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Analysis of middle cube graphs and their Spectra based on Mathematical techniques

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Abstract: In this research paper we begin with study of family of n dimensional hyper cube graphs and establish some properties related to their distance, spectra, and multiplicities and associated eigen vectors and extend to bipartite double graphs .

In a more involved way since no complete characterization was available with experiential results in several inter connection networks on this spectra our work will add an element to existing theory. In this paper we analyzed middle cube graphs and their Spectra based on Mathematical techniques.

Keywords: *Middle cube graphs, distance-regular graph, antipodal graph, bipartite double graph, extended bipartite double graph, Eigen values, Spectrum, Adjacency Matrix.*

1) Introduction: an n dimensional hyper cube Q_n [24] also called n-cube is an n dimensional analogue of Square and a Cube. It is closed compact convex figure whose 1-skelton consists of groups of opposite parallel line segments aligned in each of spaces dimensions, perpendicular to each other and of same length.

A point is a hypercube of dimension zero. If one moves this point one unit length, it will sweep out a line segment, which is the measure polytope of dimension one. If one moves this line segment its length in a perpendicular direction from itself; it sweeps out a two-dimensional square.

If one moves the square one unit length in the direction perpendicular to the plane it lies on, it will generate a three-dimensional cube. This can be generalized to any number of dimensions. For example, if one moves the cube one unit length into the fourth dimension, it generates a 4-dimensional measure polytope or tesseract.

The family of hypercubes is one of the few regular polytopes that are represented in any number of dimensions. The dual polytope of a hypercube is called a cross-polytope.

A hypercube of dimension n has $2n$ "sides" (a 1-dimensional line has 2 end points; a 2-dimensional square has 4 sides or edges; a 3-dimensional cube has 6 faces; a 4-dimensional tesseract has 8 cells). The number of vertices (points) of a hypercube is 2^n (a cube has 2^3 vertices, for instance).

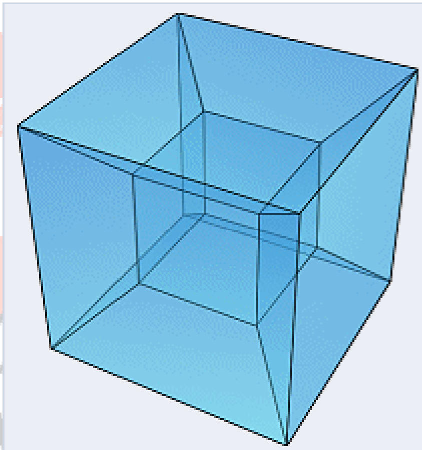
The number of m-dimensional hyper cubes on the boundary of an n-cube is

$$2^{n-m}\binom{n}{m}$$

For example, the boundary of a 4-cube contains 8 cubes, 24 squares, 32 lines and 16 vertices.

A unit hyper cube is a hyper cube whose side has length 1 unit whose corners are

$$V_{2^{n+1}} \leftarrow \begin{pmatrix} V_{2^n} & I_{2^n} \\ I_{2^n} & V_{2^n} \end{pmatrix}$$



A projection of hypercube into two-dimensional image

2^n Points in R^n with each coordinate equal to 0 or 1 termed as measure polytope.

The correct number of edges of cube of dimension n is $n * 2^{n-1}$ for example 7-cube has $7 * 2^6$ =448 edges.

A) Dimension of the cube

	1	2	3	4	5	6
No. of vertices	2	4	8	16	32	64
No. edges	1	4	12	32	80	192

Here we define adjacency matrix of n cube described in a constructive way.

