IJCRT.ORG ISSN : 2320-2882
INTERNATIONAL JOURNAL OF CREATIVE RESEARCH THOUGHTS (IJCRT)
An International Open Access, Peer-reviewed, Refereed Journal

BASIC COMMUTATIVE ALGEBRA

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Abstract
The purpose of this work is to build a graded algebra A = A(q1, q2, q3) with three shift parameters, q1, q2, and q3. By introducing a specific filtration connected with the dominance ordering among partitions, we verify the basic features of the algebra A, including commutativity and the Poincar e series. The Gordon filtration is a stratification that is characterised by a series of null conditions related with the partitions and the shift parameter qi. The elliptic algebra [EO] can be thought of as a smooth limit of our algebra. Specifically, the original algebra is built over an elliptic curve, whereas our algebra A is built over a degenerate CP1.

Keyword: Algebra, Poincar. Elliptic Algebra

1. Introduction

POLYNOMIAL RING

If R is a ring, the ring of polynomials in x with coefficients in R is denoted R[x] . It consists of all formal sums.

$$\sum_{i=0}^{\infty} a_i x^i$$

That is $R[x] = \sum_{i=0}^{i=n} a_i x^i$

QUOTIENT RING

Let R be a ring and I be an ideal of R.Then R/I is form a ring .that ring is called quotient ring .

EXAMPLE: Let K[x] is a polynomial ring and its quotient rings are

$k[x]= < x_2 + x + 1 >, k[x]= < x_3 + 2 >$ etc.

PRIME IDEAL

An ideal P is said to be prime ideal if ab 2 P then a 2 p or b 2 P.

EXAMPLE: $< x_2 + 1>$ is an prime ideal of R[x].

NOTE:

Let R be a ring and I be an ideal of R.Then R/I is said to be integral domain if and only if I is prime ideal.

MAXIMAL IDEAL

Let R be a commutative ring and ideal M of R is said to be maximal ideal of R, if there exist an ideal N of R such that M N _ R then

M=N or N=R.

EXAMPLE:

Let K[x] be a ring and $< x_2 + 1 >$ its maximal ideal.

NOTE:

1. R be a commutative ring with unity.an ideal M of R is maximal ideal of R if and only if R=M is a field.
2. Every maximal ideal is a prime ideal.

NILRADICAL

The set N of all Nilpotent element in a ring R is an ideal and R/N has no Nilpotent element (except zero) .

The Nilradical of a ring R is the intersection of all prime ideal of R.

EXAMPLE:

Let Z be a ring and pZ be its prime ideal (where p is prime) then Nilradical of this is 0.

JACOBSON RADICAL

The Jacobson radical of J of a ring R is define to be the intersection of all maximal ideals of R.

COLON IDEAL

Let R be a commutative ring. Let S be a subset of R and I be an ideal of R.Now we define the subset (I : S) = {a $\epsilon R/$ aS I} and the (I:S) is an ideal of R.This ideal is called the colon ideal or ideal quotient.

EXAMPLE: R = K[x]= < x_2 >

EXTENSION AND CONTRACTION

Let f : A→B be a ring homomorphism .If I be an ideal of A, define the extension I^e of I to be I^e =< f(I) > ideal generated in B .That is ,

I^e ={$\sum_{i=0}^{i=n} y-i\, f(x-i)/n \geq 1$, y-i ∈B, x-i ∀i }

let an ideal of B then f^{-1}(J) is an ideal of A is called the contration J^c of J .That is ,

J^c= f^{-1}(J) = {x/ f(x) ∈ B}

MODULES

DEFINITION: Let A be a commutative ring.An A module M is an abelian group written additively with scalar multiplication and a mapping f : A ×M → M with following properties

a(x+y)=ax+ay

(a+b)x=ax+bX

(ab)x=a(bx)

1x=x

where a; b ∈ A ,x ,y ∈ M

EXAMPLE:

1. An ideal I of a ring A is an A-modules.
2. If A is a _eld k=R, then A-modules=K vector space.

HOMOMORPHISM

Let M,N be A-module,A mapping f : M → N is an A-modules homomorphism if,

f(x+y)=f(x)+f(y)

f(ax)=af(x)

where for all a ∈ A and all x; y∈M.

