IJCRT.ORG ISSN : 2320-2882
INTERNATIONAL JOURNAL OF CREATIVE RESEARCH THOUGHTS (IJCRT)
An International Open Access, Peer-reviewed, Refereed Journal

On the Diophantine Equation $\frac{r}{n} = \sum_{i=1}^{m} \frac{1}{x_i}$, $r < n$, m, n are Positive Integers

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Abstract:

In this paper, the Diophantine equation $\frac{r}{n} = \sum_{i=1}^{m} \frac{1}{x_i}$ has been discussed for $r = 3, 4$ for integral solution. This is some sort of Erdos & Straus conjecture.

Key words: Diophantine equation, conjecture and integral solution.

Introduction:

Erdos & Straus (1950) conjectured that for all integers $n \geq 4$, the rational number $\frac{4}{n}$ can be expressed as sum of three unit fractions. Thus the conjecture states that for $n \geq 4$ there exist positive integers x, y and z such that

$$\frac{4}{n} = \frac{1}{x} + \frac{1}{y} + \frac{1}{z}.$$

These unit fractions of $\frac{4}{n}$ is called Egyptian fraction representation. For illustration for n=1801, there exists solution $x = 451, y = 295364$ and $z = 3249004$. Computer searches have verified the conjecture upto $n \geq 10^{14}$ but it is still open problem to prove for all values of n.

For some value of n satisfying certain congruence relation, one can obtain an expansion for $\frac{4}{n}$ as a polynomial identity. For example, for $n \equiv 2(mod 3)$, $\frac{4}{n}$ has the expansion

$$\frac{4}{n} = \frac{1}{n} + \frac{1}{\frac{(n-2)}{3}+1} + \frac{1}{n\left(\frac{(n-2)}{3}+1\right)}.$$

In the above identity, each term in the denominator of R.H.S. is a polynomial of n having integer value whenever $n \equiv 2(mod 3)$.

Jaroma (2004) presented the following expansion with one negative term:

$$\frac{4}{n} = \frac{1}{(n-1)/2} + \frac{1}{\frac{(n+1)}{2}} - \frac{1}{n(n-1)(n+1)/4}.$$

Hari Kishan, Megha Rani and Smiti Agarwal (2011) discussed the Diophantine equations of second and higher degree of the form $3xy = n(x+y)$ and $3xyz = n(xy+yz+zx)$ etc.

In this paper, The Diophantine equation $\frac{r}{n} = \sum_{i=1}^{m} \frac{1}{x_i}$ has been discussed for $r = 3, 4$ for integral solution.

Analysis:

The above Diophantine equation has been discussed in the following cases:

(A): For $r = 3$ the given Diophantine equation becomes $\frac{3}{n} = \sum_{i=1}^{m} \frac{1}{x_i}$. …(1)

Case 1: For *m*=6, the Diophantine equation (1) can be written as

$$\frac{3}{n} = \frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3} + \frac{1}{x_4} + \frac{1}{x_5} + \frac{1}{x_6}. \quad \ldots(2)$$

One has to find the integral values of x_1, x_2, x_3, x_4, x_5 and x_6. The left hand side of (2) can be written as

$$\frac{3}{n} = \frac{1}{n} + \frac{2}{n} = \frac{1}{n} + \frac{2}{n+1} + \frac{2}{n(n+1)} = \frac{1}{n} + \frac{1}{\frac{(n-1)}{2}+1} + \frac{1}{n\left(\frac{n-1}{2}+1\right)}$$

$$= \frac{1}{n+1} + \frac{1}{n(n+1)} + \frac{1}{\frac{(n-1)}{2}+1} + \frac{1}{n\left(\frac{n-1}{2}+1\right)}$$

$$= \frac{1}{n+3} + \frac{1}{(n+2)(n+3)} + \frac{1}{(n+1)(n+2)} + \frac{1}{n(n+1)} + \frac{1}{\frac{(n-1)}{2}+1} + \frac{1}{n\left(\frac{n-1}{2}+1\right)}. \quad \ldots(3)$$

Comparing equation (2) and (3), one get

$$x_1 = n+3, x_2 = (n+2)(n+3), x_3 = (n+1)(n+2), x_4 = n(n+1)$$

$$x_5 = \frac{(n-1)}{2} + 1, x_6 = n\left(\frac{n-1}{2} + 1\right).$$

Now if $n \equiv 1(mod 2)$ then x_1, x_2, x_3, x_4, x_5 and x_6 are positive integers. Therefore these are the solution of the given Diophantine equation. Few solutions are as follows:

n	x_1	x_2	x_3	x_4	x_5	x_6
1	4	12	6	2	1	1
3	6	30	20	12	2	6
5	8	56	42	30	3	15
7	10	90	72	56	4	28

