y = x_n$ *in (2.1(a)), we obtain*

$$\psi\big(d(Tx_{n-1},Tx_n)\big) \le \psi\left(\big(d(x_{n-1},x_n).d(x_{n-1},Tx_{n-1}).d(x_n,Tx_n)\big)^{\frac{1}{3}}\right)$$

$$- \phi\left(\big(d(x_{n-1},x_n).d(x_{n-1},Tx_{n-1}).d(x_n,Tx_n)\big)^{\frac{1}{3}}\right)$$

$$\psi\big(d(x_n,x_{n+1})\big) \le \psi\left(\big(d(x_{n-1},x_n).d(x_{n-1},x_n).d(x_n,x_{n+1})\big)^{\frac{1}{3}}\right)$$

$$- \phi\left(\big(d(x_{n-1},x_n).d(x_{n-1},x_n).d(x_n,x_{n+1})\big)^{\frac{1}{3}}\right) \qquad (2.2)$$

Using the monotone property of $\psi -$ *function we have*

$$d^2(x_n,x_{n+1}) \le d^2(x_{n-1},x_n)$$

Which implies

$$d(x_n,x_{n+1}) \le d(x_{n-1},x_n) \qquad (2.3)$$

It follows that the sequence $\{d(x_n,x_{n+1})\}$ *is monotone decreasing and consequently there exists* $r \ge 0$ *such that*

$$d(x_n,x_{n+1}) \to r \ \ as\ \ n \to \infty. \qquad (2.4)$$

Letting $n \to \infty$ *in (2.2) we obtain*

$$\psi(r) \le \psi(r) - \phi(r), \qquad (2.5)$$

This is a contraction unless $r = 0$.

Hence

$$d(x_n,x_{n+1}) \to 0 \ \ as\ \ n \to \infty. \qquad (2.6)$$

We next prove that $\{x_n\}$ *is a Cauchy sequence. If possible, let* $\{x_n\}$ *be not a Cauchy sequence. Then there exists* $\epsilon > 0$ *for which we can fixed subsequences* $\{x_{m(k)}\}$ *and* $\{x_{n(k)}\}$ *of* $\{x_n\}$ *with* $n(k) > m(k) > k$ *such that*

$$d\left(x_{m(k)}, x_{n(k)}\right) \geq \epsilon. \qquad (2.7)$$

Further, corresponding to $m(k)$, *we can choose* $n(k)$ *is such a way that if is the smallest integer with* $n(k) > m(k)$ *and satisfying (2.7). Then*

$$d\left(x_{m(k)}, x_{n(k)-1}\right) < \epsilon. \qquad (2.8)$$

Then we have

$$\epsilon \leq d\left(x_{m(k)}, x_{n(k)}\right) \leq d\left(x_{m(k)}, x_{n(k)-1}\right) + d\left(x_{n(k)-1}, x_{n(k)}\right) < \epsilon + d\left(x_{n(k)-1}, x_{n(k)}\right)$$

(2.9)

Letting $k \to \infty$ *and using (2.6),*

$$\lim_{k\to\infty} d\left(x_{m(k)}, x_{n(k)}\right) = \epsilon.$$

(2.10)

Again,

$$d\left(x_{m(k)}, x_{n(k)}\right) \leq d\left(x_{m(k)}, x_{m(k)-1}\right) + d\left(x_{m(k)-1}, x_{n(k)-1}\right) + d\left(x_{n(k)-1}, x_{n(k)}\right)$$

$$d\left(x_{m(k)-1}, x_{n(k)-1}\right) \leq d\left(x_{n(k)-1}, x_{n(k)}\right) + d\left(x_{m(k)}, x_{n(k)}\right) + d\left(x_{m(k)}, x_{m(k)-1}\right) \qquad (2.11)$$

Letting $k \to \infty$ *in the above two inequalities and using (2.6) and (2.10), we get*

$$\lim_{k\to\infty} d\left(x_{m(k)-1}, x_{n(k)-1}\right) = \epsilon \qquad (2.12)$$

