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ON A CLASS OF GENERALIZED CONVEX FUNCTIONS WITH SOME APPLICATIONS – II

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7.1 Introduction

J. Sandoor [8] introduced the notion of A-convexity. Under A-convexity he studied the notions of B-subadditive and anti-symmetricity. R.B. Dash et al. [4] generalized the definition of anti-symmetry from two dimensions to three dimensions and four dimensions. They applied this to prove theorems on A-convxity, subadditivity and super additivity.

In this chapter, the idea of anti-symmetry is further generalized to n dimensions and is effectively applied to prove theorems on sub-additive and A-convexity and B decreasing properties of functions.

7.2 A-Convex Functions

Definition 7.1

Let V be a vector space and $B: V\times V\times ...\times V \to V$ be a map in E^n. Then the function $f: V \to R$ is called B-subadditive (super additive) if

$$f\left(\sum_{i=1}^{n} x_i\right) \le (\ge) \sum_{i=1}^{n} f(x_i) + B(x_1, x_2, ...x_n)$$

for all $x_i \in V,\ i = 1, 2, ...n.$

Definition 7.2.

The map $B: V\times V\times ...V \to R$ is called anti-symmetric map if $B(x_1, x_2, ...x_n) = B\left(x_{\sigma(1)}, x_{\alpha(2)}, ...x_{\sigma(n)}\right)$

and $B(x_1, x_2, ...x_n) = B\left(x_{\tau(1)}, x_{\tau(2)}, ...x_{\tau(n)}\right)$

where σ and τ denote the even and odd permutations respectively.

Theorem 7.1

If B is anti-symmetric map in E^n and f is sub-additive (super additive), then f is sub-additive (super additive).

Proof: As f is sub-additive (super additive) one can write.

$$f\left(\sum_{i=1}^{n} x_i\right) \le (\ge) \sum_{i=1}^{n} f(x_i) + B\left(x_{\sigma(1)}, x_{\sigma(2)}, \dots x_{\sigma(n)}\right) \quad (7.1)$$

and $$f\left(\sum_{i=1}^{n} x_i\right) \le (\ge) \sum_{i=1}^{n} f(x_i) + B\left(x_{\tau(1)}, x_{\tau(2)}, \dots x_{\tau(n)}\right) \quad (7.2)$$

Adding all inequalities in (7.1) over even permutations, we have

$$\sum_{\sigma} f\left(\sum_{i=1}^{n} x_i\right) \le (\ge) \sum_{\sigma} \left(\sum_{i=1}^{n} f(x_i)\right) + \sum_{\sigma} B\left(x_{\sigma(1)}, x_{\sigma(2)}, \dots x_{\sigma(n)}\right) \quad (7.3)$$

Adding all inequalities in (7.2) over odd permutations, we have

$$\sum_{\tau} f\left(\sum_{i=1}^{n} x_i\right) \le (\ge) \sum_{\tau} \left(\sum_{i=1}^{n} f(x_i)\right) + \sum_{\tau} B\left(x_{\tau(1)}, x_{\tau(2)}, \dots x_{\tau(n)}\right) \quad (7.4)$$

Now, (7.3) becomes

$$f\left(\sum_{i=1}^{b} x_i\right) \sum_{\alpha} 1 \le (\ge) \sum_{i=1}^{n} f(x_i) \sum_{\sigma} 1 + \sum_{\sigma} B\left(x_{\sigma(1)}, x_{\sigma(2)}, \dots x_{\sigma(n)}\right)$$

$$\Rightarrow \frac{n!}{2} f\left(\sum_{i=1}^{n} x_i\right) \le (\ge) \frac{n!}{2} \sum_{i=1}^{n} f(x_i) + \sum_{\sigma} B\left(x_{\sigma(1)}, x_{\sigma(2)}, \dots x_{\sigma(n)}\right)$$

