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ANALYTICAL SOLUTION FOR SPACE-TIME SOLUTE DISTRIBUTION IN A HOMOGENEOUS SEMI-INFINITE AQUIFER

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Abstract: Fourier Transform Technique (FTT) was used to obtain analytical solution of solute transport modeling in groundwater system. A solute transport model is formulated with time-dependent source concentration in one-dimensional homogeneous semi-infinite aquifer with suitable initial and boundary conditions. A time-dependent source concentration is considered at the origin of the aquifer and zero at the other end. The solutions were obtained for sinusoidal and exponential form of velocity which represent the seasonal pattern in a year. The data obtained by Mat Lab were discussed and presented in different graphs. The result may be used as preliminary predictive tools in groundwater management and benchmark for numerical solutions.

***Index Terms* - Groundwater, Analytical solution, FTT, Solute transport.**

1. Introduction

Nowadays a large number of toxic compounds are utilized in various industries in developing countries, which discharge their waste into neigh boring ponds, streams, or rivers, often without any treatment. The chemical waste may be constituted of such materials as pesticides, fertilizers, mine effluents, septic tanks, etc., which often infiltrate into the groundwater. Many aquifers are being contaminated by a host of human activities, such as sewage disposal, refuse dumps, pesticides, chemical fertilizer contamination, industrial effluent discharges, and toxic waste disposal. Overall groundwater contamination has become a serious problem in many developing countries, and due to the seriousness of pollutants as a result of their spread over a large area, and that controlling them and preventing them from reaching the aquifer takes a very long time, in addition the process of treating polluted groundwater is very difficult and expensive, causing increased concern about the existence of a safe hydro environment necessary for the existence of life on the earth. Therefore, many researchers from different disciplines like groundwater hydrology, soil physics, chemical engineering bio sciences, environmental sciences, and academics, have resorted to determine the concentration and to describe the behavior of contaminants concentration in groundwater, either in an experimental or theoretical method.

Analytical solutions, numerical simulations, and field observations are used to address groundwater flow and contaminant transport problems in semi-infinite aquifers. Contaminant (solute) transport through a medium is described by a partial differential equation of parabolic type and it is usually known as advection-dispersion equation (ADE) (Jaiswal & Kumar, 2011). The ADE has been analyzed analytically using several methods, and by a group of researchers, which most of their studies concluded – despite the use of different analytical solutions – that the concentration of pollutants decreases with the expansion of the pollution area at specific time intervals due to dispersion and chemical reaction, etc. Among those analytical solutions: Van Genuchten & Alves, (1982) obtained analytical solutions of 1D convective-dispersion solute transport equation under third-type boundary condition by Laplace transform method. Kumar (1983) considered an exponentially decreasing unsteady/non-uniform flow against dispersion finite porous media time-dependent transport coefficients. An analytical solution for describing the transport of contaminants (solutes) with a distance-dependent dispersion in heterogeneous porous media was developed by Yates (1990). Fry et al. (1993) found analytical solution for describing the solute transport equation with rate-limited desorption and decay. Chen et al. (2008) proposed an analytical technique for solving advection-diffusion equation with hyperbolic asymptotic distance-dependent dispersivity, by applying the extended power series method coupled with the Laplace transform. Using the Laplace transform technique (LTT), Singh et al. (2008) obtained analytical solutions to 1D advection-diffusion equation, to predict the nature of the solute concentration along non-uniform/ unsteady groundwater flow in semi-infinite aquifer. A classical substitution was used by Guerrero et al. (2009) for studying solution of the linear ADE with constant coefficients for both steady and transient

state regimes. Using the Laplace integral transform technique (LITT), Jaiswal et al. (2009); Kumar et al. (2009), and Kumar et al. (2010), obtained analytical solutions of some dispersion problems, in which the two coefficients: dispersion parameter and velocity of the flow field were considered as either temporally dependent or spatially dependent. Jaiswal & Kumar (2011) solved a 1D advection-dispersion equation in a longitudinal domain with variable by LTT. Singh et al. (2011) derived analytical solutions for space-time distribution of contaminant concentration of along non-uniform/ unsteady groundwater flow in homogeneous semi-infinite aquifer using Fourier Transform Technique (FTT) and then compared the result with the solution obtained by LTT. Further, by using LTT, Singh et al. (2012) obtained an analytical solution for One-Dimensional (1D) solute dispersion along steady / uniform groundwater flow in a homogeneous semi-infinite porous formation with time-dependent source concentration. An analytical solution was obtained using Laplace transform, which describes the solute transport in a semi-infinite homogeneous aquifer with a fixed-point source concentration by Singh et al. (2018). Carr (2020) derived new semi-analytical solutions to the ADE for modelling solute transport multilayer porous media using Laplace transform.
The objective of this study is to derive an analytical solution for space-time distribution of the concentration of contaminants in a semi-infinite aquifer using Fourier Transform Technique (FTT).