Setting $x = x_{m(k)-1}$ *and* $y = x_{n(k)-1}$ *in (2.1(a)) and using (2.7), we obtain*

$$\psi(\epsilon) \leq \psi\left(d\left(Tx_{m(k)-1}, Tx_{n(k)-1}\right)\right)$$

$$\leq \psi\left(\left(d\left(x_{m(k)-1}, x_{n(k)-1}\right).d\left(x_{m(k)-1}, Tx_{m(k)-1}\right).d\left(x_{n(k)-1}, Tx_{n(k)-1}\right)\right)^{\frac{1}{3}}\right)$$

$$-\phi\left(\left(d\left(x_{m(k)-1}, x_{n(k)-1}\right).d\left(x_{m(k)-1}, Tx_{m(k)-1}\right).d\left(x_{n(k)-1}, Tx_{n(k)-1}\right)\right)^{\frac{1}{3}}\right)$$

(2.13)

Letting $k \to \infty$, *and using (2.10) and (2.12), we obtain*

$$\psi(\epsilon) \leq \psi(\epsilon) - \phi(\epsilon)$$

(2.14)

which is a contraction if $\epsilon > 0$.

This shows that $\{x_n\}$ *is a Cauchy sequence and hence is convergent in the complete metric space X. Let*

$$x_n \to z \text{ as } n \to \infty$$

(2.15)

Substituting $x = x_{n-1}$ *and* $y = z$ *in (2.1(a)), we obtain*

$$\psi\left(d(Tx_{n-1}, Tz)\right) \leq \psi\left(\left(d(x_{n-1}, z).d(x_{n-1}, Tx_{n-1}).d(z, Tz)\right)^{\frac{1}{3}}\right)$$

$$- \phi \left(\left(d(x_{n-1}, z). d(x_{n-1}, Tx_{n-1}). d(z, Tz) \right)^{\frac{1}{3}} \right) \quad (2.16)$$

Letting $n \to \infty$ *using (2.15) and continuity of* ϕ *and* ψ *we have*

$$\psi\big(d(z, Tz)\big) \le \psi(0) - \phi(0) = 0$$

Which implies $\psi\big(d(z, Tz)\big) = 0$*, that is*

$$d(z, Tz) = 0 \text{ or } z = Tz.$$

To prove the uniqueness of the fixed point, let us suppose that z_1 *and* z_2 *are two fixed points of T. Putting* $x = z_1$ *and* $y = z_2$ *in (2.1(a))*

$$\psi\left(d(Tz_1, Tz_2)\right) \le \psi\left(\left(d(z_1, z_2). d(z_1, Tz_1). d(z_2, Tz_2) \right)^{\frac{1}{3}} \right) - \phi\left(\left(d(z_1, z_2). d(z_1, Tz_1). d(z_2, Tz_2) \right)^{\frac{1}{3}} \right)$$

$$\psi\left(d(Tz_1, Tz_2)\right) \le 0$$

Or equivalently $d(z_1, z_2) = 0$ *that is* $z_1 = z_2$*. This proves that the uniqueness of the fixed point.*

Corollary 2.2: *Let* (X, d) *be a complete metric space and let* $T: X \to X$ *be a self mapping satisfying the inequality*

$$\psi\big(d(Tx, Ty)\big) \le k\, \psi\left(\left(d(x, y). d(x, Tx). d(y, Ty) \right)^{\frac{1}{3}} \right) \qquad (2.2(a))$$

where $\psi : [0, \infty) \to [0, \infty)$ *is continuous and monotone nondecreasing functions with* $\psi(t) = 0$ *if and only if* $t = 0$ *and* $0 < k < 1$*. Which shows that T has unique fixed point in X.*

Proof : *If we particularly take* $\phi(t) = (1 - k)\psi(t)\ \forall\, t > 0$ *where* $0 < k < 1$*, in* ***Theorem 2.1(a)*** *then we obtain the result.*

Corollary 2.3: *Let* (X, d) *be a complete metric space and let* $T: X \to X$ *be a self mapping satisfying the inequality*

$$d(Tx, Ty) \le k \left(d(x, y). d(x, Tx). d(y, Ty) \right)^{\frac{1}{3}} \qquad (2.3(a))$$

where $0 < k < 1$*. Then T has a unique fixed point.*

Proof : *If we particularly take* $\psi(t) = t\ \forall\, t > 0$ *, in* ***Corollary 2.2*** *then we obtain the result.*

Now we give another fixed point theorem satisfying rational contractive condition.

