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NUMERICAL STUDY OF ITERATIVE METHODS FOR SOLVING THE NON-LINEAR EQUATION

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Abstract: An iterative method is a mathematical procedure and part of the numerical analysis that used as an initial value to generate a sequence of improving approximate solutions for solving the non-linear equations $f(x) = 0$. The main purpose of this paper is to find out the best method out of bisection method, Regula-Falsi Method and Newton-Raphson Method for solving non-linear equations $f(x) = 0$ and also comparing them through iterative methods. The researcher used non-linear algebraic equation $f(x) = 0$ to find approximate root and explained by three iterative methods i.e bisection method, Regula-Falsi Method and Newton-Raphson Method. Their results obtained that the Newton-Raphson Method is best and fast approximation method to compare than other two iterative methods.

Key-word: *Non-linear algebraic equation, bisection Method, Regula-Falsi Method, Newton-Raphson Method*.

1. INTRODUCTION

Numerical analysis is a branch of mathematics that provides tools and methods for solving mathematical problems. Numerical analysis is the study of algorithms that use numerical approximation for the problems of mathematical analysis. The goal of the field of numerical analysis is to design and analyze the techniques to give approximate but accurate solutions to hard problems. Many mathematical models of physics, economics, engineering, science and other disciplines come up with the different types of non-linear equations. In recent time, several scientists and engineers have been focused on to solve the non-linear equations both numerically and analytically. Solving nonlinear equations is one of the important research areas in numerical analysis but finding the exact solutions of nonlinear equations is a difficult task. In several occasions, it may not be simple to get the exact solutions. Even solutions of equations exist; they may not be real rational. In this case, we need to find its rational/decimal approximation. Hence, numerical methods are helpful to find approximate solutions.

One of the most frequently occurring problems in scientific work is to find the root of non-linear equations $f(x) = 0$. We always assume that $f(x)$ is continuously differentiable real-valued function of a real variable x . The researcher focused on solving the root of nonlinear algebraic equations involving only a single variable using three iterative methods i.e bisection method, Regula-Falsi method and Newton-Raphson method.

The objective of the study is to compare the three iterative methods to determine the most appropriate methods for solving the root of non-linear algebraic given equation $f(x) = 0$ in one variable.

2. LITERATURE REVIEW

Moheuddin, M. M., Uddin, M. J., & Kowsher, M. (2019) the major goal of this study is to determine the optimal approach for solving the nonlinear equation using iterative methods. The objective of this research is to determine the rate of convergence, the correct solution, and the level of errors in the methodologies. The researcher aims at comparing existing methods in order to find the most effective method for solving nonlinear equations. The researcher discussed four iterative methods, their rates of convergence, and how they compare to graphic representation. The researcher has found that Newton-Raphson Method is the most effective and precise method for solving non-linear equations.

Ebelechukwu, O. C. (2018) The aim of this research is to find the most appropriate method for solving nonlinear equations. Four methods for solving nonlinear equations were explained in this paper. The objective of this research is to find the best outcome for using numerical methods to solve nonlinear equations. The researcher has compared the various approaches to determine which solution (or methods) is best for the particular problem. The research has found that Newton-Raphson Method is the most effective method for finding the roots of non-linear equations, because it converges to the roots of the non-linear equation is faster than the other three ways, depending on the results obtained from the four methods. In comparison to the other three methods, which take a long time to converge, it converges after a few iterations.

Hasan, A. (2016) provide a numerical analysis of some iterative methods for solving nonlinear equations in this research. The goal of the research is to compare the rates of performance (convergence) of Bisection, Newton-Raphson, and Secant as root-finding methods. The Bisection method converges at the 47th iteration, whereas the Newton-Raphson and Secant methods converge at the 4th and 5th iterations, respectively, to the exact root of 0.36042170296032 with the same error level. The Newton approach, in comparison to the Secant method, has less iterations. The Secant approach was subsequently shown to be the most successful of the three ways considered. Numerical experiments are used to illustrate that the secant approach is more efficient than other methods. Researcher concluded that the secant method is formally the most effective of the Newton method.