2. MATHEMATICAL ILLUSTRATION

Let c (x, t) be the concentration of contaminants in a homogeneous semi-infinite aquifer [ML^{-3}], D the dispersion coefficient [L^2T^{-1}] and u the groundwater velocity [LT^{-1}] at time t [T]. The problem can be mathematically formulated as follows:

$$D\frac{\partial^2 c}{\partial x^2} - u\frac{\partial c}{\partial x} = \frac{\partial c}{\partial t}; \quad (1)$$

$$u(t) = u_o\, V(t); \quad (2)$$

Where u_o is the initial groundwater velocity [LT^{-1}] at distance x[L]. The groundwater flow in aquifer is unsteady, where the velocity follows either a sinusoidal $V(t) = 1 - \sin mt$ or exponential form $V(t) = e^{-mt}$ if $mt < 1$, where m is the flow resistance coefficient [T^{-1}].
The initial and boundary conditions can be expressed as:

$$c(x,t) = c_i\,; \qquad x \geq 0, \quad t = 0 \quad (3)$$

$$c(x,t) = c_o[1 + e^{-qt}]; \qquad x = 0, \quad t > 0 \quad (4a)$$

$$c(x,t) = 0\,; \qquad x \to \infty, \quad t > 0 \quad (4b)$$

$$\frac{\partial c(x,t)}{\partial x} = 0\,; \qquad x \to \infty \quad (5)$$

Where c_i is the initial concentration [MT^{-3}] describing distribution of the contaminant at all point i.e., at x = 0, c_o is the solute concentration [MT^{-3}] and q is the parameter like a flow resistance coefficient, known as decay rate coefficients [T^{-1}]. The physical system of the previously problem formulated is shown in the Figure 1.

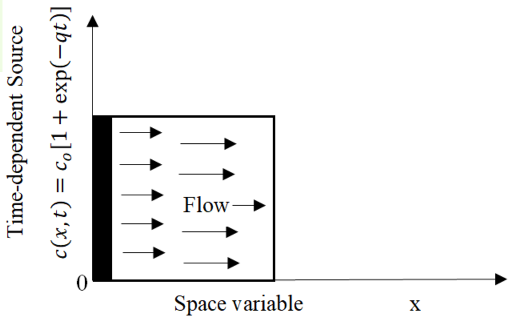


Figure 1. Sketches the physical system of the problem.

Let: $D = au$; where a is the coefficient of dimension length. Using equation (2), we get: $D = D_oV(t)$; here $D_o = au_o$ is an initial dispersion coefficient. Equation (1) can now be written as:

$$D_o\frac{\partial^2 c}{\partial x^2} - u_o\frac{\partial c}{\partial x} = \frac{1}{V(t)}\frac{\partial c}{\partial t}; \quad (6)$$

Introducing a new time variable, T^*, with the transformation (Crank, 1975) gives:

$$T^* = \int_0^t V(t)\,dt\,; \quad (7)$$

Equation (6) becomes:

$$D_o \frac{\partial^2 c}{\partial x^2} - u_o \frac{\partial c}{\partial x} = \frac{\partial c}{\partial T^*}; \tag{8}$$

The initial condition (3) and boundary conditions 4(a, b) and (5) becomes:

$$c(x, T^*) = c_i\,; \qquad x \geq 0,\ T^* = 0 \tag{9}$$

$$c(x, T^*) = c_o(2 - QT^*)\,; \qquad x = 0,\ T^* > 0 \tag{10a}$$

$$c(x, T^*) = 0\,; \qquad x \to \infty,\ T^* > 0 \tag{10b}$$

$$\frac{\partial c(x, T^*)}{\partial x} = 0\,; \qquad x \to \infty \tag{11}$$

Now, the following non-dimensional variables are introduced:

$$C = \frac{c}{c_o}, \qquad X = \frac{x\, u_o}{D_o}, \quad T = \frac{u_o^2\, T^*}{D_o}, \qquad Q = \frac{q D_o}{u_o^2}; \tag{12}$$

The P. D. E. (8) in the form of non-dimensional variable may be written as:

$$\frac{\partial^2 C}{\partial X^2} - \frac{\partial C}{\partial X} = \frac{\partial C}{\partial T}; \tag{13}$$

The initial and boundary conditions:

$$C(X, T) = \frac{c_i}{c_o}\,; \qquad X \geq 0,\ T = 0 \tag{14}$$

$$C(X, T) = 2 - QT\,; \qquad X = 0,\ T > 0 \tag{15a}$$

$$C(X, T) = 0\,; \qquad X \to \infty, T > 0 \tag{15b}$$

$$\frac{\partial C(X, T)}{\partial X} = 0\,; \qquad X \to \infty \tag{16}$$

3. Analytical Solution Using Fourier Transform Technique (FTT)

Using the Transform:

$$C(X, T) = K(X, T) \cdot e^{\left(\frac{X}{2} - \frac{T}{4}\right)}; \tag{17}$$

Differentiating equation (17) and substituting from equation (14), we get:

$$e^{\left(\frac{X}{2} - \frac{T}{4}\right)} \frac{\partial^2 K}{\partial X^2} - \frac{1}{4} K(X, T) \cdot e^{\left(\frac{X}{2} - \frac{T}{4}\right)} = -\frac{1}{4} K(X, T) \cdot e^{\left(\frac{X}{2} - \frac{T}{4}\right)} + e^{\left(\frac{X}{2} - \frac{T}{4}\right)} \frac{\partial K}{\partial T};$$