Theorem 2.4: *Let* (X, d) *be a complete metric space and let* $T: X \to X$ *be a self mapping satisfying the inequality*

$$\psi\big(d(Tx, Ty)\big) \le \psi\left(\frac{d^2(x, Tx) + d^2(y, Ty)}{1 + d(x, Tx) + d(y, Ty)} \right) - \phi\left(\frac{d^2(x, Tx) + d^2(y, Ty)}{1 + d(x, Tx) + d(y, Ty)} \right) \quad (2.4(a))$$

where $\psi, \phi : [0, \infty) \to [0, \infty)$ *are both continuous and monotone nondecreasing functions with* $\psi(t) < kt$ *for* $0 < k < 1$ *and* $\psi(t) = 0 = \phi(t)$ *if and only if* $t = 0$*. Then T has a unique fixed point.*

Proof : *For any* $x_0 \in X$*, we construct the sequence* $\{x_n\}$ *by* $x_n = Tx_{n-1}$*,* $n = 1, 2, \ldots\ldots$

Substituting $x = x_{n-1}$ *and* $y = x_n$ *in (2.4(a)), we obtain*

$$\psi\big(d(Tx_{n-1}, Tx_n)\big) \le \psi\left(\frac{d^2(x_{n-1},Tx_{n-1})+d^2(x_n,Tx_n)}{1+d(x_{n-1},Tx_{n-1})+d(x_n,Tx_n)}\right) - \phi\left(\frac{d^2(x_{n-1},Tx_{n-1})+d^2(x_n,Tx_n)}{1+d(x_{n-1},Tx_{n-1})+d(x_n,Tx_n)}\right)$$

$$\psi\big(d(x_n, x_{n+1})\big) \le \psi\left(\frac{d^2(x_{n-1},x_n)+d^2(x_n,x_{n+1})}{1+d(x_{n-1},x_n)+d(x_n,x_{n+1})}\right) - \phi\left(\frac{d^2(x_{n-1},x_n)+d^2(x_n,x_{n+1})}{1+d(x_{n-1},x_n)+d(x_n,x_{n+1})}\right) \quad (2.16)$$

Using the monotone property of $\psi -$ *function we have*

$$d(x_n, x_{n+1}) \le \frac{k}{1-k}\, d(x_{n-1}, x_n) \quad (2.17)$$

Similarly we have

$$d(x_{n-1}, x_n) \le \frac{k}{1-k}\, d(x_{n-1}, x_{n-2}) \quad (2.18)$$

Processing the same way we have

$$d(x_n, x_{n+1}) \le \frac{k^n}{(1-k)^n}\, d(x_0, x_1)$$

It follows that the sequence $\{d(x_n, x_{n+1})\}$ *is monotone decreasing and consequently there exists* $r \ge 0$ *such that*

$$d(x_n, x_{n+1}) \to r \;\; as \;\; n \to \infty. \quad (2.19)$$

Letting $n \to \infty$ *in (2.16) we obtain*

$$\psi(r) \le \psi(r) - \phi(r), \quad (2.20)$$

This is a contraction unless $r = 0$.

Hence

$$d(x_n, x_{n+1}) \to 0 \;\; as \;\; n \to \infty. \quad (2.21)$$

We next prove that $\{x_n\}$ *is a Cauchy sequence. If possible, let* $\{x_n\}$ *be not a Cauchy sequence. Then there exists* $\epsilon > 0$ *for which we can fixed subsequences* $\{x_{m(k)}\}$ *and* $\{x_{n(k)}\}$ *of* $\{x_n\}$ *with* $n(k) > m(k) > k$ *such that*

$$d\big(x_{m(k)}, x_{n(k)}\big) \ge \epsilon. \quad (2.22)$$

Further, corresponding to $m(k)$, *we can choose* $n(k)$ *is such a way that if is the smallest integer with* $n(k) > m(k)$ *and satisfying (2.22). Then*

$$d\big(x_{m(k)}, x_{n(k)-1}\big) < \epsilon. \quad (2.23)$$

Then we hgave

$$\epsilon \le d\big(x_{m(k)}, x_{n(k)}\big) \le d\big(x_{m(k)}, x_{n(k)-1}\big) + d\big(x_{n(k)-1}, x_{n(k)}\big) < \epsilon + d\big(x_{n(k)-1}, x_{n(k)}\big) \quad (2.24)$$