Case 2: For *m*=7, the Diophantine equation (1) can be written as

$$\frac{3}{n} = \frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3} + \frac{1}{x_4} + \frac{1}{x_5} + \frac{1}{x_6} + \frac{1}{x_7}. \qquad \ldots(4)$$

One has to find the integral values of $x_1, x_2, x_3, x_4, x_5, x_6$ and x_7. The left hand side of (4) can be written as

$$\frac{3}{n} = \frac{1}{n} + \frac{2}{n} = \frac{1}{n} + \frac{2}{n+1} + \frac{2}{n(n+1)} = \frac{1}{n} + \frac{1}{\frac{(n-1)}{2}+1} + \frac{1}{n\left(\frac{n-1}{2}+1\right)}$$

$$= \frac{1}{n+1} + \frac{1}{n(n+1)} + \frac{1}{\frac{(n-1)}{2}+1} + \frac{1}{n\left(\frac{n-1}{2}+1\right)}$$

$$= \frac{1}{n+4} + \frac{1}{(n+3)(n+4)} + \frac{1}{(n+2)(n+3)} + \frac{1}{(n+1)(n+2)} + \frac{1}{n(n+1)} + \frac{1}{\frac{(n-1)}{2}+1} + \frac{1}{n\left(\frac{n-1}{2}+1\right)}.$$

$$\ldots(5)$$

Comparing equation (4) and (5), one get

$$x_1 = n+4, x_2 = (n+3)(n+4), x_3 = (n+2)(n+3),$$

$$x_4 = (n+1)(n+2), x_5 = n(n+1)\ \ x_6 = \frac{(n-1)}{2}+1, x_7 = n\left(\frac{n-1}{2}+1\right).$$

Now if $n \equiv 1(mod 2)$ then x_1, x_2, x_3, x_4, x_5 and x_6 are positive integers. Therefore these are the solution of the given Diophantine equation. Few solutions are as follows:

n	x_1	x_2	x_3	x_4	x_5	x_6	x_7
1	5	20	12	6	2	1	1
3	7	42	30	20	12	2	6
5	9	72	56	42	30	3	15
7	11	110	90	72	56	4	28

(B): For $r = 4$ the given Diophantine equation becomes $\frac{4}{n} = \sum_{i=1}^{m} \frac{1}{x_i}$. …(6)

Case 3: For *m*=6, the Diophantine equation (6) can be written as

$$\frac{4}{n} = \frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3} + \frac{1}{x_4} + \frac{1}{x_5} + \frac{1}{x_6}. \quad …(7)$$

One has to find the integral values of x_1, x_2, x_3, x_4, x_5 and x_6. The left hand side of (7) can be written as

$$\frac{4}{n} = \frac{1}{n} + \frac{3}{n} = \frac{1}{n} + \frac{3}{n+1} + \frac{3}{n(n+1)} = \frac{1}{n} + \frac{1}{\frac{(n-2)}{3}+1} + \frac{1}{n\left(\frac{n-2}{3}+1\right)}$$

$$= \frac{1}{n+1} + \frac{1}{n(n+1)} + \frac{1}{\frac{(n-2)}{3}+1} + \frac{1}{n\left(\frac{n-2}{3}+1\right)}$$

$$= \frac{1}{n+3} + \frac{1}{(n+2)(n+3)} + \frac{1}{(n+1)(n+2)} + \frac{1}{n(n+1)} + \frac{1}{\frac{(n-2)}{3}+1} + \frac{1}{n\left(\frac{n-2}{3}+1\right)}. \quad …(8)$$

Comparing equation (7) and (8), one get

$$x_1 = n+3, x_2 = (n+2)(n+3), x_3 = (n+1)(n+2), x_4 = n(n+1)$$

$$x_5 = \frac{(n-2)}{3} + 1, x_6 = n\left(\frac{n-2}{3} + 1\right).$$

Now if $n \equiv 2(mod3)$ then x_1, x_2, x_3, x_4, x_5 and x_6 are positive integers. Therefore these are the solution of the given Diophantine equation. Few solutions are as follows:

n	x_1	x_2	x_3	x_4	x_5	x_6
2	5	20	12	6	1	2
5	8	56	42	30	2	10
8	11	110	90	72	3	24
11	14	182	132	156	4	44

