IJCRT.ORG ISSN : 2320-2882

INTERNATIONAL JOURNAL OF CREATIVE RESEARCH THOUGHTS (IJCRT)

An International Open Access, Peer-reviewed, Refereed Journal

ηg-closed Sets and η-Normal Spaces in Topological Spaces

Hamant Kumar
Department of Mathematics
Veerangana Avantibai Government Degree College, Atrauli-Aligarh, U. P. (India)

Abstract. The aim of this paper is to introduce and study a new class of sets called ηg-closed sets and a new class of spaces called η-normal spaces. The relationships among β*g-normal, α-normal, s-normal and η-normal spaces are investigated. Moreover, we introduce the concept of η-generalized closed functions. We also obtain some characterizations and preservation theorems of η-normal spaces, in the forms of generalized η-closed and η-generalized closed functions.

Key Words: η-open, gη-closed, and ηg-closed sets; η-normal spaces; η-closed and η-ηg-closed functions.

2020 Mathematics Subject Classification: 54A05, 54C08, 54C10, 54D15.

1. Introduction

Normality is an important topological property and hence it is of significance both from intrinsic interest as well as from applications view point to obtain factorizations of normality in terms of weaker topological properties. In 1937, Stone [**15**] introduced the notion of regular open sets. In 1963, Levine [**9**] introduced the concept of semi-open sets. In 1965, Njastad [**13**] introduced the concept of α-open sets. In 1970, Levine [**10**] initiated the study of generalized closed (briefly g-closed) sets. In 1973, Carnahan [**5**] introduced the concept of R-map. In 1974, Arya and Gupta [**1**] introduced the notion of completely continuity. In 1978, Maheshwari and Prasad [**11**] introduced the notion of s-normal spaces and obtained their characterizations. In 1990, Arya and Nour [**2**] introduced the concept of gs-closed sets. In 1994, Maki et al. [**12**] introduced the notion of αg-closed and gα-closed sets. In 2009, Benchalli and Patil [**3**] introduced the contept of α-normal spaces and obtained their characterizations. In 2019, Subbulakshmi, Sumathi and Indirani [**16, 17, 18**] introduced and investigated the notion of η-open and gη-closed sets. In 2019, Kumar, Singh and Kumar [**7**] introduced the concepts of β*g-normal spaces and obtained properties of β*g-normal spaces. In 2021, Kumar and Sharma [**8**] introduced the concepts of η-separation axioms in topological spaces and obtained some properties of η-separation axioms.

2. Preliminaries

In what follows, spaces always mean topological spaces on which no separation axioms are assumed unless explicitly stated and $f : (X, \Im) \to (Y, \sigma)$ (or simply $f : X \to Y$) denotes a function f of a space $(X, \Im)$ into a space (Y, σ). Let A be a subset of a space X. The closure and the interior of A are denoted by cl(A) and int(A), respectively.

2.1 Definition. A subset A of a space X is said to be:

(1) **regular open** [**15**] if A = int(cl(A)).
(2) **regular closed** [**15**] if A = cl(int(A)).
(3) **s-open** [**9**] if $A \subset$ cl(int(A)).
(4) **α-open** [**13**] if $A \subset$ int(cl(int(A))).
(5) **η-open** [**16**] if $A \subset$ int(cl(int(A))) $\cup$ cl(int(A)).

The complement of a s-open (resp. α-open, η-open) set is called **s-closed** (resp. **α-closed**, **η-closed**).

The intersection of all s-closed (resp. α-closed, η-closed) sets containing A is called the **s-closure** (resp. **α-closure**, **η-closure**) of A and is denoted by **s-cl(A)** (resp. **α-cl(A)**, **η-cl(A)**). The **η-interior** of A, denoted by **η-int(A)** is defined to be the union of all η-open sets contained in A.

The family of all η-open (resp. η-closed, regular open, regular closed, s-open, s-closed, α-open, α-closed) sets of a space X is denoted by **η-O(X)** (resp. **η-C(X), R-O(X), R-C(X), S-O(X), S-C(X), α-O(X), α-C(X)**).

2.2 Definition. A subset A of a space $(X, \Im)$ is said to be
(1) **g-closed** [**10**] if $cl(A) \subset U$ whenever $A \subset U$ and $U \in \Im$.
(2) **gs-closed** [**2**] if $s\text{-}cl(A) \subset U$ whenever $A \subset U$ and $U \in \Im$.
(3) **sg-closed** [**4**] if $s\text{-}cl(A) \subset U$ whenever $A \subset U$ and $U \in S\text{-}O(X)$.
(4) **αg-closed** [**12**] if $\alpha\text{-}cl(A) \subset U$ whenever $A \subset U$ and $U \in \Im$.
(5) **gα-closed** [**12**] if $\alpha\text{-}cl(A) \subset U$ whenever $A \subset U$ and $U \in \alpha\text{-}O(X)$.
(6) **gη-closed** [**17**] if $\eta\text{-}cl(A) \subset U$ whenever $A \subset U$ and $U \in \Im$.
(7) **ηg-closed** if $\eta\text{-}cl(A) \subset U$ whenever $A \subset U$ and $U \in \eta\text{-}O(X)$.