Consequently:

$$\frac{\partial^2 K}{\partial X^2} = \frac{\partial K}{\partial T}; \tag{18}$$

Using Fourier sine transform of (18) and using the notation:

$$K_s(p, T) = \sqrt{\frac{2}{\pi}} \int_0^\infty K(X, T) \sin pX \;\, dX; \tag{19}$$

$$\sqrt{\frac{2}{\pi}} \int_0^\infty \frac{\partial^2 K}{\partial X^2} \sin pX \;\, dX = \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{\partial K}{\partial T} \sin pX \;\, dX;$$

$$\sqrt{\frac{2}{\pi}} \int_0^\infty \frac{\partial^2 K}{\partial X^2} \sin pX\, dX = \frac{\partial K_s(p, T)}{\partial T};$$

$$\sqrt{\frac{2}{\pi}} \left\{ \left[\sin pX \frac{\partial K}{\partial X} \right]_0^\infty - p \int_0^\infty \frac{\partial K}{\partial X} \cos pX\, dX \right\} = \frac{\partial K_s(p, T)}{\partial T};$$

Using initial and boundary conditions:

$$K(X, T) = \frac{c_i}{c_o} e^{\left(-\frac{X}{2}\right)}\,; \qquad X \geq 0,\ T = 0 \tag{20}$$

$$K(X, T) = (2 - QT) e^{\left(\frac{T}{4}\right)}\,; \qquad X = 0,\ T > 0 \tag{21a}$$

$$K(X, T) = 0\,; \qquad X \to \infty,\ T > 0 \tag{21b}$$

$$\frac{\partial K(X,T)}{\partial X} = 0\,; \qquad X \to \infty, \;\; T \geq 0 \tag{22}$$

We get:

$$-\sqrt{\frac{2}{\pi}}\, p \int_0^\infty \frac{\partial K}{\partial X} \cos pX\, dX = \frac{\partial K_s(p,T)}{\partial T};$$

$$-\sqrt{\frac{2}{\pi}}\, p \left\{[0 - K(0,T)] + p\sqrt{\frac{\pi}{2}} K_s(p,T)\right\} = \frac{\partial K_s(p,T)}{\partial T};$$

Substituting boundary condition (21a), we obtain:

$$\sqrt{\frac{2}{\pi}}\, p\,(2 - QT)e^{\frac{T}{4}} - p^2 K_s(p,T) = \frac{\partial K_s(p,T)}{\partial T};$$

Consequently:

$$\frac{\partial K_s(p,T)}{\partial T} + p^2 K_s(p,T) = \sqrt{\frac{2}{\pi}}\, p\,(2 - QT)e^{\frac{T}{4}}; \tag{23}$$

And this is a linear differential equation in one variable, the general solution is:

$$K_s(p,T) = p\sqrt{\frac{2}{\pi}}\left[\frac{(2 - QT)}{\left(p^2 + \frac{1}{4}\right)} + \frac{Q}{\left(p^2 + \frac{1}{4}\right)^2}\right] e^{\frac{T}{4}} + c_1 e^{-p^2 T}; \tag{24}$$

To calculate c_1, we should substitute by initial condition (20) as follows:

$$K_s(p,0) = \boldsymbol{\mathcal{F}}_s[K(X,0)] = \boldsymbol{\mathcal{F}}_s\left[\frac{c_i}{c_o} e^{\left(-\frac{X}{2}\right)}\right];$$

$$K_s(p,0) = \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{c_i}{c_o} e^{\left(-\frac{X}{2}\right)} \sin pX \;\; dX;$$

By integration, we obtain

$$K_s(p,0) = \frac{1}{p}\frac{c_i}{c_o}\sqrt{\frac{2}{\pi}} - \frac{1}{4p^2}\frac{c_i}{c_o}\sqrt{\frac{2}{\pi}} \int_0^\infty e^{\left(-\frac{X}{2}\right)} \sin pX\, dX;$$

$$K_s(p,0) = \frac{1}{p}\frac{c_i}{c_o}\sqrt{\frac{2}{\pi}} - \frac{1}{4p^2} K_s(p,0);$$

$$K_s(p,0) = \frac{1}{p}\frac{c_i}{c_o}\sqrt{\frac{2}{\pi}} \frac{1}{\left(1 + \frac{1}{4p^2}\right)};$$

$$K_s(p,0) = \frac{c_i}{c_o}\sqrt{\frac{2}{\pi}} \frac{p}{\left(p^2 + \frac{1}{4}\right)}; \tag{25}$$

