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FUNDAMENTALS OF TOPOLOGY SETS AND ITS IMPACT ON MATHEMATICS

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Abstract: Topology is a discipline of mathematics in which two objects are regarded equal if they may be continually deformed into one another through motions in space such as bending, twisting, stretching, and shrinking without preventing tearing apart or gluing together sections. Differentiable equations, dynamical systems, knot theory, and Riemann surfaces in complex analysis are all examples of topology in mathematics. It's also used in physics to describe the universe's space-time structure and in string theory. The branch of mathematics known as topology studies continuity and related notions. Open and closed sets, continuity, and homeomorphism are only a few of the essential concepts that will be introduced in the near future.

***Index Terms* - Topology, Open Set, Closed Set, Mathematics**

1.1 Introduction:

Topology is a discipline of mathematics in which two objects are regarded equal if they may be continually deformed into one another through motions in space such as bending, twisting, stretching, and shrinking without preventing tearing apart or gluing together sections. Differentiable equations, dynamical systems, knot theory, and Riemann surfaces in complex analysis are all examples of topology in mathematics. It's also used in physics to describe the universe's space-time structure and in string theory.

The branch of mathematics known as topology studies continuity and related notions. Open and closed sets, continuity, and homeomorphism are only a few of the essential concepts that will be introduced in the near future. Topology as a well-defined mathematical discipline dates from the early twentieth century, but some isolated results can be traced back hundreds of years. Among these are some of Leonhard Euler's investigations into geometry. One of the first practical applications of topology was his paper on the Seven

Bridges of Königsberg, published in 1736. Euler wrote to a friend on November 14, 1750, that he had realized the importance of a polyhedron's edges. $V E + F = 2$ was his polyhedron formula as a result of this (where V, E, and F respectively indicate the number of vertices, edges, and faces of the polyhedron).

1.2 Types of Topology Sets:

1. **Open Set Topology:** If a set is a neighborhood of every point, it is called an open set in topology. While a neighborhood is defined as follows: If X is a topological space and p is a point in X, a neighborhood of p is a subset V of X that includes an open set U containing p.
2. **Closed Set Topology:** A closed set is a set whose counterpart is an open set in geometry, topology, and related disciplines of mathematics. A closed set in a topological space is a set that contains all of its limit points.

1.3 Open Set Topology v/s Closed Set Topology:

An open set is one that has no limit or boundary points. The criteria for determining if a set is open or not is whether you can draw a circle, no matter how small, around any point in the set. The closed set is the open set's inverse.

Definition: The distance between real numbers x and y is |x−y|. For example, dist(−4,3) = |(−4)−(3)| = 7.

Definition: For a real number x and > 0, B(x) = {y ∈R : dist(x,y) < }. Of course, B(x) is another way of describing the open interval (x−,x+). We also call this an epsilon neighborhood of x.

Definition: An open subset of R is a subset E of R such that for every x in E there exists > 0 such that B(x) is contained in E.

For example, the open interval (2,5) is an open set. Any open interval is an open set. Both R and the empty set are open. The union of open sets is an open set.

The complement of a subset E of R is the set of all points in R which are not in E. It is denoted R\E or E∼. (Note, the second notation requires you to know that the complement is defined relative to R.)

[2,5] is not an open set, but its complement (−∞,2)∪(5,∞) is open. This is an example of the next which provides a useful perspective on convergence.

Proposition C. Suppose E is a subset of R. The following are equivalent.

1. If a sequence of points from E converges, then the limit of the sequence is in E.

2. The complement of E is an open set.

Names. There are names for the two properties listed above:

1. E is sequentially closed.

2. E is closed.

In your future, you may see the more general situations in which the two notions need not be the same. In R, you can use either to mean "closed".

For the next summary, here is some notation. A sequence converges in E if it converges and its limit is in E. A cover of a set E is a collection C of sets whose union contains E. A subcover is a collection of some of the sets in C whose union still contains E. A finite subcover is a subcover which uses only finitely many of the sets in C. An open cover is a cover by a collection of sets all of which are open.

Proposition K. Suppose E is a subset of R. The following are equivalent.

1. E is closed and bounded.

2. Every sequence from E has a subsequence which converges in E.

3. Every open cover of E has a finite subcover.

Names. The last two properties have names:

(1) E is sequentially compact.

(2) E is compact.

In R the conditions are equivalent, and you can used "compact" and "sequentially compact" interchangeably.

The condition (3) is more abstract. You're not required to know about it now. But you should be able to prove the other two conditions are equivalent. As follows, for example.

Proof

(1) =⇒ (2)

Suppose E is closed and bounded. Suppose (an) is a sequence of points from E. Since it is bounded, it has a convergence subsequence. Since E is closed, the limit is in E. Therefore E is sequentially compact.

(2) =⇒ (1)

It suffices to show that if E is not closed and bounded, then E is not sequentially compact.