The complement of g-closed (resp. gs-closed, sg-closed, αg-closed, gα-closed, gη-closed, ηg-closed) set is said to be **g-open** (resp. **gs-open, sg-open, αg-open, gα-open, gη-open, ηg-open**).

2.3 Remark. We have the following implications for the properties of subsets:

$$\begin{tikzcd}
\text{closed} \arrow[rr] \arrow[d] & & \text{g-closed} \arrow[d] \\
\alpha\text{-closed} \arrow[r] \arrow[d] & \text{g}\alpha\text{-closed} \arrow[r] & \alpha\text{g-closed} \arrow[d] \\
\text{s-closed} \arrow[r] \arrow[d] & \text{sg-closed} \arrow[r] & \text{gs-closed} \arrow[d] \\
\eta\text{-closed} \arrow[r] & \eta\text{g-closed} \arrow[r] & \text{g}\eta\text{-closed}
\end{tikzcd}$$

Where none of the implications is reversible as can be seen from the following examples:

2.4 Example. Let X = {a, b, c, d} and $\Im$ = {ϕ, {b, d}, {a, b, d}, {b, c, d}, X}. Then
(1) closed sets in $(X, \Im)$ are ϕ, X, {a}, {c}, {a, c}.
(2) g-closed sets in $(X, \Im)$ are ϕ, X, {a}, {c}, {a, c}, {a, b, c}, {a, c, d}.
(3) s-closed sets in $(X, \Im)$ are ϕ, X, {a}, {c}, {a, c}.
(4) gs-closed sets in $(X, \Im)$ are ϕ, X, {a}, {c}, {a, c}, {a, b, c}, {a, c, d}.
(5) sg-closed sets in $(X, \Im)$ are ϕ, X, {a}, {c}, {a, c}, {a, b, c}, {a, c, d}.
(6) α-closed sets in $(X, \Im)$ are ϕ, X, {a}, {c}, {a, c}.
(7) αg-closed sets in $(X, \Im)$ are ϕ, X, {a}, {c}, {a, c}, {a, b, c}, {a, c, d}.
(8) gα-closed sets in $(X, \Im)$ are ϕ, X, {a}, {c}, {a, c}, {a, b, c}, {a, b, d}, {a, c, d}.
(9) η-closed sets in $(X, \Im)$ are ϕ, X, {a}, {c}, {a, c}.
(10) gη-closed sets in $(X, \Im)$ are ϕ, X, {a}, {c}, {a, c}, {a, b, c}, {a, b, d}, {a, c, d}, {b, c, d}.
(11) ηg-closed sets in $(X, \Im)$ are ϕ, X, {a}, {c}, {a, c}, {a, b, c}, {a, b, d}, {a, c, d}, {b, c, d}.

2.5 Example. Let X = {a, b, c, d} and $\Im$ = {ϕ, {a}, {b}, {a, b}, {a, c}, {a, b, c}, X}. Then
(1) closed sets in $(X, \Im)$ are ϕ, X, {d}, {b, d}, {c, d}, {a, c, d}, {b, c, d}.
(2) g-closed sets in $(X, \Im)$ are ϕ, X, {d}, {a, d}, {b, d}, {c, d}, {a, b, d}, {a, c, d}, {b, c, d}.
(3) s-closed sets in $(X, \Im)$ are ϕ, X, {b}, {c}, {d}, {a, c}, {b, d}, {c, d}, {a, c, d}, {b, c, d}.
(4) gs-closed sets in $(X, \Im)$ are ϕ, X, {b}, {c}, {d}, {a, c}, {a, d}, {b, d}, {c, d}, {a, b, d}, {a, c, d}, {b, c, d}.
(5) sg-closed sets in $(X, \Im)$ are ϕ, X, {b}, {c}, {d}, {a, c}, {b, d}, {c, d}, {a, c, d}, {b, c, d}.
(6) α-closed sets in $(X, \Im)$ are ϕ, X, {c}, {d}, {b, d}, {c, d}, {a, c, d}, {b, c, d}.
(7) αg-closed sets in $(X, \Im)$ are ϕ, X, {c}, {d}, {a, d}, {b, d}, {c, d}, {a, b, d}, {a, c, d}, {b, c, d}.
(8) gα-closed sets in $(X, \Im)$ are ϕ, X, {c}, {d}, {b, d}, {c, d}, {a, c, d}, {b, c, d}.

