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STUDY OF Z-TRANSFORM

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Abstract:

This paper presents a significant of Z-transform. It concerns with the review of what z-transform plays in the analysis of discrete signal in accordance with Laplace transform plays in the analysis of continuous signal and here we also represent the Region of convergence & its examples, and describe inverse Z-transform. We have also given emphasis on the relationship of Z-transform and Laplace transform. This study lays stress on the properties of Z-transform and its real-life applications.

Keyword – Z-transform, Laplace transform, Region of convergence (ROC), property, Discrete time signal, Continuous time signal

1. Introduction:

The Z-transform applied in discrete mathematics. In 1730 De-moivre states that the mode of generating functions. Then in 1947 the transformation of sampled signal or sequence was interpreted by W. Hurewicz. And it is about to solve linear constant co-efficient difference equation. In 1952, This transformation 'z transform' named by 'Ragazzini and Zadeh' in the sampled data control group at Columbia University.

The Z-transform (ZT) occupies a significant portion in digital signal processing (DSP). It processed in digital data or signals. Laplace transform (LT) and Fourier transform (FT) apply in continuous signals, while it takes to analyze discrete signal. The discrete form of LT is the part of ZT. The ZT is indicated as Z[f(m)] or designate F(z), where f(m) be the function and Z is the operator. It applies to explain difference equation. Here we define Z-transform, and described region of convergence in Z-plane, and its properties. Derive ZT and its relation to Laplace, at the end we write in real life applications.

Definition:

Z-transform (ZT) is of two distinct forms, unilateral and bilateral. The bilateral or two-way ZT mainly applied in signal processing, and unilateral and one-sided ZT gives to analysis of discrete systems.

1. BZT is written as

$$F(z) = Z\,[f(m))] = \sum_{n=-\infty}^{\infty} f(m)z^{-m} \quad (1)$$

2. UZT is written as

$$F(z) = Z\,[f(m))] = \sum_{m=0}^{\infty} f(m)z^{-m} \quad (2)$$

Where 'z' is a complex variable

2. Mode of Region of Convergence

Region of convergence has indicated as ROC. It is the area or region for which ZT exists or converges. ROC of ZT designated in the z-plane of the unit circle. ROC is not containing at the pole. Because at the pole F(z) is infinite, then it cannot be convergence. But ROC is finite or convergence. Then

the general form is

$$F(z) = \sum_{m=0}^{\infty} f(m)z^{-m} \quad (3)$$

Let the series of complete convergence is $\sum_{m=0}^{\infty} a_m$

then we get $a_m = f(m)z^{-m}$ (4)

here we use ratio and root test to find the value of the series z which is convergence.

Applying the ratio test only for a convergence series

$$\lim_{m\to\infty} \left|\frac{a_{m+1}}{a_m}\right| = A \quad (5)$$

And this series otherwise called d'Alembert ratio test or Cauchy ratio test

if A<1 then it is convergence and if A>1 then it is divergence & A=1 may convergence or divergence.

The root test series is

$$\lim_{m\to\infty} \sqrt[m]{|a_m|} = A \quad (6)$$

the series converges. if A<1,

$$\lim_{m\to\infty} \sqrt[m]{|a_m|} < 1 \quad (7)$$

The series diverges if A >1,

$$\lim_{n\to\infty} \sqrt[m]{|a_m|} > 1 \quad (8)$$

And it can converge and diverge if A=1.

$$\lim_{n\to\infty} \sqrt[m]{|a_m|} = 1$$

Compare the Z-transform of sequence Eq (3) in root test series Eq (6) becomes

$$\lim_{n\to\infty} \sqrt[m]{|f(m)z^{-m}|} < 1 \qquad \text{(since } a_m = f(m)z^{-m}\text{)}$$

$$= \lim_{n\to\infty} (|f(m)z^{-m}|)^{\frac{1}{m}} < 1$$

$$= \lim_{n\to\infty} (|\,|f(m)|\,|(z^{-1})^m|)^{\frac{1}{m}} < 1$$

$$= \lim_{n\to\infty} (|f(m)|)^{\frac{1}{m}} |z^{-1}| < 1$$

$$= \lim_{n\to\infty} (|f(m)|)^{\frac{1}{m}} < |z|$$

$$= \lim_{n\to\infty} \sqrt[m]{|f(m)|} < |z| = R \qquad (9)$$

Where R is a series of radius Convergence. Therefore, the series will have absolute convergence for all points in the origin. This area is known as Region of Convergence [ROC].

