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Algebra of α - Fuzzy Subgroup and Lagrange's Theorem

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Abstract

In this paper, we use the concept of $\alpha -$ fuzzy subgroup and introduces Lagrange's theorem for the $\alpha -$ fuzzy subgroup, order of an element in the $\alpha -$ fuzzy subgroup, and order of the $\alpha -$ fuzzy subgroup. Defined the intersection of $\alpha -$ fuzzy subgroups on different domain.

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$\boldsymbol{Keywords}$ **:** fuzzy subgroup, fuzzy sets, $\alpha -$ fuzzy subgroup, fuzzy Lagrange's theorem.

1 Introduction

L.A. Zadeh used the term fuzzy set in 1965[8]. He introduced basic set operations, namely the union of fuzzy sets, the intersection of fuzzy sets, and the complement of a fuzzy set. In 1971[6], Rosenfeld used the concept of the fuzzy set given by Zadeh to develop the theory of fuzzy group and basic properties of a fuzzy group. In 1979[2], Anthony and Sherwood redefined the concept given by Rosenfeld. In 1981[4], P.S. Das proposed level fuzzy subset and level fuzzy subgroup. In 1992[3], Bhakat and Das used the concept of fuzzy subgroup and defined fuzzy cosets. In 1994[5], J. Kim and D. Kim introduced the notion of fuzzy $p*$ subgroups and defined the order of an element in a fuzzy subgroup. In 2009[1], Abraham and Sebastian fuzzify the famous theorems of Cayley and Lagrange in group theory differently. In 2013[7], Sharma defined the $\alpha -$ fuzzy subgroup.

We defined the order of the $\alpha -$ fuzzy subgroup by using the order of fuzzy subgroup given by J.kim in [5]. Lagrange's theorem for $\alpha -$ fuzzy subgroup is defined using concept of Abraham and Sebastian in [1].

2 Preliminaries

Definition 2.1 (fuzzy set) *[8] Let X be any set, then a fuzzy set $\tilde{A}$ of X is a set of ordered pairs:*

$$\tilde{A} = \{(x, \mu_{\tilde{A}}(x): x \in X\}$$

here $\mu_{\tilde{A}}: X \longrightarrow [0,1]$ is called membership function.

Let $\tilde{A}$ and $\tilde{B}$ be two fuzzy subsets of a set X. Then the following expression are defined in

1. $\tilde{A} \subseteq \tilde{B}$ if and only if $\mu_{\tilde{A}}(x) \leq \mu_{\tilde{A}}(x), \forall \ x \in X$
2. $\tilde{A} = \tilde{B}$ if and only if $\tilde{A} \subseteq \tilde{B}$ and $\tilde{B} \subseteq \tilde{A}$
3. The complement of the fuzzy set $\tilde{A}$ is $\tilde{A}^c$ and is defined as
   $\mu_{\tilde{A}^c}(x) = 1 - \mu_{\tilde{A}}(x)$
4. $\mu_{\tilde{A} \cap \tilde{B}}(x) = \min\{\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)\}, \forall \ x \in X$
5. $\mu_{\tilde{A} \cup \tilde{B}}(x) = \max\{\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)\}, \forall \ x \in X$

Definition 2.2 (fuzzy subgroup) *[6] Let G be any group. $\tilde{A}$ is a fuzzy set of G, then $\tilde{A}$ is called fuzzy subgroup (FSG) of G if*

1. $\mu_{\tilde{A}}(xy) \geq \min\{\mu_{\tilde{A}}(x), \mu_{\tilde{A}}(y)\} \ \forall \ x, y \in G$
2. $\mu_{\tilde{A}}(x^{-1}) \geq \mu_{\tilde{A}}(x), \forall \ x \in G$

Proposition 2.3 *A fuzzy set $\tilde{A}$ of a group G is a FSG if and only if $\mu_{\tilde{A}}(xy^{-1}) \geq min\{\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(y)\}, \forall \ x, y \in G$*

Proposition 2.4 *If $\tilde{A}$ is a FSG of group G, then*

1. $\mu_{\tilde{A}}(x) \leq \mu_{\tilde{A}}(e), \forall \ x \in G$, where e is the identity element of group G
2. $\mu_{\tilde{A}}(xy^{-1}) = \mu_{\tilde{A}}(e) \Rightarrow \ \mu_{\tilde{A}}(x) = \mu_{\tilde{A}}(y), \forall \ x, y \in G$