Letting $k \to \infty$ *and using (2.21),*

$$\lim_{k \to \infty} d\big(x_{m(k)}, x_{n(k)}\big) = \epsilon. \quad (2.25)$$

Again,

$$d\left(x_{m(k)}, x_{n(k)}\right) \leq d\left(x_{m(k)}, x_{m(k)-1}\right) + d\left(x_{m(k)-1}, x_{n(k)-1}\right) + d\left(x_{n(k)-1}, x_{n(k)}\right)$$

$$d\left(x_{m(k)-1}, x_{n(k)-1}\right) \leq d\left(x_{n(k)-1}, x_{n(k)}\right) + d\left(x_{m(k)}, x_{n(k)}\right) + d\left(x_{m(k)}, x_{m(k)-1}\right) \qquad (2.26)$$

Letting $k \to \infty$ in the above two inequalities and using (2.22) and (2.25), we get

$$\lim_{k\to\infty} d\left(x_{m(k)-1}, x_{n(k)-1}\right) = \epsilon \qquad (2.27)$$

Setting $x = x_{m(k)-1}$ and $y = x_{n(k)-1}$ in (2.4(a)) and using (2.23), we obtain

$$\psi(\epsilon) \leq \psi\left(d\left(Tx_{m(k)-1}, Tx_{n(k)-1}\right)\right)$$

$$\leq \psi\left(\frac{d^2\left(x_{m(k)-1}, Tx_{m(k)-1}\right) + d^2\left(x_{n(k)-1}, Tx_{n(k)-1}\right)}{1 + d\left(x_{m(k)-1}, Tx_{m(k)-1}\right) + d\left(x_{n(k)-1}, Tx_{n(k)-1}\right)}\right) - \phi\left(\frac{d^2\left(x_{m(k)-1}, Tx_{m(k)-1}\right) + d^2\left(x_{n(k)-1}, Tx_{n(k)-1}\right)}{1 + d\left(x_{m(k)-1}, Tx_{m(k)-1}\right) + d\left(x_{n(k)-1}, Tx_{n(k)-1}\right)}\right)$$

(2.28)

Letting $k \to \infty$, and using (2.25) and (2.27), we obtain

$$\psi(\epsilon) \leq \psi(\epsilon) - \phi(\epsilon) \qquad (2.29)$$

which is a contraction if $\epsilon > 0$.

This shows that $\{x_n\}$ is a Cauchy sequence and hence is convergent in the complete metric space X. Let

$$x_n \to z \ \text{as} \ n \to \infty \qquad (2.30)$$

Substituting $x = x_{n-1}$ and $y = z$ in (2.4(a)), we obtain

$$\psi\left(d(Tx_{n-1}, Tz)\right) \leq \psi\left(\frac{d^2(x_{n-1}, Tx_{n-1}) + d^2(z, Tz)}{1 + d(x_{n-1}, Tx_{n-1}) + d(z, Tz)}\right) - \phi\left(\frac{d^2(x_{n-1}, Tx_{n-1}) + d^2(z, Tz)}{1 + d(x_{n-1}, Tx_{n-1}) + d(z, Tz)}\right) \qquad (2.31)$$

Letting $n \to \infty$ using (2.30) and continuity of ϕ and ψ we have

$$\psi\left(d(z, Tz)\right) \leq \psi(0) - \phi(0) = 0$$

Which implies $\psi\left(d(z, Tz)\right) = 0$, that is

$$d(z, Tz) = 0 \ \text{or} \ z = Tz.$$

To prove the uniqueness of the fixed point, let us suppose that z_1 and z_2 are two fixed points of T. Putting $x = z_1$ and $y = z_2$ in (2.4(a))

$$\psi\left(d(Tz_1, Tz_2)\right) \leq \psi\left(\frac{d^2(z_1, Tz_1) + d^2(z_2, Tz_2)}{1 + d(z_1, Tz_1) + d(z_2, Tz_2)}\right) - \phi\left(\frac{d^2(z_1, Tz_1) + d^2(z_2, Tz_2)}{1 + d(z_1, Tz_1) + d(z_2, Tz_2)}\right)$$

$$\psi\left(d(Tz_1, Tz_2)\right) \leq 0$$

Or equivalently $d(z_1, z_2) = 0$ that is $z_1 = z_2$. This proves that the uniqueness of the fixed point.