3. S-plane, Z-plane and the unit circle:

S-plane is the complex plane of LT and Z-plane is the complex plane of ZT. The S and Z plane are visualized two-dimensional plane. The LT $s = \sigma + J\omega$. σ is the real axis and $J\omega$ is the imaginary axis in S-plane. and the Z.T $z=re^{J\omega}$ are complex variables.

$z = e^{s} = e^{\sigma+J\omega} = e^{\sigma} \cdot e^{J\omega} = re^{J\omega}$ here $r = e^{\sigma}$ is the real part and $e^{J\omega}$is the imaginary part

In the S-plane the vertical $J\omega$-axis is the frequency axis, and the horizontal σ –axis decay rate, or growth rate and amplitude of the sinusoid (sin, cos)

- At the origin $s = 0$ in s-plane is mapped into $z = e^{0} = 1$ on the real axis in Z-plane.
- In the vertical axis Re(s) in S-plane is mapped into a circle $|z|$ at the origin in z-plane.
  - In left of the vertical axis $Re(s) = \sigma = -\infty$ is mapped to the circle $|z| = e^{-\infty} = 0$ centered at the origin
  - In right of the vertical axis $Re(s) = \sigma = \infty$ is mapped to the circle $|z| = e^{\infty} = \infty$.

In fig (1) the horizontal dashes lines in the s-plane. And the periodic processes Fig (1) represented a circle in the z-plane in Fig (2), it associated unit circle. Let imagine in s-plane the jw axis is twisted in sampled signal. The direction of left side of s-plane turned into a circle. This unit circle is called z-plane. for $\sigma < 0$ then $r = e^{\sigma}<1$, is convergence in the left side area of s-plane. Because it is mapping inside the unit circle. Then the region is stable. In the right-side area of the s-plane, for $\sigma > 0$ then $r = e^{\sigma}>1$, is divergence. Because it mapped out-side the unit circle. Then the reign is unstable. In the origin of jw-axis, for $\sigma = 0$ then $r = e^{\sigma}=1$, is on the line of unit circle.

Hence S-plane is mapping into Cartesian co-ordinate for CTS and Z-plane map into a polarization method for DTS.

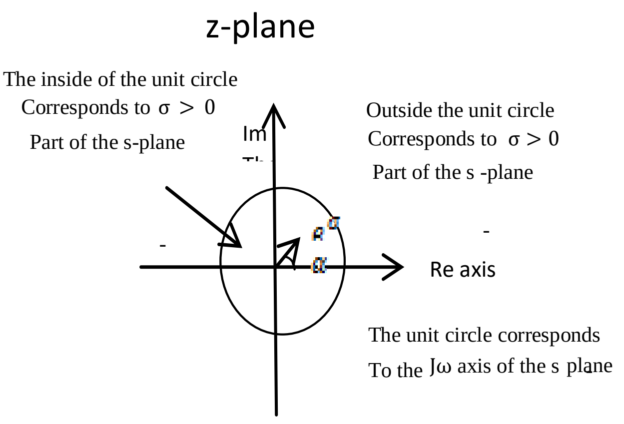

Fig (2)

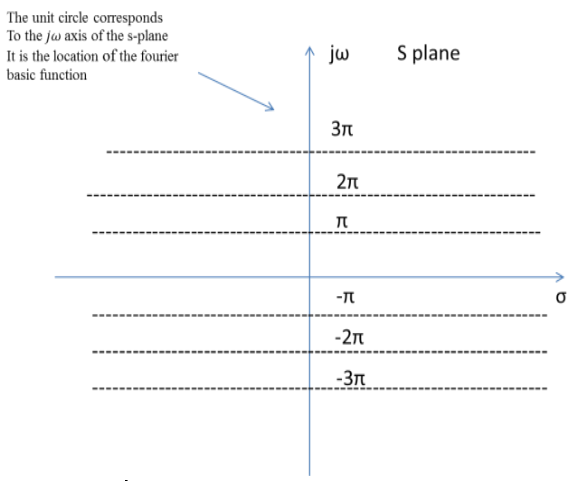

Fig (1)

4. Properties of Z-Transform:

The important properties of ZT are given below. Some properties are applied in bilateral and some properties are applied in unilateral.