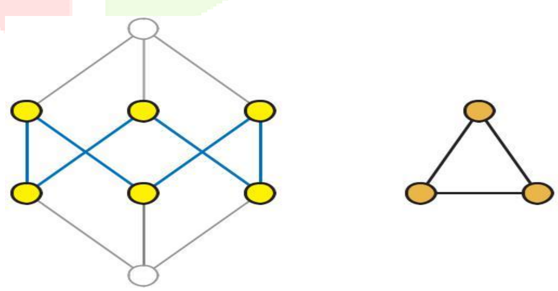
$$V_{2^{n+1}} \leftarrow \begin{pmatrix} V_{2^n} & I_{2^n} \\ I_{2^n} & V_{2^n} \end{pmatrix}$$

Since n Q_n is n regular bipartite graph of 2^n vertices characteristic vector of subsets of $[n]=\{1,2,3,...n\}$ vertices of layer L_k corresponds to subsets of cardinality k.

If n is odd n=2k-1,the middle two layers L_k, L_{k-1} of Q_n with nc_k, nc_{k-1} vertices forms middle cube graph MQ_k by induction.

As MQ_k is bipartite double graph which is a sub graph of n-cube Q_n induced by vertices whose binary representations have either k-1 or k no. Of 1's is of k-regular as shown in figures below

The middle cube graph MQ_2 as a subgraph [1]of Q_3 or as the bipartite double graph of $O_2 = K_3$.



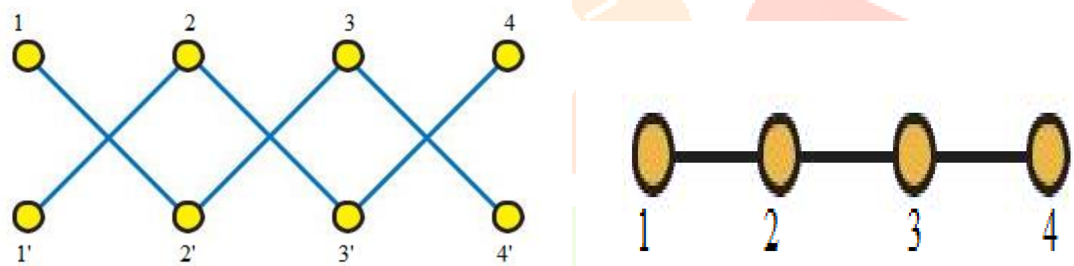
We start with spectral properties of bipartite double graphs [17][18] and extend for study of eigen values of MQ_k .

B) Bipartite double graph: Let H= (V, E) be a graph of order n, with vertex set V = {1, 2…. n}. Its bipartite double graph $1+\lambda,-1-\lambda\hat{H}$ $\bar{H} = (\bar{V},\bar{E})$ is the graph with the duplicated vertex set[2]

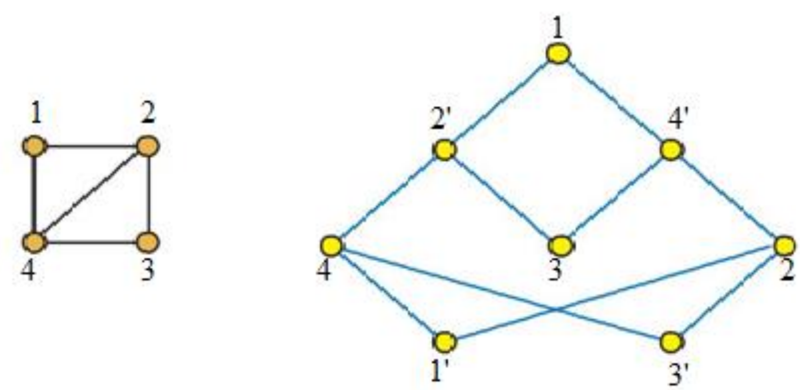
$\bar{V}$ = {1,2… n.1',2',...n' }and adjacencies induced from the adjacencies in H as follows:

$$i \square\ j \Rightarrow \begin{cases} i \overset{E}{\square}\ j' \\ j \overset{E}{\square}\ i' \end{cases}$$

Thus, the edge set of $\bar{H}$ is $\bar{E}$ = {ij' | ij $\in$ E}. From the definition, it follows that $\bar{H}$ Is a bipartite graph [24.21] with stable subsets V_1= {1, 2…. n}, and V_2= {1',2',...n' }. For example, if H is a bipartite graph, then its bipartite double graphs $\bar{H}$ consists of two non-connected copies of H.