Also (7.4) becomes,

$$f\left(\sum_{i=1}^{b} x_i\right) \sum_{\tau} 1 \le (\ge) \sum_{i=1}^{n} f(x_i) \sum_{\tau} 1 + B\left(x_{\tau(1)}, x_{\tau(2)}, \dots x_{\tau(n)}\right)$$

$$\Rightarrow \frac{n!}{2} f\left(\sum_{i=1}^{n} x_i\right) \le (\ge) \frac{n!}{2} \sum_{i=1}^{n} f(x_i) + \sum_{\tau} B\left(x_{\tau(1)}, x_{\tau(2)}, \dots x_{\tau(n)}\right)$$

Adding above equations, we have

$$n! f\left(\sum_{i=1}^{n} x_i\right) \le (\ge) n! \sum_{i=1}^{n} f(x_i)$$

as other terms are cancelled due to odd and even permutations.

$$\therefore f\left(\sum_{i=1}^{n} x_i\right) \le (\ge) \sum_{i=1}^{n} f(x_i)$$

$\Rightarrow f$ is sub-additive (super additive)

Definition 7.3

Let $B: V \times V \times V \times \dots \times V \to R$ be a map in E^n with vector space V. Then $f: V - R$ is called absolutely B-sub-additive if

$$\left| f\left(\sum_{i=1}^{n} x_i\right) - \sum_{i=1}^{n} f(x_i) \right| \le B\left(x_1, x_2, \dots, x_n\right) \text{ for all } x_i \;\; x_i \in V, \; i = 1, 2, \dots n$$

Theorem 7.2

Let $B: V \times V \times V \times \dots \times V \to R$ be a map in E^n with vector space V and $f: V \to R$ be absolutely B-sub-additive. Then there is an additive function $g: V \to R$ such that

$$\left| f(x) - g(x) \right| \le B\left(x, x, \dots, x\right) \quad \text{for all } x \in V.$$

Proof:

Let us take

$$x_1 = n^{k-1}x,\ x_2 = n^{k-1}x, \ldots x_n = n^{k-1}x$$

Now as f is absolutely B-subadditive, one can write

$$\left| f\left(\sum_{i=1}^{n} x_i\right) - \sum_{i=1}^{n} f(x_i) \right| \le B(x_1, x_2, \ldots x_n)$$

$$\Rightarrow \left| f\left(n.n^{k-1}x\right) - nf\left(n^{k-1}x\right) \right| \le B\left(n^{k-1}x, \ldots n^{k-1}x\right)$$

$$\Rightarrow \left| f\left(n^{k}x\right) - nf\left(n^{k-1}x\right) \right| \le \bar{B}(x, x, \ldots x)$$

$$\Rightarrow \left| \frac{f\left(n^{k}x\right)}{n^k} - \frac{nf\left(n^{k-1}x\right)}{n^k} \right| \le \frac{\bar{B}(x, x, \ldots x)}{n^k}$$

for k >m,

$$\left| \frac{f\left(n^{k}x\right)}{n^k} - \frac{f\left(n^{m}x\right)}{n^m} \right| \le \left| \frac{f(n^{k}x)}{n^k} - \frac{f(n^{k-1}x)}{n^{k-1}} \right| + \left| \frac{f\left(n^{k-1}x\right)}{n^{k-1}} - \frac{f\left(n^{k-2}x\right)}{n^{k-2}} \right|$$

$$+\ldots+ \left| \frac{f\left(n^{m+1}x\right)}{n^{m+1}} - \frac{f\left(n^{m}x\right)}{n^m} \right| \le B(x, x, ..x)\left(\frac{1}{n^k} + \frac{1}{n^{k-1}} + \ldots + \frac{1}{n^m}\right)$$

Above line shows hat the sequence of general term is Cauchy.