Theorem 2.5: *Let (X, d) be a complete metric space and let $T: X \to X$ be a self mapping satisfying the inequality*

$$\psi\left(d(Tx, Ty)\right) \leq \psi(\max\{d(x, y), d(x, Tx), d(y, Ty)\}) - \phi(\max\{d(x, y), d(x, Tx), d(y, Ty)\})$$

$(2.5(a))$

where $\psi, \phi : [0, \infty) \to [0, \infty)$ are both continuous and monotone nondecreasing functions with $\psi(t) = 0 = \phi(t)$ if and only if $t = 0$. Then T has a unique fixed point.

Proof: *For any* $x_0 \in X$, *we construct the sequence* $\{x_n\}$ *by* $x_n = Tx_{n-1}$, $n = 1,2, \dots$.

Substituting $x = x_{n-1}$ *and* $y = x_n$ *in (2.5(a)), we obtain*

$$\psi\big(d(Tx_{n-1}, Tx_n)\big) \le \psi(\max\{d(x_{n-1}, x_n), d(x_{n-1}, Tx_{n-1}), d(x_n, Tx_n)\})$$

$$- \phi(\max\{d(x_{n-1}, x_n), d(x_{n-1}, Tx_{n-1}), d(x_n, Tx_n)\})$$

$$\psi\big(d(x_n, x_{n+1})\big) \le \psi(\max\{d(x_{n-1}, x_n), d(x_{n-1}, x_n), d(x_n, x_{n+1})\})$$

$$- \phi(\max\{d(x_{n-1}, x_n), d(x_{n-1}, x_n), d(x_n, x_{n+1})\}) \quad (2.31)$$

Using the monotone property of $\psi -$ *function we have*

$$d(x_n, x_{n+1}) \le \max\{d(x_{n-1}, x_n), d(x_{n-1}, x_n), d(x_n, x_{n+1})\}$$

Which implies

$$d(x_n, x_{n+1}) \le d(x_{n-1}, x_n) \quad (2.32)$$

It follows that the sequence $\{d(x_n, x_{n+1})\}$ *is monotone decreasing and consequently there exists* $r \ge 0$ *such that*

$$d(x_n, x_{n+1}) \to r \text{ as } n \to \infty. \quad (2.33)$$

Letting $n \to \infty$ *in (2.31) we obtain*

$$\psi(r) \le \psi(r) - \phi(r), \quad (2.34)$$

This is a contraction unless $r = 0$.

Hence

$$d(x_n, x_{n+1}) \to 0 \text{ as } n \to \infty. \quad (2.35)$$

We next prove that $\{x_n\}$ *is a Cauchy sequence. If possible, let* $\{x_n\}$ *be not a Cauchy sequence. Then there exists* $\epsilon > 0$ *for which we can fixed subsequences* $\{x_{m(k)}\}$ *and* $\{x_{n(k)}\}$ *of* $\{x_n\}$ *with* $n(k) > m(k) > k$ *such that*

$$d\big(x_{m(k)}, x_{n(k)}\big) \ge \epsilon. \quad (2.36)$$

Further, corresponding to $m(k)$, *we can choose* $n(k)$ *is such a way that if is the smallest integer with* $n(k) > m(k)$ *and satisfying (2.36). Then*

$$d\big(x_{m(k)}, x_{n(k)-1}\big) < \epsilon. \quad (2.37)$$

Then we hgave

$$\epsilon \le d\big(x_{m(k)}, x_{n(k)}\big) \le d\big(x_{m(k)}, x_{n(k)-1}\big) + d\big(x_{n(k)-1}, x_{n(k)}\big) < \epsilon + d\big(x_{n(k)-1}, x_{n(k)}\big) \quad (2.38)$$