Path p-4 and its bipartite Double Graph



Graph H has diameter 2 and $\bar{H}$ has diameter 3

If H is a δ-regular graph, then $\bar{H}$ also, if the degree sequence of the original graph H is $\delta = (\delta_1,\delta_2,\delta_3....\delta_n)$, the degree sequence for its bipartite double graph is $\bar{\delta}$ =($\delta_1,\delta_2,\delta_3....\delta_n,\delta_1,\delta_2,\delta_3....\delta_n$)

The distance between vertices in the bipartite double graph H can be given in terms of the even

And odd distances in H[4].

$$dist_{\bar{H}}(\mathrm{i}, \mathrm{j}) = \mathrm{dist}^{+}_{H}(\mathrm{i}, \mathrm{j})$$
$$dist_{\bar{H}}(\mathrm{i}, \mathrm{j'}) = \mathrm{dist}^{-}_{H}(\mathrm{i}, \mathrm{j})$$

Involutive auto orphism without fixed edges, which interchanges vertices i and i' , the map from $\bar{H}$ Onto H defined $i' \to i, i \to i$ is a 2-fold covering.

If $\hat{H}$ is extended bipartite double graph by adding edges (i,i')f or each $i \in V$ $\bar{H} \equiv \hat{H}$.

The section of the paper as follows**,** hypercube in section 1, Graph analysis in section 2, Eigenvalues are described in section 3, Results are given in section 4, Conclusion is given in 5.

2) Graph Notations:

The order of the graph G is n = {V} and its size is m = {E}. We label the vertices with the integers 1,2,…, n. If i is adjacent to j, that is, ij $\in$ E, we write i□ j or i $\overset{(E)}{□}$ j. The distance between two vertices is denoted by dist(I,j). We also use the concepts of even distance and odd distance between vertices, denoted by dist+ and dist -, respectively. They are defined as the length of a shortest even (respectively, odd) walk between the corresponding vertices. The set of vertices which are L-apart from vertex i, with respect to the usual distance, is $\Gamma_l(i) = \{ j : dist(\mathrm{i}, \mathrm{j}) = l$, so that the degree of vertex is simply $\Gamma_l(i)$. The eccentricity of a vertex is

ecc(i)= $\max_{1 \le X_{1 \le j \le n}} dist(\mathrm{i}, \mathrm{j})$ max1j_n dist(i; j) and the diameter of the graph is D =D(G) $\max_{1 \le X_{1 \le j \le n}} dist(\mathrm{i}, \mathrm{j})$ graph G` has the same vertex set as G and two vertices are adjacent in G` if and only if they are at distance l in G. An antipodal graph G is a connected graph of diameter D for which GD is a disjoint union of cliques. The folded graph of G is the graph G whose vertices are the maximal cliques[23]. Let G = (V;E) be a graph with adjacency matrix A and λ-eigenvector v[5,6]. Then, the charge of vertex i $\in$ V is the entry vi of v, and the equation $Av = \lambda v$.$\Rightarrow$eigen values of the bipartite double graph[11,16] $\bar{G}$ and the extended bipartite double graph $\hat{G}$ as functions of the eigen values of a non-bipartite graph G.

We study some more results which are less elementary but relevant on spectra multiplicities of associated eigen vectors extended to bipartite double graphs[24].

3)Eigenvalues of the Graphs:

Definition 1. *For a matrix* $A \in R^{m*n}$ *, a number* λ *is an eigenvalue if for some vector* $x \neq 0$,

$$Ax = \lambda x.$$

The vector x is called an eigenvector corresponding to λ.