We know every Cauchy sequence is convergent. So limit exists.

$$\therefore \lim_{k\to\infty} \frac{f(n^k x)}{n^k} = g(x)$$

Now $\left| g\left(\sum_{i=1}^{n} x_i\right) - \sum_{i=1}^{n} g(x_i) \right|$

$$= \lim_{k\to\infty} \left| \frac{f\left(\sum_{i=1}^{n} n^k x_i\right)}{n^k} - \sum_{i=1}^{n} \frac{f\left(n^k x_i\right)}{n^k} \right| \le \lim_{k\to\infty} \frac{B(x_1, x_2, \ldots, x_n)}{n^k} = 0$$

$$\Rightarrow g\left(\sum_{i=1}^{n} x_i\right) = \sum_{i=1}^{n} g(x_i)$$

$\Rightarrow g$ is additive map.

Now we will show that this g is unique. For this let us assume that there is another function h such that

$$\left| f(x) - h(x) \right| \le B(x, x, \ldots, x)$$

We have

$$\left| f(x) - g(x) \right| \le B(x, x, \ldots, x)$$

So, $\left| g(x) - h(x) \right| \le 2B(x, x, \ldots, x)$

$$\Rightarrow \left|g(n^k x)-h\left(n^k x\right)\right| \le 2B\left(n^k x, n^k x,\ldots,n^k x\right)$$

$$\Rightarrow \left|g\left(x\right)-h\left(x\right)\right| \le \frac{2\bar{B}\left(x,x,\ldots,x\right)}{n^k} \to 0 \text{ as } k \to \infty$$

(note that, $g\left(n^k x\right)=n^k g\left(x\right), h\left(n^k x\right)=n^k h\left(x\right)$)

So g(x) = h(x), for all x.

$\Rightarrow g$ is unique.

Now inductivity property shows that

$$\left|f\left(n^k x\right)-n^k f\left(x\right)\right| \le B\left(n^k x, n^k x,\ldots. n^k x\right)$$

$$= n^k B\left(x,x,\ldots,x\right)$$

Dividing n^k and taking the limit as $k \to \infty$,

We get

$$\left|g(x)-f\left(x\right)\right| \le B\left(x,x,\ldots,x\right)$$

Definition 7.4

The map $g: R_+ \to R$ is called B-decreasing on R_+ if

$$x_1 > x_2 \Rightarrow g\left(x_1\right) \le g\left(x_2\right)+B\left(x_1,x_2,\ldots,x_n\right) \quad \text{for all } x_i \in R_+, i=1,2,\ldots.n.$$

Theorem 7.3

Let $f: R_+ \to R$ be a function such that the map $x \to \frac{f(x)}{x}$ is B-decreasing on R_+, then f is B_1-sub-additive,

$$B_1\left(x_1,x_2\ldots.x_n\right)=x_1 B\left(\sum_{i=1}^{n} x_i, x_1, x_1\ldots x_1\right)+ x_2 B\left(\sum_{i=1}^{n} x_i, x_2, x_2\ldots x_2\right)+\ldots..$$

$$+x_n B\left(\sum_{i=1}^{n} x_i, x_n, x_n\ldots x_n\right) \qquad \text{for all } x_i \in R_+, i=1,2,\ldots.n$$

Proof: Now $x_i \in R_+, i=1,2,\ldots.n$

Also $\sum_{i=1}^{n} x_i > x_1$

Then $$\frac{f\left(\sum_{i=1}^{n} x_i\right)}{\sum_{i=1}^{n} x_i} \le \frac{f\left(x_1\right)}{x_1}+B\left(\sum_{i=1}^{n} x_1, x_1,\ldots x_1\right)$$

Similarly, we have

$$\frac{f\left(\sum_{i=1}^{n} x_i\right)}{\sum_{i=1}^{n} x_i} \le \frac{f\left(x_2\right)}{x_2}+B\left(\sum_{i=1}^{n} x_i, x_2, x_2,\ldots x_2\right)$$

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$$\frac{f\left(\sum_{i=1}^{n} x_i\right)}{\sum_{i=1}^{n} x_i} \le \frac{f(x_n)}{x_n} + B\left(\sum_{i=1}^{n} x_i, x_n, x_n, \ldots x_n\right)$$