And from equation (24), we find:

$$K_s(p,0) = p\sqrt{\frac{2}{\pi}}\left\{\frac{2}{\left(p^2 + \frac{1}{4}\right)} + \frac{Q}{\left(p^2 + \frac{1}{4}\right)^2}\right\} + c_1;$$

And so we get

$$\frac{c_i}{c_o}\sqrt{\frac{2}{\pi}} \frac{p}{\left(p^2 + \frac{1}{4}\right)} = p\sqrt{\frac{2}{\pi}}\left\{\frac{2}{\left(p^2 + \frac{1}{4}\right)} + \frac{Q}{\left(p^2 + \frac{1}{4}\right)^2}\right\} + c_1;$$

$$c_1 = \frac{c_i}{c_o}\sqrt{\frac{2}{\pi}} \frac{p}{\left(p^2 + \frac{1}{4}\right)} - p\sqrt{\frac{2}{\pi}}\left\{\frac{2}{\left(p^2 + \frac{1}{4}\right)} + \frac{Q}{\left(p^2 + \frac{1}{4}\right)^2}\right\};$$

$$c_1 = -\sqrt{\frac{2}{\pi}}\left\{\left(2 - \frac{c_i}{c_o}\right)\frac{p}{\left(p^2 + \frac{1}{4}\right)} + \frac{Qp}{\left(p^2 + \frac{1}{4}\right)^2}\right\}; \tag{26}$$

Therefore, from equation (24) the general solution is as follows

$$K_s(p,T) = \sqrt{\frac{2}{\pi}}\left\{(2 - QT)\frac{p}{\left(p^2 + \frac{1}{4}\right)} + \frac{Qp}{\left(p^2 + \frac{1}{4}\right)^2}\right\} e^{\frac{T}{4}} - \sqrt{\frac{2}{\pi}}\left\{\left(2 - \frac{c_i}{c_o}\right)\frac{p\, e^{-p^2 T}}{\left(p^2 + \frac{1}{4}\right)} + \frac{Qpe^{-p^2 T}}{\left(p^2 + \frac{1}{4}\right)^2}\right\};$$

$$K_s(p,T) = \sqrt{\frac{2}{\pi}}[H_1(p,T) + H_2(p,T)]e^{\frac{T}{4}} - \sqrt{\frac{2}{\pi}}[H_3(p,T) + H_4(p,T)]; \quad (27)$$

Now, we should return it to its origin, i. e. as a function of (X, T) by inverse transform it by sine transform of Fourier:

$$K(X,T) = \boldsymbol{\mathcal{F}}_s^{-1}[K_s(p,T)]; \quad (28)$$

$$K(X,T) = \sqrt{\frac{2}{\pi}}\int_0^\infty K_s(p,T)\sin pX\,dp;$$

$$H_1(X,T) = \sqrt{\frac{2}{\pi}}\int_0^\infty H_1(p,T)\sin pX\,dp;$$

$$H_1(X,T) = \sqrt{\frac{2}{\pi}}\int_0^\infty (2-QT)\frac{p}{\left(p^2+\frac{1}{4}\right)}\sin pX\,dp;$$

$$H_1(X,T) = \sqrt{\frac{2}{\pi}}(2-QT)\boldsymbol{\mathcal{F}}_s^{-1}\left[\frac{p}{\left(p^2+\frac{1}{4}\right)}\right] = \sqrt{\frac{2}{\pi}}\frac{\pi e^{\left(-\frac{X}{2}\right)}}{2}(2-QT);$$

$$H_1(X,T) = \sqrt{\frac{\pi}{2}}(2-QT)e^{\left(-\frac{X}{2}\right)}; \quad (29)$$

$$H_2(X,T) = \boldsymbol{\mathcal{F}}_s^{-1}[H_2(p,T)] = \boldsymbol{\mathcal{F}}_s^{-1}\left[\frac{Qp}{\left(p^2+\frac{1}{4}\right)^2}\right];$$

$$H_2(X,T) = Q\,\boldsymbol{\mathcal{F}}_s^{-1}\left[\frac{p}{\left(p^2+\frac{1}{4}\right)^2}\right] = Q\sqrt{\frac{\pi}{2}}\frac{Xe^{\left(-\frac{X}{2}\right)}}{2\left(\frac{1}{2}\right)};$$

$$H_2(X,T) = \sqrt{\frac{\pi}{2}}\,Q\,Xe^{\left(-\frac{X}{2}\right)}; \quad (30)$$

$$H_3(X,T) = \boldsymbol{\mathcal{F}}_s^{-1}[H_3(p,T)] = \boldsymbol{\mathcal{F}}_s^{-1}\left[\left(2-\frac{c_i}{c_o}\right)\frac{p\,e^{-p^2T}}{\left(p^2+\frac{1}{4}\right)}\right];$$

$$H_3(X,T) = \left(2-\frac{c_i}{c_o}\right)\boldsymbol{\mathcal{F}}_s^{-1}\left[\left(\frac{p}{\left(p^2+\frac{1}{4}\right)}\right)\left(e^{-p^2T}\right)\right];$$