Some basic properties of eigenvalues are

- The eigenvalues are exactly the numbers λ that make the matrix $A - \lambda$ I singular, i.e. solutions of $\det(A - \lambda I) = 0$.
- All eigenvectors corresponding to λ form a subspace V_λ ; the dimension of V_λ is called the multiplicity of λ.
- In general, eigenvalues can be complex numbers. However, if *A* is a symmetric matrix $(a_{ij} = a_{ji})$, then all eigenvalues are real, and moreover there is an orthogonal basis consisting of eigenvectors.
- The sum of all eigenvalues, including multiplicities, is $\sum_{i=1}^{n} \lambda_i = Tr(A) = \sum_{i=1}^{n} a_{ij}$ the trace of A.
- The product of all eigenvalues, including multiplicities, is $\prod_{i=1}^{n} \lambda_i = \det(A)$ the determinant of A.
- The number of non-zero eigenvalues, including multiplicities, is the rank of *A*.
  For graphs, we define eigenvalues as the eigenvalues of the *adjacency matrix[8]*.

Definition 2. For a graph G, the adjacency matrix A(G) is defined as follows:

- $a_{ij} = 1 if\ (i, j) \in E(G)$
- $a_{ij} = 0 if i = j or (i, j) \notin E(G)$

Because Tr(A(G)) = 0, we get immediately the following.

The sum of all eigenvalues of a graph is always 0.

The (ordinary) spectrum of a graph is the spectrum of its (0,1) adjacency matrix.

The graph on n vertices without edges (the n-coclique, $\overline{K}_n$) has zero adjacency matrix, hence spectrum 0^n, where the exponent denotes the multiplicity

Complete bipartite graphs[11]

The complete bipartite graph $K_{m,n}$ has spectrum $\pm\sqrt{mn}, 0^{m+n-2}$

More generally, every bipartite graph has a spectrum that is symmetric w.r.t. the origin: if θ is eigenvalue, then also $-\theta$, with the same multiplicity.

The n-cube graph (called 2^n, or Q_n) is the graph with as vertices the binary vectors of length n, where two vectors are adjacent when they differ in a single position. The 0-cube is K_1, the 1-cube is K_2, the 2-cube is C_4[28].

The spectrum of 2^n consists of the eigenvalues n − 2i with multiplicity $\binom{n}{i} (0 \le i \le n)$

The complete bipartite graph $K_{m,n}$ has an adjacency matrix of rank 2, therefore we expect to have eigenvalue 0 of multiplicity *n*-2, and two non-trivial eigenvalues. These should be equal to $\pm\lambda$, because the sum of all eigenvalues is always 0.

We find λ by solving *Ax* = λ *x*. By symmetry, we guess that the eigenvector *x* should have *m* Coordinates equal to α and *n* coordinates equal to β Then,

$$Ax = (m\beta, ..., m\beta, n\alpha, ...n\alpha)$$

This should be a multiple of $x = (\alpha, ..., \alpha, \beta, ..., \beta)$. Therefore, we get $m\beta = \lambda\alpha$ and $n\alpha = \lambda\beta$ i.e. and $mn\beta = \lambda^2\beta$ and $\lambda = \sqrt{mn}$

A graph Γ is called bipartite when its vertex set can be partitioned into two disjoint parts $X_1 X_2$ such that all edges of Γ meet both X_1 and X_2. The adjacency matrix of a bipartite graph has the form $A = \begin{Bmatrix} 0 & B \\ B^T & 0 \end{Bmatrix}$. It follows that the spectrum of a bipartite graph is symmetric w.r.t. 0: if $\begin{bmatrix} u \\ v \end{bmatrix}$ is an eigenvector with eigenvalue θ, then $\begin{bmatrix} u \\ -v \end{bmatrix}$ is an eigenvector with eigenvalue −θ.

For the ranks one has rkA = 2 rk B. If n_i = |Xi| (i = 1, 2) and n1 ≥ n2, then rkA ≤ 2n2, so that Γ has eigenvalue 0 with multiplicity at least n1 − n2.