The set of all A-module homomorphism from M to N is also A-module follow:

we define, f+g and af by the rule

(f+g)x=f(x)+g(x)

(af)x=af(x)

this is also A-module and is denoted by Hom_A(M;N).

Homomorphism u : M′ → M and v : N′ →N induce mapping u′:*Hom* (M,N)→ *Hom*(M′,N) and v′ : *Hom* (M,N)→*Hom* (M,N′)

define as follow,

u′(f) = f_0u,v′(f) = v_0f these module are A-module homomorphism.

SUB-MODULES AND QUOTIENT MODULES

An sub-module N of M is a subgroup of M which is closed under the multiplicaton by element of A.That is x:n ∈ N for all x ∈ A and n ∈ N.

EXAMPLE: A be a ring and itself is a A-modules and its ideal is sub-modules.

The Abelian group M/N gives an A-modules structure from M define by a(x+N)=ax+N. The module M/N is quotient of M by N.

1. The kernel of f is the set Ker(f) = x 2 M : f(x) = 0 is sub-module of M.
2. The image of f is the set im(f)=f(M) is a sub-module of N.
3. The coker of f is coker(f)=N/im(f)

ANNIHILATOR

If N,P are sub-module of M,we de_ne (N:M) to be the set of all a such that aP _ N it is an ideal of A .In particular (0:M) is the set of all a 2 A such that aM=0,this ideal is called the annihilator of M and is also denoted by Ann(M). An A-module is faithful if Ann(M)=0.If Ann(M)=a then M is faithful as an A/a module.

DIRECT SUM AND DIRECT PRODUCT

If M,N are A-module their direct sum M _ N is the set of all pairs
(x,y) with x ∈ M; y∈ N.
$(x_1; y_1) + (x_2; y_2) = (x_1 + x_2; y_1 + y_2)$
a(x; y) = (ax; ay)

If $(M_i)_{i\in I}$ is any family of A-module,we can de_ne the direct sum (M_i) its element are families $(x_i)_{i\in I}$ such that (x_i) 2 (M_i) for each i ∈ I and at-most all (x_i) are zero. If we remove on the number of non zero X's we have the direct Product .

CO-MAXIMAL

Let R be a ring ,ideal A and B are said to be co-maximal if A+B=R.
EXAMPLE: Let Z be a ring and I=2Z and J=3Z be two co-maximal ideal.

2. Chinese Remainder Theorem

Theorem 1. Let $A_1;A_2;A_3......A_k$ be an ideals in R.The mapping $R\to R/A_1 \times R/A_2 \times ....R/A_k$ define by, $r\to(r + A_1, r + A_2,.........r + A_k)$ is a ring homomorphism with kernel $A_1 \cap A_2 \cap ...._k$. If for each i; j ∈ {1; 2; 3..... k} with $i\neq j$
the ideals A_i and A_j are co-maximal, then this map is Surjective and $A_1 \cap A_2 \cap ......_k = A_1A_2......A_k$, so $R/(A_1A_2......A_k) = R/(A_1\cap A_2 \cap ...._k) \cong R/A_1/A_2 \times....../A_k$

Proof. for k=2
We first prove this for k = 2; the general case will follow by induction.
Let $A = A_1$ and $B = A_2$.
Consider the map $f :R \to R/A\times R/B$.
defined by f(r) = (r modA; r modB), where mod A means the class in R/A containing r (that is, r+ A). when A and B are co-maximal,
f is surjective and A∩B = AB.
Since A + B = R, there are elements x∈A and y∈B s.t. x+y =
1. This equation show that f (x) = (0,1) and f (y) =(1,0).
since, for example, x is an element of A and x = 1- y ∈1 + B. If now ($r_1$modA; $r_2$modB) is an arbitrary element in R/A*R/B, then element $r_2x + r_1y$ maps to this to element.
Since $f(r_2x + r_1y) = f(r_2)f(x) + f(r_1)f(y)$
$=(r_2modA; r_2modB)(0; 1) + (r_1modA; r_1modB)(1; 0)$
$=(0; r_2modB) + (r_1modA; 0)$
$=(r_1modA; r_2modB)$.