3. MATERIALS AND METHODS

3.1 Bisection method

Assume that $f(x)$ is continuous on a given interval [a, b] and that is also satisfies $f(a).f(b) < 0$ with $f(a) \neq 0$ and $f(b) \neq 0$. The intermediate value theorem highlights the function $f(x)$ has atleast one root in [a, b]. It can be assumed that there is only one root for the non-linear equation $f(x) = 0$ in the interval [a, b].

In this method, we follow the steps:

1. Equate the given polynomial to zero i.e $f(x) = 0$.
2. Find two real number a and b such that $a < b$ and $f(a).f(b) < 0$
3. Find approximation $x_n = \frac{a+b}{2}$
4. (a) If $f(a).f(x_n) < 0$ then root lies in (a, x_n) and go to step (c) with $b = x_n$
   (b) If $f(b).f(x_n) < 0$ then root lies in (x_n, b) and go to step (c) with $a = x_n$
   (c) If $f(a).f(x_n) = 0$ then x_n is a root of the equation $f(x) = 0$.
5. Repeat steps (3) and (4) until $f(x_n) = 0$ or $|f(x_n)| \leq$ desired accuracy.

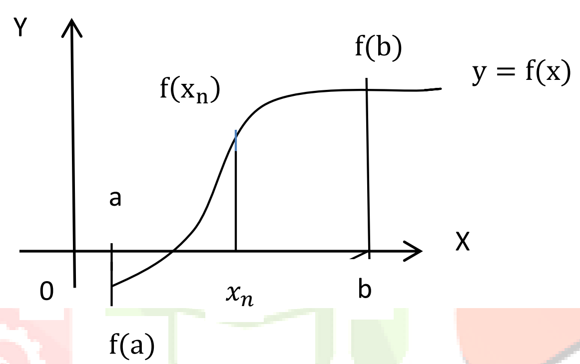

Fig.1

3.2 Regula-Falsi Method

Consider a curve $y = f(x)$ in the interval (a, b). This curve is approximated by chords or straight line. The point at which the chord intersects the X-axis is the approximate value.

In this method, we follow the steps:

1. Find the interval (a, b) such that $f(a)$ and $f(b)$ have opposite signs i.e. $f(a).f(b) < 0$.
2. Consider the point x_n such that Chord AB intersect X-axis at $(x_n, 0)$.
3. Find approximation
$$x_n = \frac{a.f(b) - b.f(a)}{f(b) - f(a)}$$
4. Calculate $f(x_n)$
5. (a) If $f(a)$ and $f(x_n)$ have opposite signs, then $f(x) = 0$ has a root between a and x_n
   (b) If $f(b)$ and $f(x_n)$ have opposite signs, then $f(x) = 0$ has a root between x_n and b
6. Repeat steps (3), (4) and (5) until $f(x_n) = 0$ or $|f(x_n)| \leq$ Desired accuracy.

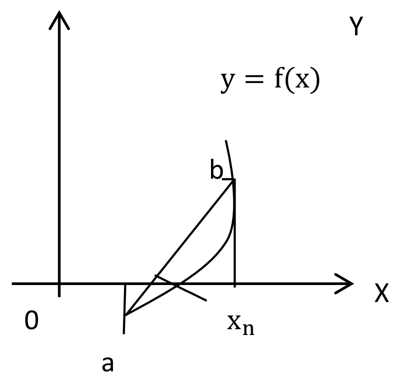

Fig.2

3.3 Newton-Raphson Method

Consider a curve $y = f(x)$, Let $(x_0, f(x_n))$ be any point on curve where $x_0$is initial root.

Now the curve is approximate by the tangent at the point$(x_0, f(x_n))$.

This tangent if meets $X-$ axis at $x = x_1$ then the new approximation is x_1 as the root.