Now $f\left(\sum_{i=1}^{n} x_i\right) = \frac{f\left(\sum_{i=1}^{n} x_i\right)}{\sum_{i=1}^{n} x_i}\left(\sum_{i=1}^{n} x_i\right) \le \sum_{i=1}^{n} f(x_i) + x_1 B\left(\sum_{i=1}^{n} x_i, x_1, x_1, \ldots x_1\right) + \ldots$

$$+ x_n B\left(\sum_{i=1}^{n} x_{i,} x_n, \ldots\ldots x_n\right) + \sum_{i=1}^{n} f(x_i) + B_1(x_1, x_2, \ldots x_n)$$

$\Rightarrow f$ is B_1-sub-additive.

Theorem 7.4

Let $A : R_+ \times R_+ \times R_+ \to R$ and $f : R_+ \to R$ be A concave function with $A_1 : C \times C \to R$ with convex vector space C. Let A_1 (. , .)= A(. , . , 0). Assume f(0) = 0. Then f is B_1-subadditive where

$$B_1(x_1 \ldots x_n) = -x_1 A_1\left(x_1 \sum_{i=1}^{n} x_i\right) - x_2 A_1\left(x_2, \sum_{i=1}^{n} x_i\right) \ldots x_n A_1\left(x_n, \sum_{i=1}^{n} x_i\right)$$

Proof:

Note that the function f is A-convex (A-Concave) if

$$\frac{f(x) - f(z)}{x - z} \le (\ge) \frac{f(y) - f(z)}{y - z} + A(x, y, z)$$

Where x < z < y

By hypothesis, f is A-concave. So one can write,

$$\frac{f(x) - f(z)}{x - z} \ge \frac{f(y) - f(z)}{y - z} + A(x, y, z)$$

$$\Rightarrow (y - z)(f(x) - f(z)) \ge (x - z)(f(y) - f(z)) + (y - z)(x - z)A(x, y, z)$$

$$\Rightarrow (y - z)f(x) - (y - z)f(z) \ge (x - z)f(y) - (x - z)f(z) + (y - z)(x - z)A(x, y, z)$$

$$\Rightarrow (y - z)f(x) \ge (x - z)f(y) - (x - z)f(z) + (y - z)f(z) + (y - z)(x - z)A(x, y, z)$$

$$\Rightarrow (y - z)f(x) \ge (x - z)f(y) + (y - z)f(z) + (y - z)(x - z)A(x, y, z)$$

$$\Rightarrow f(x) \ge \frac{x - z}{y - z} f(y) + \frac{y - x}{y - z} f(z) + (x - z)A(x, y, z)$$

Take

$$\lambda = \frac{x - z}{y - z} \in (0, 1)$$

$$1 - \lambda = \frac{y - x}{y - z}$$

Also $\lambda y+(1-\lambda)z=\frac{x-z}{y-z}y+\frac{y-x}{y-z}z=x$

So $f(x)\geq\lambda f(y)+(1-\lambda)f(z)+(x-z)A(x,y,z)$

By assumption, f(0) = 0

$\therefore\frac{f(x)-f(0)}{x-0}=\frac{f(x)}{x}$

Now from above, we observe that the function $\frac{f(x)}{x}$ is A_1 increasing. So one can write

$$\frac{f(x_1)}{x_1}\geq\frac{f\left(\sum_{i=1}^{n}x_i\right)}{\sum_{i=1}^{n}x_i}+A_1\left(x_1,\sum_{i=1}^{n}x_i\right)$$

....