To solve this, we have to use the following rule:

$$\boldsymbol{\mathcal{F}}_s^{-1}[F(p)G(p)] = \frac{1}{2}\sqrt{\frac{2}{\pi}}\int_0^\infty f_s(w)\,[g_c(w-x) - g_c(w+x)]dw$$

$$f_s(X) = \boldsymbol{\mathcal{F}}_s^{-1}\left[\frac{p}{\left(p^2+\frac{1}{4}\right)}\right] = \sqrt{\frac{\pi}{2}}e^{\left(-\frac{X}{2}\right)};$$

$$g_c(X) = \boldsymbol{\mathcal{F}}_c^{-1}\left[e^{(-p^2T)}\right] = \frac{1}{\sqrt{2T}}e^{\left(-\frac{X^2}{4T}\right)};$$

We put: $F(p) = \left(\frac{p}{\left(p^2+\frac{1}{4}\right)}\right)$ and $G(p) = e^{(-p^2T)}$, we get

$$H_3(X,T) = \frac{1}{2\sqrt{2T}}\left(2-\frac{c_i}{c_o}\right)\left\{\int_0^\infty e^{-\left(\frac{(w-x)^2}{4T}+\frac{w}{2}\right)}dw - \int_0^\infty e^{-\left(\frac{(w+x)^2}{4T}+\frac{w}{2}\right)}dw\right\};$$

Let: $y = \frac{w-x+T}{2\sqrt{T}} \to dw = 2\sqrt{T}dy$ and $Z = \frac{w+x+T}{2\sqrt{T}} \to dw = 2\sqrt{T}dZ$; at $w = o,\ y = \frac{\sqrt{T}}{2} - \frac{X}{2\sqrt{T}}$ and $Z = \frac{\sqrt{T}}{2} + \frac{X}{2\sqrt{T}}$, while at $w \to \infty$ $then$ $y \to \infty$ and $z \to \infty$, we obtain

$$H_3(X,T) = \frac{1}{\sqrt{2}}\left(2-\frac{c_i}{c_o}\right)e^{\left(\frac{T}{4}-\frac{X}{2}\right)}\left\{\int_{\frac{\sqrt{T}}{2}-\frac{X}{2\sqrt{T}}}^\infty e^{-y^2}\,dy - e^X\int_{\frac{\sqrt{T}}{2}+\frac{X}{2\sqrt{T}}}^\infty e^{-Z^2}\,dZ\right\};$$

$$H_3(X,T) = \frac{1}{2}\sqrt{\frac{\pi}{2}}\left(2-\frac{c_i}{c_o}\right)e^{\left(\frac{T}{4}-\frac{X}{2}\right)}\left\{erfc\left(\frac{\sqrt{T}}{2}-\frac{X}{2\sqrt{T}}\right) - e^X erfc\left(\frac{\sqrt{T}}{2}+\frac{X}{2\sqrt{T}}\right)\right\}; \quad (31)$$

$$H_4(X,T) = \boldsymbol{\mathcal{F}}_s^{-1}[H_4(p,T)] = \boldsymbol{\mathcal{F}}_s^{-1}\left[Q\,\frac{p\,e^{-p^2T}}{\left(p^2+\frac{1}{4}\right)^2}\right];$$

And to solve this, we have to use the following rule:

$$H_4(X,T) = Q\,\boldsymbol{\mathcal{F}}_s^{-1}\left[\left(\frac{p}{\left(p^2+\frac{1}{4}\right)^2}\right)\left(e^{(-p^2T)}\right)\right];$$

$$\boldsymbol{\mathcal{F}}_s^{-1}[F(p)G(p)] = \frac{1}{2}\sqrt{\frac{2}{\pi}}\int_0^\infty f_s(w)\,[g_c(w-x)-g_c(w+x)]dw;$$

$$f_s(X) = \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{p}{\left(p^2+\frac{1}{4}\right)^2}\sin pX\,dX = \boldsymbol{\mathcal{F}}_s^{-1}\left[\frac{p}{\left(p^2+\frac{1}{4}\right)^2}\right] = \sqrt{\frac{\pi}{2}}\,Xe^{\left(-\frac{X}{2}\right)};$$

$$g_c(X) = \sqrt{\frac{2}{\pi}}\int_0^\infty e^{(-p^2T)}\cos pX\,dX = \boldsymbol{\mathcal{F}}_c^{-1}\left[e^{(-p^2T)}\right] = \frac{1}{\sqrt{2T}}e^{\left(-\frac{X^2}{4T}\right)};$$