In this method, we follow the steps:

a) Find points a and b such that $a < b$ and $f(a).f(b) < 0$.

b) Take the initial value x_0 value in the interval $[a, b]$

c) Find $f(x_n)$ and $f'(x_n)$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

d) If $f(x_n) = 0$ then x_n is an exact root, else $x_0 = x_n$

e) Repeat steps (c) and (d) until $f(x_n) = 0$ or$|f(x_n)| \leq$ desired accuracy

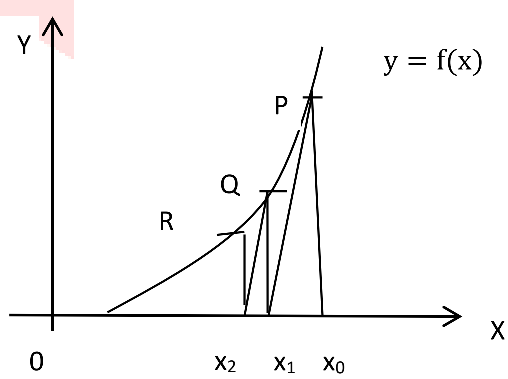

Fig.3

4. Results and Discussion

Researcher has hypothetical example for proving the objective of the research.

Numerical Example: For all three methods

Find approximate root of $x^3 - 4x - 9 = 0$ correct up to four decimal places.

3.1 Bisection method

Solution: Given, $x^3 - 4x - 9 = 0$

Let $f(x) = x^3 - 4x - 9$

By using bisection method

$$x_n = \frac{a+b}{2}$$

Take $a = 2$ and $b = 3$

$f(a) = f(2) = (2)^3 - 4(2) - 9 = -9 < 0,\ f(b) = f(3) = (3)^3 - 4(3) - 9 = 6 > 0$

$\therefore f(a).f(b) = f(2).f(3) < 0$

$\therefore f(x) = 0$ has a root lies betweena $= 2$ and b $=3$

First iteration:

$$x_1 = \frac{a+b}{2} = \frac{2+3}{2} = 2.5$$

$\therefore f(x_1) = f(2.5) = (2.5)^3 - 4(2.5) - 9 = -3.375 < 0$

$\therefore f(x_1).f(b) = f(2.5).f(3) < 0$

$\therefore f(x) = 0$ has a root lies between $x_1 = 2.5$ and b $=3$

Second iteration:

$$x_2 = \frac{x_1+b}{2} = \frac{2.5+3}{2} = 2.75$$

$\therefore f(x_2) = f(2.75) = (2.75)^3 - 4(2.75) - 9 = 0.79688 > 0$

$\therefore f(x_1).f(x_2) = f(2.5).f(2.75) < 0$

$\therefore f(x) = 0$ has a root lies between $x_1 = 2.5$ and $x_2 = 2.75$

Third iteration:

$$x_3 = \frac{x_1 + x_2}{2} = \frac{2.5 + 2.75}{2} = 2.625$$

$$\therefore f(x_3) = f(2.625) = (2.625)^3 - 4(2.625) - 9 = -1.41211 < 0$$

$$\therefore f(x_3).f(x_2) = f(2.625).f(2.75) < 0$$

$\therefore f(x) = 0$ has a root lies between $x_3 = 2.625$ and $x_2 = 2.75$

Fourth iteration:

$$x_4 = \frac{x_3 + x_2}{2} = \frac{2.625 + 2.75}{2} = 2.6875$$

$$\therefore f(x_4) = f(2.6875) = (2.6875)^3 - 4(2.6875) - 9 = -0.33911 < 0$$

$$\therefore f(x_4).f(x_2) = f(2.6875).f(2.75) < 0$$

$\therefore f(x) = 0$ has a root lies between $x_4 = 2.6875$ and $x_2 = 2.75$

Fifth iteration:

$$x_5 = \frac{x_4 + x_2}{2} = \frac{2.6875 + 2.75}{2} = 2.71875$$

$$\therefore f(x_5) = f(2.71875) = (2.71875)^3 - 4(2.71875) - 9 = 0.22092 > 0$$

$$\therefore f(x_4).f(x_5) = f(2.6875).f(2.71875) < 0$$

$\therefore f(x) = 0$ has a root lies between $x_4 = 2.6875$ and $x_5 = 2.71875$

Sixth iteration:

$$x_6 = \frac{x_4 + x_5}{2} = \frac{2.6875 + 2.71875}{2} = 2.70313$$

$$\therefore f(x_6) = f(2.70313) = (2.70313)^3 - 4(2.70313) - 9 = -0.06099 < 0$$

$$\therefore f(x_6).f(x_5) = f(2.70313).f(2.7187) < 0$$

$\therefore f(x) = 0$ has a root lies between $x_6 = 2.70313$ and $x_5 = 2.71875$