(9) η-closed sets in $(X, \Im)$ are ϕ, X, {b}, {c}, {d}, {a, c}, {b, c}, {b, d}, {c, d}, {a, c, d}, {b, c, d}.
(10) gη-closed sets in $(X, \Im)$ are ϕ, X, {b}, {c}, {d}, {a, c}, {b, c}, {b, d}, {c, d}, {a, b, d}, {a, c, d}, {b, c, d}.
(11) ηg-closed sets in $(X, \Im)$ are ϕ, X, {b}, {c}, {d}, {a, c}, {b, c}, {b, d}, {c, d}, {a, c, d}, {b, c, d}.

3. η-normal Spaces

3.1 Definition. A space X is said to be η-normal if for any pair of disjoint closed sets A and B, there exist disjoint η-open sets U and V such that $A \subset U$ and $B \subset V$.

3.2 Definition. A space X is said to be α-normal [**3**] (resp. s-normal [**11**], β^*g-normal [**11**]) if for any pair of disjoint closed sets A and B, there exist disjoint α-open (resp. s-open, β^*g-open) sets U and V such that $A \subset U$ and $B \subset V$.

3.3 Remark. The following diagram holds for a topological space $(X, \Im)$:

normal $\rightarrow$ β^*g-normal $\rightarrow$ α-normal $\rightarrow$ s-normal $\rightarrow$ η-normal

None of these implications is reversible as shown by the following examples.

3.4 Example. Let X = {a, b, c} and $\Im$ = {ϕ, {a}, {b, c}, X}. Then the space $(X, \Im)$ is normal as well as η-normal.

3.5 Example. Let X = {a, b, c, d} and $\Im$ = {ϕ, {a}, {b}, {a, b}, {a, b, c}, {a, b, d}, X}. Let A = {c} and B = {d} be disjoint closed sets, there exist disjoint s-open sets U = {a, c} and V = {b, d} such that $A \subset U$ and $B \subset V$. Then the space $(X, \Im)$ is s-normal as well as η-normal, since every s-open set is η-open. But it is neither normal nor α-normal, because U and V are neither open nor α-open sets.

3.6 Example. Let X = {a, b, c, d} and $\Im$ = {ϕ, {a}, {c}, {a, b}, {a, c}, {b, d}, {c, d}, {a, b, d}, {b, c, d}, X}. Let A = {a} and B = {c} be disjoint closed sets, there exist disjoint open sets U = {a} and V = {c} such that $A \subset U$ and $B \subset V$. Then the space $(X, \Im)$ is normal as well as α-normal, s-normal, η-normal, since every open set is α-open, s-open and η-open.

3.7 Example. Let X = {a, b, c, d} and $\Im$ = {ϕ, {a}, {c], {a, c}, {b, c, d}, X}. Then the space $(X, \Im)$ is normal as well as α-normal, s-normal, η-normal, since every open set is α-open, s-open and η-open.

3.8 Theorem. For a space X the following are equivalent:
(1) X is η-normal,
(2) For every pair of open sets U and V whose union is X, there exist η-closed sets A and B such that $A \subset U$, $B \subset V$ and $A \cup B = X$,
(3) For every closed set H and every open set K containing H, there exists an η-open set U such that $H \subset U \subset \eta\text{-cl}(U) \subset K$.

Proof. (1) $\Rightarrow$ (2) : Let U and V be a pair of open sets in an η-normal space X such that $X = U \cup V$. Then $X - U$, $X - V$ are disjoint closed sets. Since X is η-normal, there exist disjoint η-open sets U_1 and V_1 such that $X - U \subset U_1$ and $X - V \subset V_1$. Let $A = X - U_1$, $B = X - V_1$. Then A and B are η-closed sets such that $A \subset U$, $B \subset V$ and $A \cup B = X$.

(2) $\Rightarrow$ (3) : Let H be a closed set and K be an open set containing H. Then $X - H$ and K are open sets whose union is X. Then by (2), there exist η-closed sets M_1 and M_2 such that $M_1 \subset X - H$ and $M_2 \subset K$ and $M_1 \cup M_2 = X$. Then $H \subset X - M_1$, $X - K \subset X - M_2$ and $(X - M_1) \cap (X - M_2) = \phi$. Let $U = X - M_1$ and $V = X - M_2$. Then U and V are disjoint η-open sets such that $H \subset U \subset X - V \subset K$. As $X - V$ is η-closed set, we have $\eta\text{-cl}(U) \subset X - V$ and $H \subset U \subset \eta\text{-cl}(U) \subset K$.