Definition 2.5 (left fuzzy coset) *[3] Let $\tilde{A}$ is a FSG of group G. For any $x \in G$, the fuzzy set $x\tilde{A}$ defined by $\mu_{x\tilde{A}}(y) = \mu_{\tilde{A}}(x^{-1}y) \ \forall \ y \in G$ is called a left fuzzy coset of $\tilde{A}$.*

Definition 2.6 (right fuzzy coset) *[3] Let $\tilde{A}$ is a FSG of group G. For any $x \in G$, the fuzzy set $\tilde{A}x$ defined by $\mu_{\tilde{A}x}(y) = \mu_{\tilde{A}}(yx^{-1}) \ \forall \ y \in G$ is called a right fuzzy coset of $\tilde{A}$.*

Definition 2.7 (fuzzy normal subgroup) *[6] If $\tilde{A}$ is a FSG of group G, then $\tilde{A}$ is called a fuzzy normal subgroup (FNSG) of G if*

$$\mu_{\tilde{A}}(xyx^{-1}) \geq \mu_{\tilde{A}}(y) \ \forall \ x, y \in G$$

Definition 2.8 (*t*- level subset) *[4] Let $\tilde{A}$ be a fuzzy set of a group G. For $t \in [0,1]$, the t - level subset of $\tilde{A}$ is the set*

$$t_{\tilde{A}} = \{x \in G: \mu_{\tilde{A}}(x) \geq t\}.$$

Definition 2.9 (*t*- level subgroup) *[4] Let $\tilde{A}$ be a FSG of group G. For $t \in [0,1]$ with $t \leq \mu_{\tilde{A}}(e)$ the subgroup $t_{\tilde{A}}$ of G is called t - level subgroup of $\tilde{A}$.*

Definition 2.10 ($\alpha -$ Fuzzy Subset) *Let A be a fuzzy subset of a group G. Let $\alpha \in [0,1]$. Then the fuzzy set $\tilde{A}^{\alpha}$ of G is called the $\alpha -$ fuzzy subset of G (with respect to fuzzy set $\tilde{A}$) and is defined as*

$$\mu_{\tilde{A}^{\alpha}}(x) = \min\{\mu_{\tilde{A}}(x), \alpha\}, \forall \ x \in G$$

Definition 2.11 ($\alpha -$ Fuzzy Subgroup) *If $\tilde{A}$ is a $\alpha -$ fuzzy subset of group G. $\tilde{A}$ is called $\alpha -$ fuzzy subgroup ($\alpha -$ FSG) of G if $\tilde{A}^{\alpha}$ is a fuzzy subgroup of G. i.e. if the following conditions hold*

1. $\mu_{\tilde{A}^{\alpha}}(xy) \geq \min\{\mu_{\tilde{A}^{\alpha}}(x), \mu_{\tilde{A}^{\alpha}}(y)\}, \forall \ x, y \in G$
2. $\mu_{\tilde{A}^{\alpha}}(x^{-1}) = \mu_{\tilde{A}^{\alpha}}(x), \forall \ x \in G.$

Definition 2.12 (fuzzy order of an element) *[5] Let $\tilde{A}$ be a fuzzy subgroup of a group G. Given $x \in G$, the smallest positive integer n such that $\mu_{\tilde{A}}(x^n) = \mu_{\tilde{A}}(e)$ is called the fuzzy order of x with respect to $\tilde{A}$. If no such n exists, x is said to have infinite fuzzy order with respect to $\tilde{A}$. The fuzzy order of x with respect to $\tilde{A}$ is denoted by $FO_{\tilde{A}}(x)$.*

Definition 2.13 (order of a fuzzy subgroup) *[5] Let $\tilde{A}$ be a fuzzy subgroup of a group G. The least positive integer n such that $\mu_{\tilde{A}}(x^n) = \mu_{\tilde{A}}(e), \forall \ x \in G$, is called the order of $\tilde{A}$ and denoted by $O(\tilde{A})$. If no such n exists, $\tilde{A}$ is said to have an infinite order.*

Theorem 2.14 (Lagrange's theorem for fuzzy subgroups) *[1] Let H be a subgroup of a group G and let n be the order of a fuzzy subgroup $\tilde{A}$ of G. then $O(\tilde{A}|_H)|O(\tilde{A})$.*