1. Linearity property:

The linear property is also known as linear transformation

Let R_1, R_2 are

A and b are two scalars and f_1 & f_2 are the sampled sequence

Then $Z[af_1(m) + bf_2(m)] = \ = a\,[F_1(z)] + b\,[F_2(z)]$

Proof: $Z[af_1(m) + bf_2(m)]$

$= Z[a \sum_{n=0}^{\infty} f_1(m)\, z^{-m}) + b \sum_{m=0}^{\infty} f_2(m)\, z^{-m}]$

$= Z[a \sum_{m=0}^{\infty} f_1(m)\, z^{-m})] + Z[b \sum_{n=0}^{\infty} f_2(m)\, z^{-m}]$

$= a\,[F_1(z)] + b\,[F_2(z)] \quad$ for $|z| > \max\{R_1, R_2\}$

Here $ROC = R_1 \cap R_2$ and $R_1 \& R_2$ are two different transforms

2. Time Shifting property:

Let R is the ROC of Z(f(m)),

if k is any integer,

(1) Right shifting property:

ZT of right shifting property states that If $k > 0$, then $Z\,[f(m-k)] = z^{-k} Z[\,f(m)]$.

Proof: By the definition of ZT

$Z\,[f(m-k)] = \sum_{m=0}^{\infty} f(m-k)\, z^{-m} \qquad (m+k = r)$

$= \sum_{r+k=0}^{\infty} f(r)\ z^{-(r+k)}$

$= \sum_{r=-k}^{\infty} f(r)\ z^{-r} z^{-k}$

$= z^{-k} \sum_{r=k}^{\infty} f(r)\ z^{-r}$

$= z^{-k} \sum_{r=-k}^{\infty} f(r)\ z^{-r}$

$= z^{-k} \sum_{r=-k}^{\infty} f(r)\ z^{-r}$

(2) Left shifting property:

ZT of left shifting property states that If $k > 0$,

then $Z\,[f(m+k)] = z^k\, Z\, f(m) - \sum_{r=0}^{k-1} f(r)\ z^{k-r} \quad |z| > R$

Proof

$Z\,[f(m+k)] = \sum_{m=0}^{\infty} f(m+k)\, z^{-m}$ (m+k = r)

$= \sum_{r-k=0}^{\infty} f(r)\ z^{-(r-k)}$

$= \sum_{r=k}^{\infty} f(r)\ z^{-r} z^{k}$ ROC, except $|z| = 0$ if k>0 and $|z| = \infty$

$= z^{k} \sum_{r=k}^{\infty} f(r)\ z^{-r}$

$= z^{k} \sum_{r=0}^{\infty} f(r)\ z^{-r} - z^{k} \sum_{r=0}^{k-1} f(r)\ z^{-r}$

$= z^{k}[\sum_{r=0}^{\infty} f(r)\ z^{-r} - \sum_{r=0}^{k-1} f(r)\ z^{-r}]$

$= z^{k}[F(z) - \sum_{r=0}^{k-1} f(r)\ z^{-r}]$

In ROC, R is lying except z=0, z= ∞

3. Time scaling or exponential property:

The time scaling property of ZT is

$Z\,[a^{m} f(m)] = F(a^{-1}z) = F(z/a)$

Proof:

$Z\,[a^{m} f(m)] = \sum_{m=-\infty}^{\infty} a^{m} f(m) z^{-m}$

$= \sum_{m=-\infty}^{\infty} f(m)(a^{-1}z)^{-m}$

$= F(a^{-1}z) = F(z/a)$

ROC: $r < |a^{-1}z| < R$

4. Reversal property:

$Z[f(-m)] = F(z^{-1})$

proof

$Z[f(-m)] = \sum_{m=-\infty}^{\infty} f(-m) z^{-m}$ (-m = k)

$= \sum_{k=-\infty}^{\infty} f(k) z^{k}$

$= \sum_{k=-\infty}^{\infty} f(k)(z^{-1})^{-k} = F(z^{-1})$

ROC: z^{-1} in R_1 and ROC= $1/R_1 < |Z|$

5. Conjugation:

$Z[f^*(m)] = F^*(z^*)$

The conjugation of f(m) is f*(m)

ROC: R_1

Proof:

$Z[f^*(m)] = \sum_{m=-\infty}^{\infty} f*(m) z^{-m}$

$= \sum_{m=-\infty}^{\infty} [f(m) z*^{-m}]*$

$= [\sum_{m=-\infty}^{\infty} f(m) z*^{-m}]*$

$= F^*(z^*)$

6. Accumulation:

equivalent to the integral time -∞ to t and t→ i

$$\sum_{r=0}^{k-1} f(k) = \frac{1}{1-z^{-1}} F(z)$$

Differentiation:

Let Z(f(m)) = F(m)

The differentiation of ZT is

$$m^k f(m) = (-1)^k z^k \frac{d^k F(z)}{dz^k} \quad \text{for } k = 1,2,3, \ldots\ldots\ldots\ldots$$

For m=1 $mf(m) = -z \frac{d\,F(z)}{dz}$ is 1st derivative

For m=2 $m^2 f(m) = (-1)^2 z^2 \frac{d^2 F(z)}{dz^2} = z^2 \frac{d^2 F(z)}{dz^2}$

ROC: R_1

7. **Convolutional property or Convolutional theorem:**

This property is the most important property of ZT.