Case 4: For *m*=7, the Diophantine equation (6) can be written as

$$\frac{4}{n} = \frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3} + \frac{1}{x_4} + \frac{1}{x_5} + \frac{1}{x_6} + \frac{1}{x_7}. \quad \ldots(9)$$

One has to find the integral values of $x_1, x_2, x_3, x_4, x_5, x_6$ and x_7. The left hand side of (9) can be written as

$$\frac{4}{n} = \frac{1}{n} + \frac{3}{n} = \frac{1}{n} + \frac{3}{n+1} + \frac{3}{n(n+1)} = \frac{1}{n} + \frac{1}{\frac{(n-2)}{3}+1} + \frac{1}{n\left(\frac{n-2}{3}+1\right)}$$

$$= \frac{1}{n+1} + \frac{1}{n(n+1)} + \frac{1}{\frac{(n-2)}{3}+1} + \frac{1}{n\left(\frac{n-2}{3}+1\right)}$$

$$= \frac{1}{n+4} + \frac{1}{(n+3)(n+4)} + \frac{1}{(n+2)(n+3)} + \frac{1}{(n+1)(n+2)} + \frac{1}{n(n+1)} + \frac{1}{\frac{(n-2)}{3}+1} + \frac{1}{n\left(\frac{n-2}{3}+1\right)}.$$

…(10)

Comparing equation (9) and (10), one get

$$x_1 = n+4, x_2 = (n+3)(n+4), x_3 = (n+2)(n+3),$$

$$x_4 = (n+1)(n+2), x_5 = n(n+1)\ \ x_6 = \frac{(n-2)}{3}+1, x_7 = n\left(\frac{n-2}{3}+1\right).$$

Now if $n \equiv 2(mod3)$ then x_1, x_2, x_3, x_4, x_5 and x_6 are positive integers. Therefore these are the solution of the given Diophantine equation. Few solutions are as follows:

n	x_1	x_2	x_3	x_4	x_5	x_6	x_7
2	6	30	20	12	6	1	2
5	9	72	56	42	30	2	10
8	12	132	110	90	72	3	24
11	15	210	182	143	132	4	44

Case 5: For *m*=8, the Diophantine equation (6) can be written as

$$\frac{4}{n} = \frac{1}{x_1}+\frac{1}{x_2}+\frac{1}{x_3}+\frac{1}{x_4}+\frac{1}{x_5}+\frac{1}{x_6}+\frac{1}{x_7}+\frac{1}{x_8}. \quad \ldots(11)$$

One has to find the integral values of $x_1, x_2, x_3, x_4, x_5, x_6, x_7$ and x_8. The left hand side of (11) can be written as

$$\frac{4}{n} = \frac{1}{n}+\frac{3}{n} = \frac{1}{n}+\frac{3}{n+1}+\frac{3}{n(n+1)} = \frac{1}{n}+\frac{1}{\frac{(n-2)}{3}+1}+\frac{1}{n\left(\frac{n-2}{3}+1\right)}$$

$$= \frac{1}{n+1}+\frac{1}{n(n+1)}+\frac{1}{\frac{(n-2)}{3}+1}+\frac{1}{n\left(\frac{n-2}{3}+1\right)}$$

$$= \frac{1}{n+5}+\frac{1}{(n+4)(n+5)}+\frac{1}{(n+3)(n+4)}+\frac{1}{(n+2)(n+3)}+\frac{1}{(n+1)(n+2)}.$$

$$+\frac{1}{n(n+1)}+\frac{1}{\frac{(n-2)}{3}+1}+\frac{1}{n\left(\frac{n-2}{3}+1\right)}. \quad \ldots(12)$$

Comparing equation (11) and (12), one get

$$x_1 = n+5, x_2 = (n+4)(n+5), x_3 = (n+3)(n+4),$$

$$x_4 = (n+2)(n+3),\ x_5 = (n+1)(n+2),\ x_6 = n(n+1)$$

$$x_7 = \frac{(n-2)}{3} + 1, x_8 = n\left(\frac{n-2}{3} + 1\right).$$

Now if $n \equiv 2(mod 3)$ then $x_1, x_2, x_3, x_4, x_5, x_6, x_7$ and x_8 are positive integers. Therefore these are the solution of the given Diophantine equation. Few solutions are as follows:

n	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8
2	7	42	30	20	12	6	1	2
5	10	90	72	56	42	30	2	10
8	13	156	132	110	90	72	3	24
11	16	240	210	182	156	132	4	44

Concluding Remarks:

Here the given Diophantine equation has been discussed for $r = 3$ and 4. m is considered equal to 6, 7 and 8. The problem can further be discussed for other values of r and m.

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