One cannot, in general, recognize bipartiteness from the Laplace or signless Laplace spectrum. For example, $K_{1,3}$ and $K_1 + K_3$ have the same signless Laplace spectrum and only the former is bipartite[29,30].

However, by Proposition below, a graph is bipartite precisely when its

Laplace spectrum and signless[14] Laplace spectrum coincide.

3.A) Elementary Graphs associated Eigen values:

label	picture	A	L	Q	R
0.1					
1.1		0	0	0	0
2.1		1,-1	0.2	2.0	-1,1
2.2		0,0	0,0	0,0	-1,1
3.1		2,-1,-1	0,3,3	4,1,1	-2,1,1
3.2		$\sqrt{2},0,-\sqrt{2}$	0,1,3	3,1,0	-1,-1,2
3.3		1,0,-1	0,0,2	2,0,0	-2,1,1
3.4		0,0,0	0,0,0	0,0,0	-1,-1,2

Table 2.2

3.D Characteristic polynomial:

Let Γ be a directed graph on n vertices. For any directed subgraph C of Γ that is a union of directed cycles, let c(C) be its number of cycles. Then the characteristic polynomial

pA(t) =det(tI −A) of Γ can be expanded as $\sum C_i t^{n-i}$ where $C_i = \sum_C (-1)^{c(C)}$ with C running over all regular directed sub graphs with in- and outdegree 1 on i vertices.

(Indeed, this is just a reformulation of the definition of the determinant as $\det M = \sum_\sigma \mathrm{sgn}(\sigma) M_{1\sigma(1)} \ldots M_{n\sigma(n)}$ Note that when the permutation σ with n−i fixed points is written as a product of non-identity cycles, its sign is $(-1)^e$ .where e is the number of even cycles in this product. Since the number of odd non-identity cycles is congruent to i(mod 2), we have $sgn(\sigma) = (-1)^{i+c(\sigma).}$) [15]

For example, the directed triangle has $c_0 = 1, c_3 = -1$. Directed edges that do not occur in directed cycles do not influence the (ordinary) spectrum.

The same description of $p_A(t)$ holds for undirected graphs (with each edge viewed as a pair of opposite directed edges).

Since $\frac{d}{dt}\det(tI-A)=\sum_x \det(tI-A_x)$ where A_x is the submatrix of A obtained by deleting row and column x, it follows that $p'_A(t)$ is the sum of the characteristic polynomials of all single-vertex-deleted subgraphs of Γ .

The spectrum of the complete bipartite graph $K_{m,n} is \pm\sqrt{mn},\ 0^{m+n-2}$. The Laplace spectrum is $0^1,\ m^{n-1},\ n^{m-1},(m+n)^1$

The largest eigenvalue of a graph is also known as its spectral radius or index. The basic information about the largest eigenvalue of a (possibly directed) graph is provided by Perron-Frobenius theory as follows.

3.E Proposition:

Each graph Γ has a real eigenvalue θ_0 with nonnegative real

Corresponding eigenvector, and such that for each eigenvalue θ we have |θ| ≤ θ_0 .

The value θ_0 (Γ) does not increase when vertices or edges are removed from Γ .

Assume that Γ is strongly connected. Then

(i) θ_0 has multiplicity 1.

(ii) If Γ is primitive (strongly connected, and such that not all cycles have a length that is a multiple of some integer d > 1), then |θ| < θ_0 for all eigenvalues θ different from θ_0 .

(iii) The value θ_0 (Γ) decreases when vertices or edges are removed from Γ

Now let Γ be undirected. By Perron-Frobenius theory and interlacing we find an upper and lower bound for the largest eigenvalue of a connected graph.

(Note that A is irreducible if and only if Γ is connected.)

Among the connected graphs Γ , those with imprimitive A are precisely the bipartite graphs (and for these, A has period 2) is illustrated from the following proposition.

3.6 f. Proposition :

(i) A graph Γ is bipartite if and only if for each eigenvalue θ of Γ, also −θ is an eigenvalue, with the same multiplicity.