This shows that f is indeed surjective. Finally, the ideal AB is always contained in A∩B. If A and B are comaximal and x and y are as above, then for any $c \in A \cap B$; $c = c_1 = cx + cy \in AB$.
The general case follows easily by induction from the case of two ideals using $A = A_1$ and $B = A_2.......A_K$ once we show that A_1 and $A_2........A_k$ are co-maximal .By hypothesis for each i ∈ {2, 3, 4 ......k} there are elements $x_i \in A_1$ and $y_i \in A_i$ s.t.
$x_i + y_i = 1$. Since $x_i+y_i \equiv y_i \bmod A_1$, it follows that $1 = (x_2+y_2)................(xk+yk)$
is an element in $A_1 + (A_2...........A_k)$.

3. Hilbert Basis Theorem

We First describe some general Finiteness condition. Let R be a ring and let M be a left R-module.
Definition:

1.The left R-module M is said to be a Noetherian R-module or to satisfy the ascending chain condition on submodules (or A.C.C on submodules) if there are no infinite increasing chain of submodules is an increasing chain of submodules of M,then there is a positive integer m such that for all k≥m;$M_k = M_m$ (of the chain becomes stationary at stage

m: $M_m = M_{m+1} = M_{m+2} = ...$

2. The ring R is said to be Noetherian if it is Noetherian as a left module over itself, i.e if there is no infinite increasing chains of left ideals in R.

EXAMPLE:

Any field is a Noetherian ring.

Any Principal ideal domain is also a Notherian ring. So, the integers, considered as a module over the ring of integers, is a Noetheria module.

Theorem 2.

Let R be a ring and let M be a left R-module. then the following are equivalent,

(1) M is a Noetherian R-module.

(2) Every nonempty set of submodules of M contain a maximal element under inclusion.

(3) Every submodule of M is finitely generated.

Proof. First we proof that (1) ⇒ (2)

Assume M is Noetherian and let P be any nonempty collection of submodules of M.

choose any $M_1 \in \sum$.

If M_1 is a maximal element of $\sum$.(2) holds,so assume M_1 is not maximal.

Then there is some $M_2 \in \sum$. such that $M_1 \subset M_2$.

If M_2 is maximal in $\sum$.(2) holds, so we may assume there is an $M_3 \in \sum$. properly containing M_2.

proceeding in this way one see that if (2) fails we can produce by axiom of choice an infinite strictly increasing chain of elements of $\sum$.contray to (1).

Now we proof (2) ⇒ (3)

Assume (2) holds and let N be any submodules of M.

Let $\sum$. be the collection of all finitely generated submoules of N.Since $0 \in \sum$. this collection is nonempty.

By (2) $\sum$. contains a maximal element N′.

If N N′ ,Let x N N′.

since $N' \in \sum$, the submodule N′ is finitely geneated by assumption,hence also the submodule generated N′ and x is finitely generated.

This contradicts the maximality of N′,so N = N′ is finitely generated.

Now we proof that 3 ⇒ 1

Assume (3) holds and let M_1 _ M_2 _ M_3........be a chain of submodules of M.

Let N =$\bigcup_{i=1}^{\infty} M_i$ and note that N is a submodule.

By (3) N is finitely generated by ,say $a_1, a_2, ...a_n$. since $a_i \in N$ for all i,each a_i lies in one of the submodule in the chain,say $M_{j,i}$

Let m = max {$j_1, j_2, j_3 ....j_n$}.

Then $a_i \in M_m$ for all i so the module they generate is contained in M_m i.e, N _ M_m.

This implies $M_m = N = M_k$ for all k ≥ m, which proof 1.

Corollary 1. R is a Noetherian ring if and only if every ideal of R is finitely generated.

Corollary 2. If R is a P.I.D, then every nonempty set of ideal of R has a maximal element and R is a Noetherian ring.

Proof. The P.I.D ,R satisfies condition of the above theorem with M=R.

Recall that even if M itself is a finitely generated R-module, submodule of M need not be finitely generated,so the condition that M be a noetherian R-module is ingeneral stronger than the condition that M be a finitely generated R-module.

Proposition Let R be an integral domain and Let M be a free R-module of rank

n < 1.Then any n+1 element of M are R linearly dependent i.e for any $y_1,y_2,.....y_n + 1$ ∈M there are elements $r_1,r_2........r_{n+1}$ 2 R not all zero,such that $r_1y_1 + r_2y_2 + ...+ r_{n+1}y_{n+1} = 0$

Proof. The quickest way of proving this is to embed R in its quotient field F.

since R is an integral domain and observe that since M ≅R ⊕R ⊕..... R (n times).

the latter is an n-dimensioal vector space over F,so any n+1 element of M are F linearly dependent.