Here $*$ is the convolutional operator. $f_1(m)$ & $f_2(m)$ are two functions,

Then $Z[f_1(m) * f_2(m)] = F_1(z)\, F_2(z)$

$$Z\,(f_1(m) * f_2(m)) = Z\,[\sum_{p=-\infty}^{\infty} f_1(p) . f_2(m-p)]$$

$$= \sum_{m=-\infty}^{\infty} (\sum_{p=\infty}^{\infty} f_1(p) . f_2(m-p)\,)\, z^{-m}$$

$$= \sum_{m=-\infty}^{\infty} f_1(p) .\ (\sum_{p=-\infty}^{\infty} f_2(m-p)\, z^{-p} z^{-m}$$

$$= \sum_{m=-\infty}^{\infty} f_1(p) .\ [\sum_{p=-\infty}^{\infty} f_2(m-p)\, z^{-m}\,] z^{-p}$$

$$= \sum_{m=-\infty}^{\infty} f_1(p) .\ F_2(m)\, z^{-p}$$

$$= F_2(z) \sum_{m=-\infty}^{\infty} f_1(p) . z^{-p}$$

$$= F_2(z)\ .\ F_1(z)$$

This is the convolutional method of ZT.

ROC is at least $R_1 \cap R_2$

8. Correlation property:

$$Z[r_{f_1 f_2}(m)] = R_{f_1 f_2}(z)$$

Proof:

$$Z[r_{f_1 f_2}(m)]$$

$$Z\,[f_1(-m) * f_2(m)]$$

By applying convolution and reversal properties, we get

$$= Z\,[f_1(-m) * f_2(m)]$$

$= \sum_{m=0}^{\infty} f_1(-m)\, f_2(m)]\, z^{-m}$

$= F_1\,(1/z)\, F_2(z)$

9. **Initial value theorem**:

If f(m) is casual, then F (0) = $\lim_{z\to\infty} F(z)$

Proof:

by definition of Z-transform

$F(z) = \sum_{m=0}^{\infty} f(m) z^{-m} = f(0) + \frac{f(1)}{z} + \frac{f(2)}{z^2} + \cdots \ldots \ldots$

If m=0, f (0) = 0 then F (0) = $\lim_{z\to\infty} F(z)$

10. **Final value theorem**:

If a signal is characterized by Z-transform as F(z) and the poles are all inside the circle, then its ultimate value signified f(m) or F (∞) as it can be written as

$F(\infty) = \lim_{z\to 1} F(z)(z\text{-}1)$

If the rules of (z-1) F(z) are inside the unit circle

Then $\lim_{z\to 1} F(z)(z\text{-}1)$

It can be applying the property of unilateral Z-transform.

5. Inverse Z-transform:

The inverse Z-transform is the reverse or conjugate of ZT. There are three distinct procedures of inverse ZT.

1] Integral method
2] power series expansion
3] partial-fraction

1) Integration method:

The Cauchy integral is states that $\frac{1}{2\pi i} \oint_C z^{k-1}\, dz = \begin{cases} 1 & \text{if } k = 0 \\ 0 & \text{if } k \neq 0 \end{cases}$

Where c is the path at the origin lying in the ROC.

By definition of z-transform, we have

$F(z) = \sum_{n=-\infty}^{\infty} f(m) z^{-m}$

Now multiplying integral and z^{k-1}on both sides

$$\oint_C F(z)\, z^{k-1}\, dz = \oint_C \sum_{n=-\infty}^{\infty} f(m) z^{-m} z^{k-1}\, dz$$

$$= \oint_C \sum_{n=-\infty}^{\infty} f(m)\, z^{-m+k-1}\, dz$$

$$= \sum_{n=-\infty}^{\infty} f(m) \oint_C z^{-m+k-1}\, dz$$

from the Cauchy integral, we get

$$\oint_C F(z) z^{m-1} = 2\pi i\, f(m)$$

$$f(m) = \frac{1}{2\pi i} \oint_C F(z) z^{m-1} dz$$

where c is counter clockwise of ROC, the residue of F(z) z^{m-1} at the pole inside C