We put: $F(p) = \left(\frac{p}{\left(p^2+\frac{1}{4}\right)^2}\right)$ *and* $G(p) = e^{(-p^2T)}$, we get:

$$H_4(X,T) = \frac{Q}{2\sqrt{2T}}\left\{\int_0^\infty w\,e^{-\left(\frac{(w-x)^2}{4T}+\frac{w}{2}\right)}dw - \int_0^\infty w\;e^{-\left(\frac{(w+x)^2}{4T}+\frac{w}{2}\right)}\,dw\right\};$$

Let: $y = \frac{w-x+T}{2\sqrt{T}} \to dw = 2\sqrt{T}dy$ and $Z = \frac{w+x+T}{2\sqrt{T}} \to dw = 2\sqrt{T}dZ$; at $w = o,\ y = \frac{\sqrt{T}}{2} - \frac{X}{2\sqrt{T}}$ *and* $Z = \frac{\sqrt{T}}{2} + \frac{X}{2\sqrt{T}}$, while at $w \to \infty$ *then* $y \to \infty$ and $z \to \infty$, we obtain

$$H_4(X,T) = \frac{Q}{\sqrt{2}}\left\{e^{\left(\frac{T}{4}-\frac{X}{2}\right)}\int_{\frac{\sqrt{T}}{2}-\frac{X}{2\sqrt{T}}}^{\infty}\left[2\sqrt{T}\,y+(X-T)\right]e^{-y^2}\,dy \;-e^X e^{\left(\frac{T}{4}-\frac{X}{2}\right)}\int_{\frac{\sqrt{T}}{2}+\frac{X}{2\sqrt{T}}}^{\infty}\left[2\sqrt{T}\,Z-(X+T)\right]e^{-Z^2}\,dZ\right\};$$

$$H_4(X,T) = -\sqrt{\frac{\pi}{2}}\frac{Q}{2}e^{\left(\frac{T}{4}-\frac{X}{2}\right)}\left\{T\left[erfc\left(\frac{\sqrt{T}}{2}-\frac{X}{2\sqrt{T}}\right)-e^X erfc\left(\frac{\sqrt{T}}{2}+\frac{X}{2\sqrt{T}}\right)\right]-\left[Xerfc\left(\frac{\sqrt{T}}{2}-\frac{X}{2\sqrt{T}}\right)+2\sqrt{\frac{T}{\pi}}\,e^{-\left(\frac{\sqrt{T}}{2}-\frac{X}{2\sqrt{T}}\right)^2}\right]\right.$$
$$\left. - e^X\left[Xerfc\left(\frac{\sqrt{T}}{2}+\frac{X}{2\sqrt{T}}\right)-2\sqrt{\frac{T}{\pi}}\,e^{-\left(\frac{\sqrt{T}}{2}+\frac{X}{2\sqrt{T}}\right)^2}\right]\right\};$$

Suppose that:

$$F_1(X,T) = erfc\left(\frac{\sqrt{T}}{2}-\frac{X}{2\sqrt{T}}\right)-e^X erfc\left(\frac{\sqrt{T}}{2}+\frac{X}{2\sqrt{T}}\right);$$

$$F_2(X,T) = Xerfc\left(\frac{\sqrt{T}}{2}-\frac{X}{2\sqrt{T}}\right)+2\sqrt{\frac{T}{\pi}}\,e^{-\left(\frac{\sqrt{T}}{2}-\frac{X}{2\sqrt{T}}\right)^2};$$

$$F_3(X,T) = Xerfc\left(\frac{\sqrt{T}}{2}+\frac{X}{2\sqrt{T}}\right)-2\sqrt{\frac{T}{\pi}}\,e^{-\left(\frac{\sqrt{T}}{2}+\frac{X}{2\sqrt{T}}\right)^2};$$

So,

$$H_4(X,T) = -\sqrt{\frac{\pi}{2}}\frac{Q}{2}e^{\left(\frac{T}{4}-\frac{X}{2}\right)}\{TF_1(X,T)-F_2(X,T)-e^X F_3(X,T)\}; \qquad (32)$$

By substituting from equations (29), (30), (31), (32) in eq. (28) by using eq. (27), we obtain

$$K(X,T) = \sqrt{\frac{2}{\pi}}\left\{\sqrt{\frac{\pi}{2}}(2-QT)e^{\left(-\frac{X}{2}\right)}+Q\sqrt{\frac{\pi}{2}}Xe^{\left(-\frac{X}{2}\right)}\right\}e^{\left(\frac{T}{4}\right)}$$
$$-\sqrt{\frac{2}{\pi}}\left\{\sqrt{\frac{\pi}{2}}\frac{1}{2}\left(2-\frac{c_i}{c_o}\right)e^{\left(\frac{T}{4}-\frac{X}{2}\right)}F_1(X,T)-\sqrt{\frac{\pi}{2}}\frac{Q}{2}e^{\left(\frac{T}{4}-\frac{X}{2}\right)}[TF_1(X,T)-F_2(X,T)-e^X F_3(X,T)]\right\};$$

$$K(X,T) = e^{\left(\frac{T}{4}-\frac{X}{2}\right)}\left\{(2-QT)+QX-\frac{1}{2}\left(2-\frac{c_i}{c_o}\right)F_1(X,T) + \frac{Q}{2}\left[TF_1(X,T)-F_2(X,T)-e^X F_3(X,T)\right]\right\}; \quad (33)$$