(3) $\Rightarrow$ (1) : Let H_1 and H_2 be any two disjoint closed sets of X. Put $K = X - H_2$, then $H_2 \cap K = \phi$. $H_1 \subset K$, where K is an open set. Then by (3), there exists an η-open set U of X such that $H_1 \subset U \subset \eta\text{-cl}(U) \subset K$. It follows that $H_2 \subset X - \eta\text{-cl}(U) = V$, say, then V is η-open and $U \cap V = \phi$. Hence H_1 and H_2 are separated by η-open sets U and V. Therefore X is η-normal.

4. η-normal Spaces with Some Related Functions

4.1 Definition . A function $f : X \rightarrow Y$ is called
(1) **R-map** [**5**] if $f^{-1}(V)$ is regular open in X for every regular open set V of Y ,
(2) **completely continuous** [**1**] if $f^{-1}(V)$ is regular open in X for every open set V of Y ,
(3) **rc-continuous** [**6**] if for each regular closed set F in Y, $f^{-1}(F)$ is regular closed in X.

4.2 Definition . A function $f : X \to Y$ is called
(1) **strongly η-open** if $f(U) \in \eta\text{-}O(Y)$ for each $U \in \eta\text{-}O(X)$,
(2) **strongly η-closed** if $f(U) \in \eta\text{-}C(Y)$ for each $U \in \eta\text{-}C(X)$,
(3) **almost η-irresolute** if for each x in X and each η-neighbourhood V of f(x), $\eta\text{-}cl(f^{-1}(V))$ is a η-neighbourhood of x.

4.3 Theorem . A function $f : X \to Y$ is strongly η-closed if and only if for each subset A in Y and for each η-open set U in X containing $f^{-1}(A)$, there exists an η-open set V containing A such that $f^{-1}(V) \subset U$.

Proof. (⇒) : Suppose that f is strongly η-closed. Let A be a subset of Y and $U \in \eta\text{-}O(X)$ containing $f^{-1}(A)$. Put $V = Y - f(X - U)$, then V is an η-open set of Y such that $A \subset V$ and $f^{-1}(V) \subset U$.

(⇐) : Let K be any η-closed set of X. Then $f^{-1}(Y - f(K)) \subset X - K$ and $X - K \in \eta\text{-}O(X)$. There exists an η-open set V of Y such that $Y - f(K) \subset V$ and $f^{-1}(V) \subset X - K$. Therefore, we have $f(K) \supset Y - V$ and $K \subset f^{-1}(Y - V)$. Hence, we obtain $f(K) = Y - V$ and f(K) is η-closed in Y . This shows that f is strongly η-closed.

4.4 Lemma. For a function $f : X \to Y$, the following are equivalent:
(1) f is almost η-irresolute,
(2) $f^{-1}(V) \subset \eta\text{-}int(\eta\text{-}cl(f^{-1}(V)))$ for every $V \in \eta\text{-}O(Y)$.

4.5 Theorem. A function $f : X \to Y$ is almost η-irresolute if and only if $f(\eta\text{-}cl(U)) \subset \eta\text{-}cl(f(U))$ for every $U \in \eta\text{-}O(X)$.

Proof. (⇒) : Let $U \in \eta\text{-}O(X)$. Suppose $y \notin \eta\text{-}cl(f(U))$. Then there exists $V \in \eta\text{-}O(Y)$ such that $V \cap f(U) = \phi$. Hence, $f^{-1}(V) \cap U = \phi$. Since $U \in \eta\text{-}O(X)$, we have $\eta\text{-}int(\eta\text{-}cl(f^{-1}(V))) \cap \eta\text{-}cl(U) = \phi$. Then by **Lemma 4.4**, $f^{-1}(V) \cap \eta\text{-}cl(U) = \phi$ and hence $V \cap f(\eta\text{-}cl(U)) = \phi$. This implies that $y \notin f(\eta\text{-}cl(U))$.

(⇐) : If $V \in \eta\text{-}O(Y)$, then $M = X - \eta\text{-}cl(f^{-1}(V)) \in \eta\text{-}O(X)$. By hypothesis, $f(\eta\text{-}cl(M)) \subset \eta\text{-}cl(f(M))$ and hence $X - \eta\text{-}int(\eta\text{-}cl(f^{-1}(V))) = \eta\text{-}cl(M) \subset f^{-1}(\eta\text{-}cl(f(M))) \subset f^{-1}(\eta\text{-}cl(f(X - f^{-1}(V)))) \subset f^{-1}(\eta\text{-}cl(Y - V)) = f^{-1}(Y - V) = X - f^{-1}(V)$. Therefore, $f^{-1}(V) \subset \eta\text{-}int(\eta\text{-}cl(f^{-1}(V)))$. By **Lemma 4.4**, f is almost η-irresolute.