....

$$\frac{f(x_n)}{x_n}\geq\frac{f\left(\sum_{i=1}^{n}x_i\right)}{\sum_{i=1}^{n}x_i}+A_n\left(x_n,\sum_{i=1}^{n}x_i\right)$$

On simplification, we have

$$\sum_{i=1}^{n}x_i\geq x_1\frac{f\left(\sum_{i=1}^{n}x_i\right)}{\sum_{i=1}^{n}x_i}+\dots+x_n\frac{f\sum_{i=1}^{n}x_i}{\sum_{i=1}^{n}x_i}+x_1A_1\left(x_1,\sum_{i=1}^{n}x_i\right)+\dots+x_nA_n\left(x_n,\sum_{i=1}^{n}x_i\right)$$

$$=f\left(\sum_{i=1}^{n}x_i\right)-B_1(x_1,\dots x_n)$$

$$\Rightarrow f\left(\sum_{i=1}^{n}x_i\right)\leq\sum_{i=1}^{n}f(x_i)+B_1(x_1,\dots x_n)$$

$\Rightarrow$ f is B_1 sub-additive with given B_1 above.

Theorem 7.5

Let $f:R_+\to R$ be convex and B-sub-additive. Then the function $\frac{f(\cdot)}{x}$ is C-increasing for $C:R_+\times R_+\to R$.

Proof: Take $\lambda=\frac{(n-1)x}{(n-1)x+h}\in(0,1),h>0.$

$$\lambda x+(1-\lambda)(nx+h)$$

$$=\frac{(n-1)x}{(n-1)x+h}.x+\left[1-\frac{(n-1)x}{(n-1)x+h}\right](nx+h)$$

$$=\frac{(n-1)x}{(n-1)x+h}.x+\frac{(n-1)x+h-(n-1)x}{(n-1)x+h}(nx+h)$$

$$=\frac{(n-1)x^2+nxh+h^2}{(n-1)x+h}$$

$$=x+h$$

As f is B-sub additive, we have

$$f(nx+h)=f(x+...+x+x+h)\le(n-1)f(x)+f(x+h)+B(x,x,...x,x+h)$$

As f is convex function, one can write

$$f(x+h)\le\lambda f(x)+(1-\lambda)f(nx+h)$$

$$=\lambda f(x)+(1-\lambda)\{(n-1)f(x)+f(x+h)+B(x,x,...x,x+h)\}$$

$$=\lambda f(x)+(n-1)f(x)-(n-1)\lambda f(x)+f(x+h)-\lambda f(x+h)$$

$$+(1-\lambda)B(x,x,....,x,x+h)$$

$$\Rightarrow\lambda f(x+h)\le(n-1)f(x)-(n-2)\lambda f(x)+(1-\lambda)B(x,x,...x,x+h)$$

$$\Rightarrow\frac{(n-1)x}{(n-1)x+h}f(x+h)$$

$$\le(n-1)f(x)-\frac{(n-2)(n-1)x}{(n-1)x+h}f(x)+\left(1-\frac{(n-1)x}{(n-1)x+h}\right)B(x,x,....x,x+h)$$

$$\Rightarrow\frac{(n-1)x}{(n-1)x+h}f(x+h)\le(n-1)f(x)-\frac{(n-2)(n-1)x}{(n-1)x+h}f(x)+\frac{h}{(n-1)x+h}B(x,x,...x,x+h)$$

$$=\frac{(n-1)(x+h)}{(n-1)x+h}f(x)+\frac{h}{(n-1)x+h}B(x,x,...x+h)$$

$$\Rightarrow(n-1)xf(x+h)\le(n-1)(x+h)f(x)+hB(x,x,...x,x+h)$$

$$\Rightarrow\frac{(n-1)xf(x+h)}{(n-1)x(x+h)}\le\frac{f(x)}{x}+\frac{hB(x,x,...x,x+h)}{(n-1)x(x+h)}$$

$$\Rightarrow\frac{f(x+h)}{x+h}\le\frac{f(x)}{x}+C(x,h)$$

where $C(x,h)=-\frac{hB(x,x,...x,x+h)}{(n-1)x(x+h)}$

So the function $\frac{f(x)}{x}$ is C-increasing.

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