Proof. $O(\tilde{A}) = n$. Then $\mu_{\tilde{A}}(x^n) = \mu_{\tilde{A}}(e), \forall\ x \in G$.
Now $\mu_{\tilde{A}|_H}(x) = \mu_{\tilde{A}}(x), \forall\ x \in H$
$\Rightarrow O(\tilde{A}|_H) \leq O(\tilde{A})$.
If $O(\tilde{A}|_H) = n$, then $O(\tilde{A}|_H)|O(\tilde{A})$.
If $O(\tilde{A}|_H) < n$, let $O(\tilde{A}|_H) = m$. Then $\mu_{\tilde{A}|_H}(x^m) = \mu_{\tilde{A}|_H}(e), \forall\ x \in H$
$\Rightarrow \mu_{\tilde{A}}(x^m) = \mu_{\tilde{A}}(e), \forall\ x \in H$
$\Rightarrow m|n, i.e. O(\mu_{\tilde{A}|_H})|O(\mu_{\tilde{A}})$.

3 Lagrange's theorem for $\alpha -$ fuzzy subgroups

Definition 3.1 ($\alpha -$ fuzzy order of an element) *Let $\tilde{A}$ be a fuzzy subset of a group G. Let $\alpha \in [0,1]$ such that $\tilde{A}^\alpha$ is a $\alpha -$ FSG of G with respect to $\tilde{A}$. Given $x \in G$, the smallest positive integer n such that $\mu_{\tilde{A}^\alpha}(x^n) = \mu_{\tilde{A}^\alpha}(e)$ is called the $\alpha -$ fuzzy order of x with respect to $\tilde{A}$. if no such n exists, x is said to have infinite $\alpha -$ fuzzy order with respect to $\tilde{A}$. The $\alpha -$ fuzzy order of x with respect to $\tilde{A}$ is denoted by $FO_{\tilde{A}^\alpha}(x)$.*

Definition 3.2 (order of a $\alpha -$ fuzzy subgroup) *Let $\tilde{A}$ be a fuzzy subset of a group G. Let $\alpha \in [0,1]$ such that $\tilde{A}^\alpha$ is a $\alpha -$ FSG of G with respect to $\tilde{A}$. The least positive integer n such that $\mu_{\tilde{A}^\alpha}(x^n) = \mu_{\tilde{A}^\alpha}(e), \forall\ x \in G$, is called the order of $\tilde{A}^\alpha$ with respect to $\tilde{A}$ and denoted by $O(\tilde{A}^\alpha)$. If no such n exists, $\tilde{A}^\alpha$ is said to have an infinite order.*

Some Results :

1. If $\tilde{A}$ be a $\alpha -$ FSG of a group G and H be a subgroup of G, then $\tilde{A}|_H$ is a $\alpha -$ FSG of H.
2. If $\tilde{A}$ is a FSG of a group G, then $\tilde{A}$ is also a $\alpha -$ FSG of G.
3. Intersection of two $\alpha -$ FSG's of a group G is also $\alpha -$ FSG of G.
4. If $\tilde{A}$ and $\tilde{B}$ be two fuzzy subset of X. Then
$$(\tilde{A} \cap \tilde{B})^\alpha = \tilde{A}^\alpha \cap \tilde{B}^\alpha$$

Theorem 3.3 *Let H be a subgroup of a group G and let n be the order of a $\alpha -$ fuzzy subgroup $\tilde{A}$ of G. then $O(\tilde{A}^\alpha|_H)|O(\tilde{A}^\alpha)$.*

Proof. $O(\tilde{A}^\alpha) = n$. Then $\mu_{\tilde{A}^\alpha}(x^n) = \mu_{\tilde{A}^\alpha}(e), \forall\ x \in G$.
Now $\mu_{\tilde{A}^\alpha|_H}(x) = \mu_{\tilde{A}^\alpha}(x), \forall\ x \in H$
$\Rightarrow O(\tilde{A}^\alpha|_H) \leq O(\tilde{A}^\alpha)$.
If $O(\tilde{A}^\alpha|_H) = n$, then $O(\tilde{A}^\alpha|_H)|O(\tilde{A})$.
If $O(\tilde{A}^\alpha|_H) < n$, let $O(\tilde{A}^\alpha|_H) = m$. Then $\mu_{\tilde{A}^\alpha|_H}(x^m) = \mu_{\tilde{A}^\alpha|_H}(e), \forall\ x \in H$
$\Rightarrow \mu_{\tilde{A}^\alpha}(x^m) = \mu_{\tilde{A}^\alpha}(e), \forall\ x \in H$
$\Rightarrow m|n, i.e. O(\mu_{\tilde{A}^\alpha|_H})|O(\mu_{\tilde{A}^\alpha})$.