4.6 Theorem. If $f : X \to Y$ is a strongly η-open continuous almost η-irresolute function from a η-normal space X onto a space Y, then Y is η-normal.

Proof. Let A be a closed subset of Y and B be an open set containing A. Then by continuity of f, $f^{-1}(A)$ is closed and $f^{-1}(B)$ is an open set of X such that $f^{-1}(A) \subset f^{-1}(B)$. As X is η-normal, there exists an η-open set U in X such that $f^{-1}(A) \subset U \subset \eta\text{-}cl(U) \subset f^{-1}(B)$ by **Theorem 3.8**. Then, $f(f^{-1}(A)) \subset f(U) \subset f(\eta\text{-}cl(U)) \subset f(f^{-1}(B))$. Since f is strongly η-open almost η-irresolute surjection, we obtain $A \subset f(U) \subset \eta\text{-}cl(f(U)) \subset B$. Then again by **Theorem 3.8,** the space Y is η-normal.

4.7 Theorem. If $f : X \to Y$ is an strongly η-closed continuous function from an η-normal space X onto a space Y , then Y is η-normal.

Proof. Let M_1 and M_2 be disjoint closed sets. Then $f^{-1}(M_1)$ and $f^{-1}(M_2)$ are closed sets. Since X is η-normal, then there exist disjoint η-open sets U and V such that $f^{-1}(M_1) \subset U$ and $f^{-1}(M_2) \subset V$. By **Theorem 4.3**, there exist η-open sets A and B such that $M_1 \subset A$, $M_2 \subset B$, $f^{-1}(A) \subset U$ and $f^{-1}(B) \subset V$. Also, A and B are disjoint. Thus, Y is η-normal.

5. η-generalized Closed Functions

5.1 Definition. A function $f : X \to Y$ is said to be
(1) **η-closed** [**8**] if f(A) is η-closed in Y for each closed set A of X,
(2) **ηg-closed** if f(A) is ηg-closed in Y for each closed set A of X,
(3) **gη-closed** [**18**] if f(A) is gη-closed in Y for each closed set A of X.

5.2 Definition. A function $f : X \to Y$ is said to be
(1) **quasi η-closed** if f(A) is closed in Y for each $A \in \eta\text{-}C(X)$,
(2) **η-ηg-closed** if f(A) is ηg-closed in Y for each $A \in \eta\text{-}C(X)$,
(3) **η-gη-closed** if f(A) is gη-closed in Y for each $A \in \eta\text{-}C(X)$,
(4) **almost gη-closed** if f(A) is gη-closed in Y for each $A \in R\text{-}C(X)$.

5.3 Definition. A function $f : X \to Y$ is said to be **η-gη-continuous** if $f^{-1}(K)$ is gη-closed in X for every $K \in \eta\text{-}C(Y)$.

5.4 Definition. A function $f : X \to Y$ is said to be **η-irresolute** [**8**] if $f^{-1}(V) \in \eta\text{-}O(X)$ for every $V \in \eta\text{-}O(Y)$.

5.5 Theorem. Let $f : X \to Y$ and $g : Y \to Z$ be functions. Then

(1) the composition gof : $X \to Z$ is η-gη-closed if f is η-gη-closed and g is continuous η-gη-closed.
(2) the composition gof : $X \to Z$ is η-gη-closed if f is strongly η-closed and g is η-gη-closed.
(3) the composition gof : $X \to Z$ is η-gη-closed if f is quasi η-closed and g is gη-closed.

5.6 Theorem. Let $f : X \to Y$ and $g : Y \to Z$ be functions and let the composition gof : $X \to Z$ be η-gη-closed. If f is a η-irresolute surjection, then g is η-gη-closed.

Proof. Let $K \in$ η-C(Y). Since f is η-irresolute and surjective, $f^{-1}(K) \in$ η-C(X) and (gof)($f^{-1}(K)$) = g(K). Hence, g(K) is gη-closed in Z and hence g is η-gη-closed.

5.7 Lemma. A function $f : X \to Y$ is η-gη-closed if and only if for each subset B of Y and each U $\in$ η-O(X) containing $f^{-1}(B)$, there exists a gη-open set V of Y such that $B \subset V$ and $f^{-1}(V) \subset U$.

Proof. ($\Rightarrow$) : Suppose that f is η-gη-closed. Let B be a subset of Y and U $\in$ η-O(X) containing $f^{-1}(B)$. Put $V = Y - f(X - U)$, then V is a gη-open set of Y such that $B \subset V$ and $f^{-1}(V) \subset U$.