Seventh iteration:

$$x_7 = \frac{x_6 + x_5}{2} = \frac{2.70313 + 2.71875}{2} = 2.71094$$

$$\therefore f(x_7) = f(2.71094) = (2.71094)^3 - 4(2.71094) - 9 = 0.07947 > 0$$

$$\therefore f(x_6).f(x_7) = f(2.70313).f(2.71094) < 0$$

$\therefore$ $f(x) = 0$ has a root lies between $x_6 = 2.70313$ and $x_7 = 2.71094$

Eighth iteration:

$$x_8 = \frac{x_6 + x_7}{2} = \frac{2.70313 + 2.71094}{2} = 2.70704$$

$$\therefore f(x_8) = f(2.70704) = (2.70704)^3 - 4(2.70704) - 9 = 0.00921 > 0$$

$$\therefore f(x_6).f(x_8) = f(2.70313).f(2.70704) < 0$$

$\therefore$ $f(x) = 0$ has a root lies between $x_6 = 2.70313$ and $x_8 = 2.70704$

Ninth iteration:

$$x_9 = \frac{x_6 + x_8}{2} = \frac{2.70313 + 2.70704}{2} = 2.70509$$

$$\therefore f(x_9) = f(2.70509) = (2.70509)^3 - 4(2.70509) - 9 = -0.02583 < 0$$

$$\therefore f(x_9).f(x_8) = f(2.70509).f(2.70704) < 0$$

$\therefore$ $f(x) = 0$ has a root lies between $x_9 = 2.70509$ and $x_8 = 2.70704$

Tenth iteration:

$$x_{10} = \frac{x_9 + x_8}{2} = \frac{2.70509 + 2.70704}{2} = 2.70607$$

$$\therefore f(x_{10}) = f(2.70607) = (2.70607)^3 - 4(2.70607) - 9 = -0.00823 < 0$$

$$\therefore f(x_{10}).f(x_8) = f(2.70607).f(2.70704) < 0$$

$\therefore$ $f(x) = 0$ has a root lies between $x_{10} = 2.70607$ and $x_8 = 2.70704$

Eleventh iteration:

$$x_{11} = \frac{x_{10} + x_8}{2} = \frac{2.70607 + 2.70704}{2} = 2.70656$$

$$\therefore f(x_{11}) = f(2.70656) = (2.70656)^3 - 4(2.70656) - 9 = 0.00058 > 0$$

$$\therefore f(x_{10}).f(x_{11}) = f(2.70607).f(2.70656) < 0$$

$\therefore$ $f(x) = 0$ has a root lies between $x_{10} = 2.70607$ and $x_{11} = 2.70656$

Twelfth iteration:

$$x_{12} = \frac{x_{10} + x_{11}}{2} = \frac{2.70607 + 2.70656}{2} = 2.70632$$

$$\therefore f(x_{12}) = f(2.70632) = (2.70632)^3 - 4(2.70632) - 9 = -0.00374 < 0$$

$\therefore f(x_{12}).f(x_{11}) = f(2.70607).f(2.70656) < 0$

$\therefore f(x) = 0$ has a root lies between $x_{12} = 2.70632$ and $x_{11} = 2.70656$

Thirteen iteration:

$$x_{13} = \frac{x_{12} + x_{11}}{2} = \frac{2.70632 + 2.70656}{2} = 2.70644$$

$\therefore f(x_{13}) = f(2.70644) = (2.70644^3 - 4(2.70644) - 9 = -0.00158 < 0$

$\therefore f(x_{13}).f(x_{11}) = f(2.70644).f(2.70656) < 0$

$\therefore f(x) = 0$ has a root lies between $x_{13} = 2.70644$ and $x_{11} = 2.70656$

Fourteen iteration:

$$x_{14} = \frac{x_{13} + x_{11}}{2} = \frac{2.70644 + 2.70656}{2} = 2.70650$$

$\therefore f(x_{13}) = f(2.70650) = (2.70650)^3 - 4(2.70650) - 9 = -0.00050 < 0$

$\therefore f(x_{14}).f(x_{11}) = f(2.70650).f(2.70656) < 0$

$\therefore f(x) = 0$ has a root lies between $x_{14} = 2.70650$ and $x_{11} = 2.70656$

Fifteen iteration:

$$x_{15} = \frac{x_{14} + x_{11}}{2} = \frac{2.70650 + 2.70656}{2} = 2.70653$$

$\therefore x_{14}$ and x_{15} both are correct four decimal places.