2) Power series method:

This method can be found to be the inverse form of ZT. The idea of power series of the form

$$F(z) = \sum_{m=0}^{\infty} f_1(m)\, z^{-m}$$

Which is convergence in ROC,

Example:

Find out the inverse of ZT

$$F(z)= \cos z, \ |z|>0$$

Sol:

The Taylor series expansion of cos z= $\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)} z^{2n}$

We know the general form $\sum_{m=0}^{\infty} f_1(m)\, z^{-m}$

So, $\sum_{m=0}^{\infty} f_1(m)\, z^{-m} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)} z^{2n}$

Comparing the power of z on both sides becomes

$$-m= 2n \quad \text{or } n=\frac{-m}{2}$$

Where -m is even then -m=-2,-4,-6,…….

Thus x(m) will be written as

$$X(m)= \frac{(-1)^n}{(2n)} = \frac{(-1)^{\frac{-m}{2}}}{(2\frac{-m}{2})} = \frac{(-1)^{\frac{-m}{2}}}{(m)}, \quad m=-2, -4, -6\ldots\ldots$$

3) Partial fraction method:

If F(z) is rational function. The way of partial fraction finds out the inverse of ZT. Now it is written as

$F(z)= a_1 F_1(z) + a_2F_2(z) + \cdots \ldots + a_kF_k(z)$

Where each $F_i(z)$ has inverse ZT of $f_i(m)$, for i=1,2, 3….

If F(z) is rational of z then F(z) can be written as

$$F(z)=\frac{A(z)}{B(z)} = \frac{\sum_{m=0}^{\infty} a_i(m)\, z^{-m}}{\sum_{m=0}^{\infty} b_i(m)\, z^{-m}}$$

Example:

Find the inverse of ZT

$$F(z)=\frac{z^2+3z}{z^2-3z+2} \text{ if ROC is } |z|<2$$

Sol,

The partial fraction written as

$$\frac{F(z)}{z}=\frac{z+3}{z^2-3z+2}=\frac{z+3}{(z-2)(z-1)}$$

$$=\frac{A}{(z-2)}+\frac{B}{(z-1)}$$

$$\frac{A}{(z-2)} = \frac{F(z)}{z}\Big|_{z=2}$$

$$A = (z-2)\frac{F(z)}{z}\Big|_{z=2}=5$$

And $\frac{B}{(z-1)} = \frac{F(z)}{z}\Big|_{z=1}$

$$B = (z-1)\frac{F(z)}{z}\Big|_{z=1} = -4$$

So, $F(z) = 5\frac{z}{(z-2)} - 4\frac{z}{(z-1)}$

Since the ROC of F(z) is |z|<2 here the sequence is anti-casual.

so, $f(m) = (-5(2)^m + 4)\, u(-m-1))$

6. Uses and Applications:

The ZT is used in DSP, in population science, for control theory, applied in pharmacokinetics.

It is mainly used in DSP, the discrete time domain signal transform into discrete frequency domain signal. It has the long range in application of mathematics and DSP. And it is mainly used to analyze the data of the DSP.

In the applications of DSP are of three different classes

1st one is low-cost high volume surrounded system for examples modems and cellular phones and 2nd is computer related systems like audio or video compressor and music synthesis. 3rd is the high frequency performance of large volume of complex data like Radar, sonar, seismic imaging, speech recognition. It is

In the microphone the input signal write head amplified into read head then the output signal amplified audio signal to the speaker which is produced sound wave. Here the electromagnetic wave converted by the microphone

The applications of DSP in Z transform are

- To analysis of digital filters
- For automatic controlling telecommunication
- It helps the system design and check the stability of the system
- Examine the linear discrete system
- Analysis of discrete signal.

The applications of DSP in our real life, that includes speech processing, image processing, Radar signal processing, Digital communications, digital communications, Sonar signal processing, spectral analysis.

7. Conclusion:

This paper is deliberated about ZT and its applications. In this paper here find out the ROC in different form. The inverse ZT are used in three methods, integration method finding the value of x(n), where power series finding discrete order power function and partial fraction finding the ZT. The basic uses are the transform of discrete time signal into frequency domain. The comparison of Laplace and Z-transform also explained in statements. And it is also containing explain the properties of Z-transform. And it is clear that Z-transform is used in applied mathematics and here we find the standard formulation of ZT. the work has thoroughly focused on z-transform & the good looks of z-transform like Laplace transform. The relationship between mathematical tools (Laplace and Z-transform) established.

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