($\Leftarrow$) : Let K be any η-closed set of X. Then $f^{-1}(Y - f(K)) \subset X - K$ and $X - K \in$ η-O(X). There exists a gη-open set V of Y such that $Y - f(K) \subset V$ and $f^{-1}(V) \subset X - K$. Therefore, we have $f(K) \supset Y - V$ and $K \subset f^{-1}(Y - V)$. Hence, we obtain $f(K) = Y - V$ and f(K) is gη-closed in Y . This shows that f is η-gη-closed.

5.8 Theorem. If $f : X \to Y$ is continuous η-gη-closed, then f(H) is gη-closed in Y for each gη-closed set H of X.

Proof. Let H be any gη-closed set of X and V an open set of Y containing f(H). Since $f^{-1}(V)$ is an open set of X containing H, η-cl(H) $\subset f^{-1}(V)$ and hence f(η-cl(H)) $\subset$ V . Since f is η-gη-closed and η-cl(H) $\in$ η-C(X), we have η-cl(f(H)) $\subset$ η-cl(f(η-cl(H))) $\subset$ V . Therefore, f(H) is gη-closed in Y .

5.9 Remark. Every η-irresolute function is η-gη-continuous but not conversely.

5.10 Theorem. A function $f : X \to Y$ is η-gη-continuous if and only if $f^{-1}(V)$ is gη-open in X for every V $\in$ η-O(Y).

5.11 Theorem. If $f : X \to Y$ is closed η-gη-continuous, then $f^{-1}(K)$ is gη-closed in X for each gη-closed set K of Y .

Proof. Let K be a gη-closed set of Y and U an open set of X containing $f^{-1}(K)$. Put $V = Y - f(X - U)$, then V is open in Y, $K \subset V$, and $f^{-1}(V) \subset U$. Therefore, we have η-cl(K) $\subset$ V and hence $f^{-1}(K) \subset f^{-1}$(η-cl(K)) $\subset f^{-1}(V) \subset U$. Since f is η-gη-continuous, f^{-1}(η-cl(K)) is gη-closed in X and hence η-cl($f^{-1}(K)$) $\subset$ η-cl(f^{-1}(η-cl(K))) $\subset$ U. This shows that $f^{-1}(K)$ is gη-closed in X.

5.12 Corollary. If $f : X \to Y$ is closed η-irresolute, then $f^{-1}(K)$ is gη-closed in X for each gη-closed set K of Y .

5.13 Theorem. If $f : X \to Y$ is an open η-gη-continuous bijection, then $f^{-1}(K)$ is gη-closed in X for every gη-closed set K of Y .

Proof. Let K be a gη-closed set of Y and U an open set of X containing $f^{-1}(K)$. Since f is an open surjective, $K = f(f^{-1}(K)) \subset f(U)$ and f(U) is open. Therefore, η-cl(K) $\subset$ f(U). Since f is injective, $f^{-1}(K) \subset f^{-1}$(η-cl(K)) $\subset f^{-1}(f(U)) = U$. Since f is η-gη-continuous, f^{-1}(η-cl(K)) is gη-closed in X and hence η-cl($f^{-1}(K)$) $\subset$ η-cl(f^{-1}(η-cl(K))) $\subset$ U. This shows that $f^{-1}(K)$ is gη-closed in X.

5.14 Theorem. Let $f : X \to Y$ and $g : Y \to Z$ be functions and let the composition gof : $X \to Z$ be η-gη-closed. If g is an open η-gη-continuous bijection, then f is η-gη-closed.

Proof. Let H $\in$ η-C(X). Then (gof)(H) is gη-closed in Z and g^{-1}((gof)(H)) = f(H). By **Theorem 5.13**, f(H) is gη-closed in Y and hence f is η-gη-closed.

5.15 Theorem. Let $f : X \to Y$ and $g : Y \to Z$ be functions and let the composition gof : $X \to Z$ be η-gη-closed. If g is a closed η-gη-continuous injection, then f is η-gη-closed.

Proof. Let H $\in$ η-C(X). Then (gof)(H) is gη-closed in Z and g^{-1}((gof)(H)) = f(H). By **Theorem 5.11**, f(H) is gη-closed in Y and hence f is η-gη-closed.