(ii) If Γ is connected with largest eigenvalue θ_1, then Γ is bipartite if and only if $-\theta_1$ is an eigenvalue of Γ.

Proof. For connected graphs all is clear from the Perron-Frobenius theorem. That gives (ii) and (by taking unions) the 'only if' part of (i). For the 'if' part of (i), let θ_1 be the spectral radius of Γ. Then some connected component of Γ Has eigenvalues θ_1 and $-\theta_1$, and hence is bipartite. Removing its contribution to the spectrum of Γ, we see by induction on the number of components that all Components are bipartite.

We establish some more theorems extended on spectra and multiplicities and associated eigen which are extended to bipartite double graphs.

Theorem: Let F be a field and let R be a commutative sub ring of F^{n*n}, the set of all n *n Matrices over F. Let M $\in R^{m*m}$, then

$$\det_F(M) = \det_F(\det_R(M))$$

$$\therefore \quad \det_F(M) = \det_F(AD - BC).$$ for a bipartite double graph characteristic polynomial. [13]

We prove the following theorems showing geometric multiplicities of eigen value λ of H $\Rightarrow$ geometric multiplicities of eigen values λ and $-\lambda$ of $\bar{H}$

$$1+\lambda, -1-\lambda \text{ of } \hat{H}$$

Theorem: Let H be a graph on n vertices, with the adjacency matrix A and characteristic

$\bar{u} = u_i^+ v_i (1+\lambda)$

$(u^+)_{i'} = \sum_{j \sim i'}^{E} u \overset{+}{j} = \sum_{j \sim i}^{E} v_j = \lambda v_i = \lambda u_i^+$ polynomial $\varnothing_H$ (x). Then, the characteristic polynomials of $\bar{H}$ and $\hat{H}$ are, respectively,

$$\varnothing_{\bar{H}}(x) = (-1)^n \varnothing_H(x) \varnothing_H(-x),$$

$$\varnothing_{\hat{H}}(x) = (-1)^n \varnothing_H(x-1) \varnothing_H(-x-1).$$

Adjacency matrices are, respectively,

$$\bar{A} = \begin{pmatrix} 0 & A \\ A & 0 \end{pmatrix} \text{and } \hat{A} = \begin{pmatrix} O & A+I \\ A+I & O \end{pmatrix}.$$

By above corollary

$$\varnothing_{\bar{H}}(x) = \det(xI_{2n} - \bar{A}) = \det\begin{pmatrix} xI_n & -A \\ -A & xI_n \end{pmatrix} = \det(x^2 I_n - A^2)$$

$$= \det(xI_n - A)\det(xI_n + A) = (..1)^n \varnothing_H(x)\varnothing_H(-x);$$

Whereas, the characteristic polynomial of $\hat{H}$ is

$$\varnothing_{\hat{H}}(x) = \det(xI_{2n} - \hat{A}) = \det\begin{pmatrix} xI_n & -A-I_n \\ -A-I_n & xI_n \end{pmatrix}$$

$$= \det(x^2 I_n - (A+I_n)^2) = \det(xI_n - (A+I_n))\det(xI_n + (A+I_n))$$

$$= \det((x-1)I_n - A)(-1)^n \det(-(x+1)I_n - A)$$

$$= (-1)^n \varnothing_H(x-1)\varnothing_H(-x-1).$$

Theorem: Let H be a graph and v a λ-eigenvector H. Let us consider the vector u+ with Components $u_i^+ = u_{i'}^+ = v_i$, u- with components $u_i^- = v_i$ and $u_{i'}^- = -v_i$ for $1 \le i, i' \le n$

Then,

u^+ λ-eigenvector $\bar{H}$ and $(1+\lambda)$ eigenvector $\hat{H}$

$\bar{u}$ - λ-eigenvector $\bar{H}$ and $(-1-\lambda)$ eigenvector $\hat{H}$

Given vertex i, $1 \le i \le n$, all its adjacent vertices are of type j', with i (E) ⊔ j.