Substituting from equation (33) in equation (17), we get:

$$C(X,T) = 2-QT+QX-\frac{1}{2}\left(2-\frac{c_i}{c_o}\right)F_1(X,T)+\frac{Q}{2}\left[TF_1(X,T)-F_2(X,T)-e^X F_3(X,T)\right]; \quad (34)$$

4. RESULTS AND DISCUSSION

The analytical solutions of equation (34) are illustrated with the help of set of input data to understand the concentration distribution behavior in the sinusoidal and exponential forms of velocity expressions, which are valid for transient groundwater flow too (Kumar,1983; Banks and Jerasate,1962). The new sinusoidal and exponential forms of velocity in equation (2) can be written as follows:

$$u(t) = u_o(1-\sin mt); \quad (35a)$$

$$u(t) = u_o e^{(-mt)}, \quad mt < 1 \quad (35b)$$

The non-dimensional time variable T can be written as follows:

$$T = \frac{u_o^2}{mD_o}[mt-(1-\cos mt)]; \; T = \frac{u_o^2}{mD_o}\left[1-e^{(-mt)}\right]$$

where $mt = 3\cdot k + 2 \ldots\ldots\ldots\ldots k = 0$ to n

The values of mt represents the groundwater level and velocity minimum during June and maximum during December just after six months (Approximately 182 days) in one year and $6 \le K \le 13$. If the flow resistance coefficient m= 0.0165 day^{-1}, equation (35a) yields approximately t (days) = 182·k+121. So, we get the time t (days) in 4th, 5th, 6th and 7th, as t (days) =1,213; 1,395; 1,577; 1,759; 1,941; 2,123; 2,305; 2,487 and mt = 20; 23; 26; 29; 32; 35; 38; 41. For the same set of inputs except m = 0.0002 day^{-1} as mt <1, we get mt = 0.243; 0.279; 0.315; 0.352; 0.388; 0.425; 0.461; 0.497.

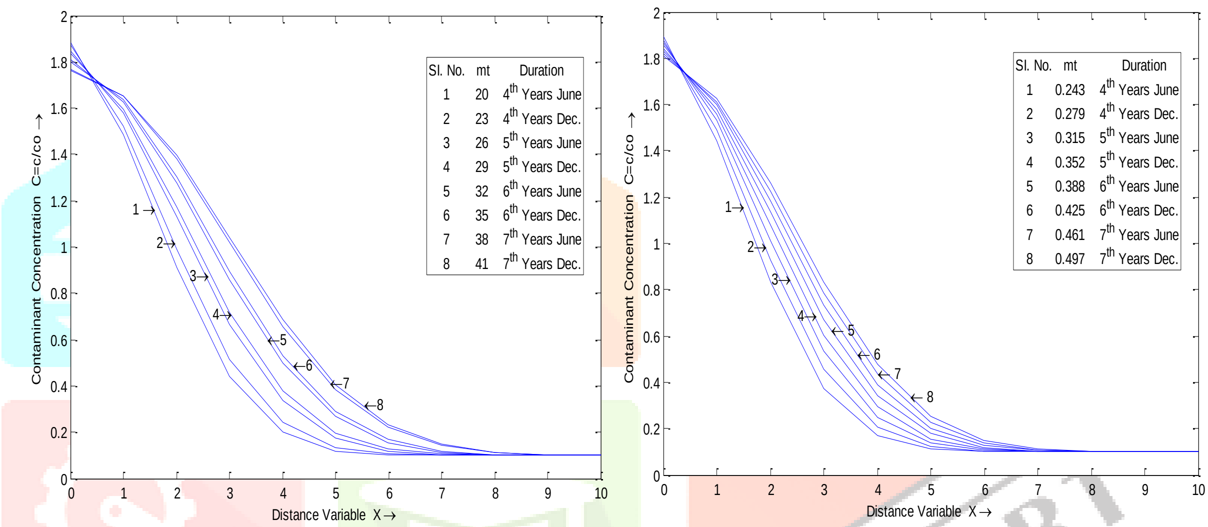
The analytical solution of equation (34) is solved and computed (MatLab) for the same values of initial concentration, c_i = 0.1, solute concentration c_o = 1.0, initial groundwater velocity u_o = 0.01 km/day, initial dispersion coefficient D_o = 0.1 km^2/day, decay rate coefficients q = 0.0001 day^{-1} and distance x = 100 km, to depict the variations of the contaminant concentration along uniform groundwater flow with the sinusoidal and the exponential form of the temporally dependent dispersion equation (35a-b).The concentration values are summered in Tables (1and 2) and graphically depicted in Figure (2a-b). It is observed that the concentration at the origin of the aquifer is decreasing for the both unsteady velocities with time. Nearby the origin, it starts slightly decreasing and emerging at the common point. It is also observed that after emergence, it goes on decreasing for sinusoidal as well as exponential form of velocity. The rate of decreasing tendency for sinusoidal form of velocity is slightly slower than the exponential form of velocity. In both cases, the decreasing tendency of the concentration levels of contaminants with time and distance travelled, may help to rehabilitate the contaminated aquifer.