6. Characterizations of η-normal Spaces and Some Preservation Theorems

6.1 Theorem. For a topological space X, the following are equivalent :
(a) X is η-normal,
(b) for any pair of disjoint closed sets A and B of X, there exist disjoint gη-open sets U and V of X such that $A \subset U$ and $B \subset V$,
(c) for each closed set A and each open set B containing A, there exists a gη-open set U such that $cl(A) \subset U \subset \eta\text{-}cl(U) \subset B$,
(d) for each closed A and each g-open set B containing A, there exists an η-open set U such that $A \subset U \subset \eta\text{-}cl(U) \subset int(B)$,
(e) for each closed A and each g-open set B containing A, there exists a gη-open set G such that $A \subset G \subset \eta\text{-}cl(G) \subset int(B)$,
(f) for each g-closed set A and each open set B containing A, there exists an η-open set U such that $cl(A) \subset U \subset \eta\text{-}cl(U) \subset B$,
(g) for each g-closed set A and each open set B containing A, there exists a gη-open set G such that $cl(A) \subset G \subset \eta\text{-}cl(G) \subset B$.

Proof. (a) ⇔ (b) ⇔ (c) : Since every η-open set is gη-open, it is obvious.

(d) ⇒ (e) ⇒ (c) and (f) ⇒ (g) ⇒ (c) : Since every closed (resp. open) set is g-closed (resp. g-open), it is obvious.

(c) ⇒ (e) : Let A be a closed subset of X and B be a g-open set such that $A \subset B$. Since B is g-open and A is closed, $A \subset int(A)$. Then, there exists a gη-open set U such that $A \subset U \subset \eta\text{-}cl(U) \subset int(B)$.

(e) ⇒ (d) : Let A be any closed subset of X and B be a g-open set containing A. Then there exists a gη-open set G such that $A \subset G \subset \eta\text{-}cl(G) \subset int(B)$. Since G is gη-open, $A \subset \eta\text{-}int(G)$. Put $U = \eta\text{-}int(G)$, then U is η-open and $A \subset U \subset \eta\text{-}cl(U) \subset int(B)$.

(c) ⇒ (g) : Let A be any g-closed subset of X and B be an open set such that $A \subset B$. Then $cl(A) \subset B$. Therefore, there exists a gη-open set U such that $cl(A) \subset U \subset \eta\text{-}cl(U) \subset B$.

(g) ⇒ (f) : Let A be any g-closed subset of X and B be an open set containing A. Then there exists a gη-open set G such that $cl(A) \subset G \subset \eta\text{-}cl(G) \subset B$. Since G is gη-open and $cl(A) \subset G$, we have $cl(A) \subset \eta\text{-}int(G)$, put $U = \eta\text{-}int(G)$, then U is η-open and $cl(A) \subset U \subset \eta\text{-}cl(U) \subset B$.

6.2 Theorem. If $f : X \to Y$ is a continuous quasi η-closed surjection and X is η-normal, then Y is normal.

Proof. Let M_1 and M_2 be any disjoint closed sets of Y. Since f is continuous, $f^{-1}(M_1)$ and $f^{-1}(M_2)$ are disjoint closed sets of X. Since X is η-normal, there exist disjoint $U_1, U_2 \in \eta\text{-}O(X)$ such that $f^{-1}(M_i) \subset U_i$ for i = 1, 2. Put $V_i = Y - f(X - U_i)$, then V_i is open in Y, $M_i \subset V_i$ and $f^{-1}(V_i) \subset U_i$ for i = 1, 2. Since $U_1 \cap U_2 = \phi$ and f is surjective; we have $V_1 \cap V_2 = \phi$. This shows that Y is normal.

6.3 Lemma [17]. A subset A of a space X is gη-open if and only if $F \subset \eta\text{-}int(A)$ whenever F is closed and $F \subset A$.

6.4 Theorem. Let $f : X \to Y$ be a closed η-gη-continuous injection. If Y is η-normal, then X is η-normal.

Proof. Let N_1 and N_2 be disjoint closed sets of X, Since f is a closed injection, $f(N_1)$ and $f(N_2)$ are disjoint closed sets of Y. By the η-normality of Y, there exist disjoint $V_1, V_2 \in \eta\text{-}O(Y)$ such that $f(N_i) \subset V_i$ for i = 1, 2. Since f is η-gη-continuous, $f^{-1}(V_1)$ and $f^{-1}(V_2)$ are disjoint gη-open sets of X and $N_i \subset f^{-1}(V_i)$ for i = 1, 2. Now, put $U_i = \eta\text{-}int(f^{-1}(V_i))$ for i = 1, 2. Then, $U_i \in \eta\text{-}O(X)$, $N_i \subset U_i$ and $U_1 \cap U_2 = \phi$. This shows that X is η-normal.

6.5 Corollary. If $f : X \to Y$ is a closed η-irresolute injection and Y is η-normal, then X is η-normal.
Proof. This is an immediate consequence since every η-irresolute function is -gη-continuous.