The approximate root of the non-linear algebraic equation $x^3 - 4x - 9 = 0$ is 2.7065 correct up to four decimal places.

3.2 Regula-Falsi (False position) method

Solution: Given, $x^3 - 4x - 9 = 0$

Let $f(x) = x^3 - 4x - 9$

By Regula-Falsi (False position) method

$$x_n = \frac{a.f(b) - b.f(a)}{f(b) - f(a)}$$

Take $a = 2$ and $b = 3$

$f(a) = f(2) = (2)^3 - 4(2) - 9 = -9 < 0,\ f(b) = f(3) = (3)^3 - 4(3) - 9 = 6 > 0$

$\therefore f(a).f(b) = f(2).f(3) < 0$

$\therefore f(x) = 0$ has a root lies between $a = 2$ and $b = 3$

First iteration:

$$x_1 = \frac{a.f(b) - b.f(a)}{f(b) - f(a)} = \frac{2(6) - 3(-9)}{6 - (-9)} = \frac{12 + 27}{6 + 9} = \frac{39}{15} = 2.6$$

$\therefore f(x_1) = f(2.6) = (2.6)^3 - 4(2.6) - 9 = -1.824 < 0$

$\therefore f(x_1).f(b) = f(2.6).f(3) < 0$

$\therefore f(x_1) = 0$ has a root lies between $x_1 = 2.6$ and $b = 3$

Second iteration:

$$x_2 = \frac{x_1.f(b) - b.f(x_1)}{f(b) - f(x_1)} = \frac{2.6(6) - 3(-1.824)}{6 - (-1.824)} = \frac{15.6 + 5.472}{6 + 1.824} = \frac{21.072}{7.824} = 2.69325$$

$\therefore f(x_2) = f(2.69325) = (2.69325)^3 - 4(2.69325) - 9 = -0.23725 < 0$

$\therefore f(x_2).f(b) = f(2.69325).f(3) < 0$

$\therefore f(x) = 0$ has a root lies between $x_2 = 2.69325$ and $b = 3$

Third iteration:

$$x_3 = \frac{x_2.f(b) - b.f(x_2)}{f(b) - f(x_2)} = \frac{2.69325\ (6) - 3(-0.23725)}{6 - (-0.23725)} = \frac{16.15950 + 0.7092}{6 + 0.23725} = \frac{16.87125}{6.23725} = 2.70492$$

$\therefore f(x_3) = f(2.70492) = (2.70492)^3 - 4(2.70492) - 9 = -0.02888 < 0$

$\therefore f(x_3).f(b) = f(2.70492).f(3) < 0$

$\therefore f(x) = 0$ has a root lies between $x_3 = 2.70492$ and $b = 3$

Fourth iteration:

$$x_4 = \frac{x_3.f(b) - b.f(x_3)}{f(b) - f(x_3)} = \frac{2.70492\ (6) - 3(0.02888)}{6 - (0.02888)} = \frac{16.22952 + 0.08664}{6 + 0.02888} = \frac{16.31616}{6.02888} = 2.70633$$

$\therefore f(x_4) = f(2.70633) = (2.70633)^3 - 4(2.70633) - 9 = -0.00356 < 0$

$\therefore f(x_4).f(b) = f(2.70633).f(3) < 0$

$\therefore f(x) = 0$ has a root lies between $x_4 = 2.70633$ and $b = 3$

Fifth iteration:

$$x_5 = \frac{x_4.f(b) - b.f(x_4)}{f(b) - f(x_4)} = \frac{2.70633\ (6) - 3(-0.00356)}{6 - (-0.00356)} = \frac{16.23798 + 0.01068}{6 + 0.00356} = \frac{16.24866}{6.00356} = 2.7065$$