If E is not closed, then there is a sequence (an) of points from E which converges to a point outside E. Since every subsequence of (an) has the same limit, no subsequence of (an) can converge in E. Therefore E is not sequentially compact.

If E is not bounded, then there for each $n \in N$ we may pick a point an from E such that $|an| > n$. Then no subsequence of (an) is bounded, so no subsequence of (an) can converge, and therefore E is not sequentially compact.

1.4 The Open and Closed Sets of a Topological Space Examples:

If (X,τ) is a topological space then a set $A \subseteq X$ is said to be open if $A \in \tau$ and A is said to be closed if $Ac \in \tau$. Furthermore, if A is both open and closed, then we say that A is clopen.

We will now look at some examples of identifying the open, closed, and clopen sets of a topological space (X,τ).

Example 1: Let $X=\{a,b,c,d\}$ and consider the topology $\tau=\{\emptyset,\{c\},\{a,b\},\{c,d\},\{a,b,c\},X\}$. What are the open, closed, and clopen sets of X with respect to this topology?

The open sets of X are those sets forming τ:

$$\text{opensetsofX}=\{\emptyset,\{c\},\{a,b\},\{c,d\},\{a,b,c\},X\} \quad (1)$$

The closed sets of X are the complements of all of the open sets:

$$\text{closedsetsofX}=\{\emptyset,\{a,b,d\},\{c,d\},\{a,b\},\{d\},X\} \quad (2)$$

The clopen sets of X are the sets that are both open and closed:

$$\text{clopensetsofX}=\{\emptyset,\{a,b\},\{c,d\},X\} \quad (3)$$

Example 2: Prove that if X is a set and every $A \subseteq X$ is clopen with respect to the topology τ then τ is the discrete topology on X.

Let X be a set and let every $A \subseteq X$ be clopen. Then every $A \subseteq X$ is open, i.e., every subset of X is open, so $\tau=P(X)$. Hence τ is the discrete topology on X.

Example 3: Consider the topological space (Z,τ) where τ is the cofinite topology. Determine whether the set of even integers is open, closed, and/or clopen. Determine whether the set $Z\setminus\{1,2,3\}$ is open, closed, and/or clopen. Determine whether the set $\{-1,0,1\}$ is open, closed, and/or clopen. Show that any nontrivial subset of Z is never clopen.

Recall that the cofinite topology τ is described by:

$\tau=\{U \subseteq X: U=\emptyset \text{or} Uc \text{isfinite}\}$

We first consider the set of even integers which we denote by $E=\{\dots,-2,0,2,\dots\}$. We see that Ec is the set of odd integers, i.e., $Ec=\{\dots,-3,-1,1,3,\dots\}$ which is an infinite set. Therefore $E \notin \tau$ so E is not open. Furthermore, we have that $(Ec)c=E$ is an infinite set and $Ec \notin \tau$ so E is not closed either.

We now consider the set $Z\setminus\{1,2,3\}$. We have that $(Z\setminus\{1,2,3\})c=\{1,2,3\}$ which is a finite set. Therefore $Z\setminus\{1,2,3\} \in \tau$, so $Z\setminus\{1,2,3\}$ is open. Now consider the complement $(Z\setminus\{1,2,3\})c=\{1,2,3\}$. The complement of this set is $(\{1,2,3\})c=Z\setminus\{1,2,3\}$ which is an infinite set, so $(\{1,2,3\})c \notin \tau$. Hence $Z\setminus\{1,2,3\}$ is not closed.

Lastly we consider the set $\{-1,0,1\}$. We have that $(\{-1,0,1\})c=\mathbb{Z}\setminus\{-1,0,1\}$ which is an infinite set, so $\{-1,0,1\}\notin\tau$ so $\{-1,0,1\}$ is not open. Now consider the complement $(\{-1,0,1\})c=\mathbb{Z}\setminus\{-1,0,1\}$. The complement of this set is $\{-1,0,1\}$ is finite, seo $(\{-1,0,1\})c\in\tau$. Hence $\{-1,0,1\}$ is closed.

Lastly, let $A\subseteq\mathbb{Z}$ be a nontrivial subset of Z, i.e., $A\neq\emptyset$ and $A\neq\mathbb{Z}$. Suppose that A is clopen. Then A is both open and closed. Hence by definition, A and Ac are both open. Hence Ac and A are both finite. But $\mathbb{Z}=A\cap Ac$ which implies that Z is a finite set - which is preposterous since the set of integers is an infinite set! Hence A cannot be clopen.

1.5 Conclusion:

A topological space is a set with a topology that allows for the definition of continuous deformation of subspaces and, more broadly, all kinds of continuity. Any distance or metric defines a topology, so Euclidean spaces and, more broadly, metric spaces are examples of topological spaces. Homeomorphisms and homotopies are two types of deformations that are taken into account in topology. A topological property is one that is invariant under such deformations.

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