6.6 Lemma. A function $f : X \to Y$ is almost gη-closed if and only if for each subset B of Y and each $U \in R\text{-}O(X)$ containing $f^{-1}(B)$, there exists a gη-open set V of Y such that $B \subset V$ and $f^{-1}(V) \subset U$.

6.7 Lemma. If $f : X \to Y$ is almost gη-closed, then for each closed set M of Y and each $U \in R\text{-}O(X)$ containing $f^{-1}(M)$, there exists $V \in \eta\text{-}O(Y)$ such that $M \subset V$ and $f^{-1}(V) \subset U$.

6.8 Theorem. Let $f : X \to Y$ be a continuous almost gη-closed surjection. If X is normal, then Y is η-normal.
Proof. Let M_1 and M_2 be any disjoint, closed sets of Y. Since f is continuous, $f^{-1}(M_1)$ and $f^{-1}(M_2)$ are disjoint closed sets of X. By the normality of X, there exist disjoint open sets U_1 and U_2 such that $f^{-1}(M_i) \subset U_i$, where i = 1, 2. Now, put $G_i = int(cl(U_i))$ for i = 1, 2, then $G_i \in R\text{-}O(X)$, $f^{-1}(M_i) \subset U_i \subset G_i$ and $G_1 \cap G_2 = \phi$. By **Lemma 6.7**, there exists $V_i \in \eta\text{-}O(Y)$ such that $M_i \subset V_i$ and $f^{-1}(V_i) \subset G_i$, where i = 1, 2. Since $G_1 \cap G_2 = \phi$ and f is surjective, we have $V_1 \cap V_2 = \phi$. This shows that Y is η-normal.

6.9 Corollary. If $f : X \to Y$ is a continuous η-closed surjection and X is normal, then Y is η-normal.

References

1. S. P. Arya and R. Gupta, On strongly continuous functions, Kyungpook Math. J., 14 (1974), 131-141.
2. S. P. Arya and T. Nour, Characterizations of s-normal spaces, Indian J. Pure Appl. Math., 21 (1990), 717-719.
3. S. S. Benchalli and P. G. Patil, Some new continuous maps in topological spaces, Journal of Advanced Studies in Topology, 2/1-2, (2009), 53-63.
4. P. Bhattacharyya and B. K. Lahiri, Semi generalized closed sets in topology, Indian J. Math., 29 (1987), 375-382.
5. D. Carnahan, Some properties related to compactness in topological spaces, PhD thesis, University of Arkansas, 1973.
6. D. S. Jankovic´, A note on mappings of extremally disconnected spaces, Acta Math. Hungar., 46 (1-2) (1985), 83-92.
7. H. Kumar and B. Singh and J. Kumar, β^*g-normal spaces in topological spaces, Jour. of Emerging Tech. and Innov. Res. Vol. 6, Issue 6, (2019), 516-524.
8. H. Kumar and M. C. Sharma, η-separation axioms in topological spaces, Jour. of Emerging Tech. and Innov. Res. Vol. 8, Issue 5, (2021), 516-524.
9. N. Levine, Semiopen sets and semicontinuity in topological spaces, Amer. Math. Monthly, 70 (1963), 36-41.
10. N. Levine, Generalized closed sets in topology, Rend. Circ. Mat. Palermo, 19 (2) (1970), 89-96.
11. S. N. Maheshwari and R. Prasad, On s-normal spaces, Bull. Math. Soc. Sci. Math. R. S. Roumanie (N. S.), 22 (68) (1978), 27-29.
12. H. Maki, R. Devi and K. Balachandram, Associated topologies of generalized α-open and α-generalized closed sets, Mem. Fac. Sci. Kochi Univ. Math.**1**(1994), 51-63.
13. O. Njastad, On some classes of nearly open sets, Pacific J. Math., 15 (1965), 961-970.
14. T. Noiri, Semi-normal spaces and some functions, Acta Math. Hungar., 65 (3) (1994), 305-311.
15. M. H. Stone, Applications of the theory of Boolean rings to general topology, Trans. Amer. Math. Soc., **41**(1937), 375-381.
16. D. Subbulakshmi, K. Sumathi, K. Indirani, η-open sets in topological spaces, International Jour. of Innovative and Exploring Engineering, Vol. 8, Issue 10S, (2019), 276-282.
17. D. Subbulakshmi, K. Sumathi, K. Indirani, gη-closed sets in topological spaces, International Jour. of Recent Technology and Engineering, Vol. 8, Issue 3, (2019), 8863-8866.
18. D. Subbulakshmi, K. Sumathi, K. Indirani, gη-homeomorphism in topological spaces, Advanced in Mathematics : Scientific Journal, Vol. 8, Issue 3, (2019), 705-713.