By clearing the denominators of the scalar ,we obtain an R-linear dependence relation among the n+1 elements of M.

If R is any integral domain and M is any R-module recall that

Torsion (M)={x ∈ M/rx= 0 for some nonzero r ∈ R}

Theorem . R is a Noetherian if and only if every prime ideal is finite generated.

proof

Assume R is Northerian .

T ={ I is an ideal of R which is NOT finitely generated}

By assumption T $\neq \varphi$

Now we used zorn's lemma to show that T has a maximal element.

Let {I∝ be a chain ,

Let I=U Iα.

Then I is an ideal.

we claim I is not Finite generated becaue if it is cheak,then

I = $Rx_1 + Rx_2 + ...Rx_n$ for x_i ∈ I αi

Set r = max α_j , where j=1,2,......n

where {α1,.........,αj} = totally ordered finite set.

x_i∈ I_r for all i

By zorn's lemma T as a maximal element say P.

we will show P is prime which will be a contradiction .

suppose ∃x∈ R/P; y ∈ R/P with xy∈ P.

Look at (x,P) is not proper superset of P and P:x is not prpper super set of P.

By maximality of P,(p,x),(P:x) are finitely generated

=> (P; x) = $Rx_1 + Rx_2 + ... + Rx_n + ...:Rx$

Assume $x_1; x_2;$ .... ∈P.

But P = $Rx_1 + Rx_2 + ...+ Rx_n +(P : x)x$

because if z P∈

=> Z ∈ (P; x) = $Rx_1 + Rx_2 + ...Rx_n + Rx$

=> z =$\lambda_1x_1 + \lambda_2x_2 + ...\lambda_nx_n + \lambda x$

=> λx ∈P

=> λ∈ (P : x).

4. Hilbert Basis Theorem

If R is a Noetherian Ring ,then the polynomial ring $R[x_1; x_2,.........x_n]$ is Noetherian.

Proof.

we may assume that n=1,

we show that R is Noetherian

=> R[x] is Noetherian.

I ⊂ R[x] be an R[x]-ideal.

Set

I_n = {a∈ R∃f∈(x) ∈ I with degree n and leading coefficient a }U{0}.

Then I_n is an R-ideal.

(For all a; b ∈ I_{n}∃ two polynomial f; g of order n such that the leading coe_cient of f; g are a; b respectively .

=> f –g is also a polynomial of degree n and leading coefcient of f - g in a - b

=> a -b ∈I_n

Thereforr(6≠ 0) ∈R; a ∈ I_n , rf is a polynomial of degree n and leading coeffcient is ra .

=> ra ∈I_n

I_n is an R-ideal.)

(∀ a ∈ I_j ∃ a polynomial f of degree j such that leading coeffcient of f is a .Now consider xf. This is a polynomial of degree j + 1 and leading coeffcient is a. This implies a ∈I_{j+1})

Since R is Noetherian.

Let f_{ni} ; $1 \leq i \leq l_n$; $0 \leq n \leq r$ be polynomial such that degf_{ni} = n and leading coe_cient of f_{ni} ; $1 \leq i \leq l_n$ generated I_n.

claim:{ f_{ni} ; $1 \leq i \leq l_n$; $0 \leq n \leq r$} = W generates I

By definition the leading coeffcient of f_{ni} generates I_n; ∀i.

Now let g(x) ∈I; g ≠ 0. By induction on deg g,

We show g ∈(W)R[x].

If degree of g equals 0 then g ∈I UR CI_0 = Rf_{01} + _ _ _ + Rf_{0l0}

Assume by induction that the result is true for all deg ≤ n -1.

Now let deg g = n. Since the leading coeffcient of g is I_n; ∃λi ∈R with deg(g-ΣλiFn-1)≤ $n-1$

By induction we know Fn-1∈ $(W)T[x]$.

REFERENCE

1- Yengui ,making the use of maximal ideals constructive.Theoretical computer Science.392(2008)
2- Yengui,The Hermite ring conjugate in dimension one.Journal of Algebra.320(2008).
3- Swan R.Factorization of Polynomials over Finite Fields,pacific Journal of Mathematics.12(3)(1962).
4- Coquand T,Lombardi H.Some remarks on normal ring,dans [CPMPCS].