$\therefore f(x_5) = f(2.7065) = (2.7065)^3 - 4(2.7065) - 9 = -0.0005 < 0$

$\therefore f(x_5).f(b) = f(2.7065).f(3) < 0$

$\therefore f(x) = 0$ has a root lies between $x_4 = (2.7065)$ and $b = 3$

Sixth iteration:

$$x_6 = \frac{x_5.f(b) - b.f(x_5)}{f(b) - f(x_5)} = \frac{2.7065\ (6) - 3(-0.0005)}{6 - (-0.0005)} = \frac{16.239 + 0.0015}{6 + 0.0005} = \frac{16.2405}{6.0005} = 2.70652$$

$\therefore x_5$ and x_6 both are correct upto four decimal places.

The approximate root of the non-linear algebraic equation $x^3 - 4x - 9 = 0$ is 2.7065 correct up to four decimal places.

3.3 Newtons-Raphson Method

Solution: Given, $x^3 - 4x - 9 = 0$

Let $f(x) = x^3 - 4x - 9$

$f'(x) = 3x^2 - 4$

By Newtons-Raphson method

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Put $x = 2,\ f(2) = (2) = (2)^3 - 4(2) - 9 = -9 < 0$

Put $x = 3,\ f(3) = (3)^3 - 4(3) - 9 = 6 > 0$

$f(2).f(3) < 0$

Take $x_0 = 2,\ f(x_0) = f(2) = (2) = (2)^3 - 4(2) - 9 = -9$,
$f'(x_0) = f'(2) = 3(2)^2 - 4 = 8$

First iteration:

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 2 - \frac{(-9)}{8} = 2 + \frac{9}{8} = 2 + 1.125 = 3.125$$

$f(x_1) = f(3.125) = (3.125)^3 - 4(3.1250) - 9 = 9.01758$

$f'(x_1) = f(3.125) = 3(3.125)^2 - 4 = 25.29688$

Second iteration:

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 3.125 - \frac{9.01758}{25.29688} = 3.125 - 0.35647 = 2.76853$$

$f(x_2) = f(2.76853) = (2.76853)^3 - 4(2.76853) - 9 = 1.14599$

$f'(x_2) = f(2.76853) = 3(2.76853)^2 - 4 = 18.99428$

Third iteration:

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 2.76853 - \frac{1.14599}{18.99428} = 2.76853 - 0.06033 = 2.7082$$

$f(x_3) = f(2.7082) = (2.7082)^3 - 4(2.7082) - 9 = 0.03008$

$f'(x_3) = f(2.7082) = 3(2.7082)^2 - 4 = 18.00304$

Fourth iteration:

$$x_4 = x_3 - \frac{f(x_3)}{f'(x_3)} = 2.7082 - \frac{0.03008}{18.00304} = 2.7082 - 0.00167 = 2.70653$$

$f(x_4) = f(2.70653) = (2.70653)^3 - 4(2.70653) - 9 = 0.00004$

$f'(x_4) = f(2.70653) = 3(2.70653)^2 - 4 = 17.97591$

Fifth iteration:

$$x_5 = x_4 - \frac{f(x_4)}{f'(x_4)} = 2.70653 - \frac{0.00004}{17.97591} == 2.70653 - 0.000002 = 2.70652$$

$\therefore x_4$ and x_5 both are correct upto four decimal places.

The approximate real root of the non-linear algebraic equation $x^3 - 4x - 9 = 0$ is 2.7065 correct up to four decimal places.

5. Conclusion

In this research paper, the researcher has analyzed and compared the three iterative methods for solving the root of non-linear equation $f(x) = 0$. The result obtained from the analysis of three iterative methods by using above numerical examples is that, the bisection method takes more numbers of iteration to solve the root of non-liner equation as compare to, Regula falsi method and Newtons-Raphson Method. Regula falsi method takes more numbers of iteration as compare to Newtons-Raphson Method and less numbers of iteration as compare to the bisection method. Newtons-Raphson Method is the fastest method to solve the root of non-linear equation $f(x) = 0$ as it takes less numbers of iteration as compared to bisection method and Regula falsi method.

In overall three iterative methods the researcher has found that the Newtons-Raphson Method is the best and fastest approximation method for solving the root of non-linear equation f(x) = 0.

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