Letting $k \to \infty$ *and using (2.35),*

$$\lim_{k\to\infty} d\big(x_{m(k)}, x_{n(k)}\big) = \epsilon. \quad (2.39)$$

Again,

$$d\big(x_{m(k)}, x_{n(k)}\big) \le d\big(x_{m(k)}, x_{m(k)-1}\big) + d\big(x_{m(k)-1}, x_{n(k)-1}\big) + d\big(x_{n(k)-1}, x_{n(k)}\big)$$

$$d\big(x_{m(k)-1}, x_{n(k)-1}\big) \leq d\big(x_{n(k)-1}, x_{n(k)}\big) + d\big(x_{m(k)}, x_{n(k)}\big) + d\big(x_{m(k)}, x_{m(k)-1}\big)$$

(2.40)

Letting $k \to \infty$ *in the above two inequalities and using (2.35) and (2.39), we get*

$$\lim_{k\to\infty} d\big(x_{m(k)-1}, x_{n(k)-1}\big) = \epsilon \qquad (2.41)$$

Setting $x = x_{m(k)-1}$ *and* $y = x_{n(k)-1}$ *in (2.5(a)) and using (2.7) , we obtain*

$$\psi(\epsilon) \leq \psi\Big(d\big(Tx_{m(k)-1}, Tx_{n(k)-1}\big)\Big)$$

$$\leq \psi\big(\max\{d\big(x_{m(k)-1}, x_{n(k)-1}\big), d\big(x_{m(k)-1}, Tx_{m(k)-1}\big), d\big(x_{n(k)-1}, Tx_{n(k)-1}\big)\}\big)$$

$$-\phi\big(\max\{d\big(x_{m(k)-1}, x_{n(k)-1}\big), d\big(x_{m(k)-1}, Tx_{m(k)-1}\big), d\big(x_{n(k)-1}, Tx_{n(k)-1}\big)\}\big) \qquad (2.42)$$

Letting $k \to \infty$, *and using (2.39) and (2.41), we obtain*

$$\psi(\epsilon) \leq \psi(\epsilon) - \phi(\epsilon) \qquad (2.43)$$

which is a contraction if $\epsilon > 0$.

This shows that $\{x_n\}$ *is a Cauchy sequence and hence is convergent in the complete metric space X. Let*

$$x_n \to z \text{ as } n \to \infty \qquad (2.44)$$

Substituting $x = x_{n-1}$ *and* $y = z$ *in (2.5(a)), we obtain*

$$\psi\left(d(Tx_{n-1}, Tz)\right) \leq \psi\left(\max\{d(x_{n-1}, z), d(x_{n-1}, Tx_{n-1}), d(z, Tz)\}\right)$$

$$- \phi\left(\max\{d(x_{n-1}, z), d(x_{n-1}, Tx_{n-1}), d(z, Tz)\}\right) \qquad (2.45)$$

Letting $n \to \infty$ *using (2.44) and continuity of* ϕ *and* ψ *we have*

$$\psi\big(d(z, Tz)\big) \leq \psi(0) - \phi(0) = 0$$

Which implies $\psi\big(d(z, Tz)\big) = 0$, *that is*

$$d(z, Tz) = 0 \text{ or } z = Tz.$$

Next we show that uniqueness of fixed point for this let z_1 *and* z_2 *be two different fixed points of T that is* $z_1 \neq z_2$. *On taking* z_1 *in place of* x *and* z_2 *in place of* y *in (2.5(a)) then we get*

$$\psi\left(d(Tz_1, Tz_2)\right) \leq \psi\left(\max\{d(z_1, z_2). d(z_1, Tz_1). d(z_2, Tz_2)\}\right) - \phi\left(\max\{d(z_1, z_2). d(z_1, Tz_1). d(z_2, Tz_2)\}\right)$$

$$\psi\left(d(Tz_1, Tz_2)\right) \leq 0.$$

Which contradiction our hypothesis. So $d(z_1, z_2) = 0$ *that is* $z_1 = z_2$ *which show that the fixed point is unique.*

Conclusion: *In this paper we prove a new type contractive condition which generalized previously known results in this direction. We also introduced the concept of symmetric rational contractive condition by using the notion of altering distance function in metric space.*

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