Then

$$(Au^+)_i = \sum_{\substack{E \\ j \sim i'}} u \overset{+}{j} = \sum_{\substack{E \\ j \sim i}} v_j = \lambda v_i = \lambda u_i^+$$

Given vertex I', $1 \le i \le n$, all its adjacent vertices are of type j, with i (E) ⊔ j.

Then

$$(Au^+)_{i'} = \sum_{\substack{E \\ j \sim i'}} u \overset{+}{j} = \sum_{\substack{E \\ j \sim i'}} v_j = \lambda v_i = \lambda u_i^+$$

By a similar reasoning with u^-, we obtain

$$(Au^-)_i = \sum_{\substack{E \\ j \sim i'}} u \overset{+}{j}' = -\sum_{\substack{E \\ j \sim i}} v_j = -\lambda u_{i-} \text{ and}$$

$$(Au^-)_{i'} = \sum_{\substack{E \\ j \sim i'}} u \overset{-}{j} = \sum_{\substack{E \\ j \sim i}} v_j = -\lambda u_i'$$

$m(\lambda_0) = m(\lambda_5) = m(\theta_0^{\pm}) = 1,$

$m(\lambda_1) = m(\lambda_4) = m(\theta_1^{\pm}) = 4,$ $\therefore$ u^- is -λ-eigenvector of bipartite double graph $\overline{H}$.

$m(\lambda_2) = m(\lambda_3) = m(\theta_2^{\pm}) = 5,$

Also 1+λ,-1-λ are eigen values for u^+, u^- eigen vectors of $\hat{H}$

From the above figures realizing an isomorphism [8, 2] defined by

$$f : V[\tilde{\mathrm{O}}_k] \to \mathrm{V}[\mathrm{MQ}_k]$$

$$\mathrm{u} \mapsto \mathrm{u}$$

$$\mathrm{u}' \mapsto \overline{\mathrm{u}}$$

is clearly bijective, according to the definition of bipartite double graph , if u and v' Are two vertices of $\tilde{\mathrm{O}}_k$.[20,21]

The middle cube graph $[\mathrm{MQ}_k]$ with D=2k-1 by above corollary is isomorphic to $\tilde{\mathrm{O}}_k$.

We prove spectrum of Q_{2k-1} contains all eigen values of $[\mathrm{MQ}_k]$,

$\theta_i^+ = (-1)^i (k-i)$ and = $\theta_i^- = -\theta_i^+$ for $0 \le i \le k-1$

With multiplicities $m(\theta_i^+) = \mathrm{m}(\theta_i^-) = \dfrac{k-1}{k}\dbinom{2k}{i}$

4) Results:

In Verification of the above results,

$$spMQ_3 = \{\pm 2, \pm 1^2\}$$

$$spMQ_5 = \{\pm 3, \pm 2^4, \pm 1^5\}$$

$$spMQ_7 = \{\pm 4, \pm 3^6, \pm 2^{14}, \pm 1^{14}\}$$

$$spMQ_9 = \{\pm 5, \pm 4^8, \pm 3^{27}, \pm 2^{48}, \pm 1^{42}\}$$

For highest degree Distance polynomials of $[\mathrm{MQ}_k]$

p5 (3) = p5 (1) = p5 (-1) = 1 and p5 (2) =p5 (-1) = p5 (-3) = -1. Then,

$$m(\lambda_0) = m(\lambda_5) = m(\theta_0^{\pm}) = 1,$$

$$m(\lambda_1) = m(\lambda_4) = m(\theta_1^{\pm}) = 4,$$

$$m(\lambda_2) = m(\lambda_3) = m(\theta_2^{\pm}) = 5,$$

5) Conclusion:

In this paper we describe graph theory, Elementary Graphs associated Eigen values.

We described middle cube graphs and their spectra clearly.

We have analyzed various eigen values and eigen matrix in a detailed manner.

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