Table 1. Contaminant Concentration values in sinusoidal form of velocity u_o = 0.01km/day and D_o= 0.1km^2/day using Fourier Transform Technique.

X	0	1	2	3	4	5	6	7	8	9	10
				mt = 20		t = 1213 days					
C Fourier	1.8824	1.4839	0.9103	0.4404	0.1995	0.1198	0.1027	0.1002	0.1000	0.1000	0.1000
				mt = 23		t = 1395 days					
C Fourier	1.8699	1.5223	0.9909	0.5157	0.2403	0.1336	0.1057	0.1007	0.1001	0.1000	0.1000
				mt = 26		t = 1577 days					
C Fourier	1.8446	1.5794	1.1283	0.6635	0.3373	0.1755	0.1179	0.1032	0.1004	0.1000	0.1000
				mt = 29		t = 1759 days					
C Fourier	1.8348	1.5956	1.1728	0.7172	0.3778	0.1962	0.1254	0.1051	0.1008	0.1001	0.1000
				mt = 32		t = 1941 days					
C Fourier	1.8071	1.6291	1.2793	0.8587	0.4981	0.2678	0.1562	0.1149	0.1031	0.1005	0.1001
				mt = 35		t = 2123 days					
C Fourier	1.7994	1.6357	1.3040	0.8944	0.5315	0.2901	0.1673	0.1190	0.1042	0.1007	0.1001
				mt = 38		t = 2305 days					
C Fourier	1.7700	1.6526	1.3833	1.0184	0.6579	0.3836	0.2195	0.1414	0.1118	0.1027	0.1005
				mt = 41		t = 2487 days					
C Fourier	1.7636	1.6548	1.3977	1.0427	0.6845	0.4051	0.2327	0.1478	0.1142	0.1035	0.1007

Table 2. Contaminant Concentration values in exponential form of velocity u_o = 0.01km/day and D_o = 0.1km^2/day using Fourier Transform Technique.

X	0	1	2	3	4	5	6	7	8	9	10
mt = 0.2426						**t = 1213 days**					
C Fourier	1.8933	1.4426	0.8315	0.3747	0.1688	0.1113	0.1012	0.1001	0.1000	0.1000	0.1000
mt = 0.2790						**t = 1395 days**					
C Fourier	1.8798	1.4923	0.9275	0.4558	0.2073	0.1223	0.1031	0.1003	0.1000	0.1000	0.1000
mt = 0.3154						**t = 1577 days**					
C Fourier	1.8670	1.5302	1.0086	0.5332	0.2506	0.1375	0.1066	0.1008	0.1001	0.1000	0.1000
mt = 0.3518						**t = 1759 days**					
C Fourier	1.8547	1.5593	1.0771	0.6054	0.2965	0.1565	0.1119	0.1018	0.1002	0.1000	0.1000
mt = 0.3882						**t = 1941 days**					
C Fourier	1.8431	1.5820	1.1352	0.6717	0.3433	0.1784	0.1189	0.1034	0.1005	0.1000	0.1000
mt = 0.4246						**t = 2123 days**					
C Fourier	1.8321	1.5997	1.1846	0.7320	0.3895	0.2025	0.1278	0.1057	0.1009	0.1001	0.1000
mt = 0.4610						**t = 2305 days**					
C Fourier	1.8217	1.6136	1.2268	0.7865	0.4343	0.2280	0.1382	0.1088	0.1016	0.1002	0.1000
mt = 0.4974						**t = 2487 days**					
C Fourier	1.8119	1.6245	1.2628	0.8355	0.4770	0.2542	0.1498	0.1126	0.1025	0.1004	0.1000



(a) **(b)**

Figure 2. Time-dependent contaminant source concentrations in semi-infinite aquifer using FTT along (a) sinusoidal (b) exponentially decreasing form of velocity.

5. CONCLUSION

The Fourier Transform Technique (FTT) was used to formulate a solute transport model for time-dependent source concentration in one-dimensional homogeneous semi-infinite aquifer with suitable initial and boundary conditions. The solutions are obtained for sinusoidal and exponential form of velocity which represent the seasonal pattern in a year. The output of the problems may help to know the position and the time period of harmless concentration levels of the contaminated aquifer.

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