4 some results on α fuzzy subgroup

Theorem 4.1 *Let $\tilde{A}$ be a FSG of a group G. For $\alpha \in [0,1]$ with $\mu_{\tilde{A}}(e) \leq \alpha$, $\tilde{A}^\alpha = \tilde{A}$*

Proof. For $\alpha \in [0,1]$ and $\mu_{\tilde{A}}(e) \leq \alpha$. Since $\tilde{A}$ is a FSG, therefore $\mu_{\tilde{A}}(xy) \geq \min\{\mu_{\tilde{A}}(x), \mu_{\tilde{A}}(y)\}$ and $\mu_{\tilde{A}}(x^{-1}) \geq \mu_{\tilde{A}}(x)$ for all $x, y \in G$.
Now for any $x \in G$
$$\mu_{\tilde{A}}(xx^{-1}) \geq \min\{\mu_{\tilde{A}}(x), \mu_{\tilde{A}}(x^{-1})\}$$
$$\Rightarrow \mu_{\tilde{A}}(e) \geq \mu_{\tilde{A}}(x)$$
Thus

$$\mu_{\tilde{A}}(x) \le \mu_{\tilde{A}}(e) \quad \forall \quad x \in G \tag{1}$$

$$\Rightarrow \mu_{\tilde{A}}(x) \le \alpha \quad \forall \quad x \in G \tag{2}$$

Now for any $x \in G$

$$\mu_{\tilde{A}^\alpha}(x) = \min\{\mu_{\tilde{A}}(x), \alpha\}$$

$$\Rightarrow \mu_{\tilde{A}^\alpha}(x) = \mu_{\tilde{A}}(x)$$

therefore $\mu_{\tilde{A}^\alpha}(x) = \mu_{\tilde{A}}(x) \quad \forall \quad x \in G$. Thus

$$\tilde{A}^\alpha = \tilde{A} \tag{3}$$

Corollary 4.2 *Let $\tilde{A}$ be a FSG of a group G. For $\alpha \in [0,1]$ with $\mu_{\tilde{A}}(e) \le \alpha$, $O(\tilde{A}^\alpha) = O(\tilde{A})$.*

Corollary 4.3 *Let $\tilde{A}$ be a FSG of a group G. For any $x \in G$ and $\alpha \in [0,1]$ with $\mu_{\tilde{A}}(e) \le \alpha$, $FO_{\tilde{A}^\alpha}(x) = FO_{\tilde{A}}(x)$.*

5 $\alpha -$ fuzzy subgroup on different domain

Definition 5.1 *Let $\tilde{A}$ and $\tilde{B}$ are fuzzy set of any set X and Y respectively. If $X \cap Y \ne \phi$, then $\tilde{A} \cap \tilde{B}$ is a fuzzy set of $X \cap Y$.*

Here $\mu_{\tilde{A}\cap\tilde{B}}(x) = \min\{\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)\}, \forall \quad x \in X \cap Y$

Theorem 5.2 *Let G be any group. H and K are subgroups of G such that $H \cap K \ne 0$. If for $\alpha \in [0,1]$, $\tilde{A}$ and $\tilde{B}$ are the $\alpha -$ fuzzy subgroup of H and K respectively, then $\tilde{A} \cap \tilde{B}$ is a $\alpha -$ fuzzy subgroup of $X \cap Y$.*

Proof. For any $x, y \in X \cap Y$

$$\mu_{(\tilde{A}\cap\tilde{B})^\alpha}(xy) \ge \min\{\mu_{(\tilde{A}\cap\tilde{B})^\alpha}(x), \mu_{(\tilde{A}\cap\tilde{B})^\alpha}(y)\} \tag{4}$$

and

$$\mu_{(\tilde{A}\cap\tilde{B})^\alpha}(x^{-1}) \ge \mu_{(\tilde{A}\cap\tilde{B})^\alpha}(x) \tag{5}$$

Hence $\tilde{A} \cap \tilde{B}$ is a $\alpha -$ FSG of $H \cap K$.

6 Conclusion

In this paper, we have introduced the concept of order of the α-fuzzy subgroup, and Lagrange's theorem for $\alpha -$ fuzzy subgroup. Concept of the intersection of two $\alpha -$ fuzzy subgroup defined on different domain is discussed.

Further work is based on the intersection of $\alpha -$ fuzzy set in different domain.

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