IJCRT.ORG ISSN : 2320-2882

INTERNATIONAL JOURNAL OF CREATIVE RESEARCH THOUGHTS (IJCRT)

An International Open Access, Peer-reviewed, Refereed Journal

AREA DISTORTION UNDER CERTAIN CLASS OF QUASI CONFORMAL MAPPING

B.AMRINTAJ[1], V.PRIYADHARSHINI[2], V.ANGESHWARI[3]

[1]ASSISTANT PROFESSOR, [2] ASSISTANT PROFESSOR,[3] ASSISTANT PROFESSOR

DEPARTMENT OF MATHEMATICS

ADHIYAMAN ARTS AND SCIENCE COLLEGE FOR WOMEN, UTHANGARAI

KRISHNAGIRI, INDIA

ABSTRACT

In the paper we study hyperbolic and Euclidean area distortion of measurable sets under classes of K-Quasiconformal mapping, angular and radial Quasiconformal mapping from the upper half plane and the unit disk onto themselves respectively.

INTRODUCTION

Quasiconformal mapping are generalization of conformal mapping. They can be considered not only Riemann surface but also on Riemannian manifolds in all dimension and even an arbitrary metric space quasiconformal mapping occur nationally in various mathematical and often a priori unrelated contexts.In 1956 Beurling and Ahlfors solved the boundary value problem for quasiconformal mapping. If M ≥ 1 they gave an explicit formula for the extension of an M-quasisymmetric function h:$\mathbb{R} \to \mathbb{R}$ to a quasiconformal mapping $f = u + iv$ from $\mathbb{H}$ onto itself, where $\mathbb{H}$ denotes the upper half-plane. The mapping f is called the Beurling – Ahlfors extension of h. In particular f satisfies $\frac{1}{cy^2} \leq \frac{J_f(Z)}{v^2} \leq \frac{c}{y^2}$

Where,$J_f$is denoted the Jacobian of f and $c = c(K) > 0$ depends on$K = K(M)$. Let $f{:}\,\Omega \to \mathbb{C}$ be an ACL (absolutely continuous on lines) homeomorphism in a domain $\Omega \subset \mathbb{C}$ that preserves orientation. If f satisfies,$D_f = \frac{|f_z|+|f_{\bar{z}}|}{|f_z|-|f_{\bar{z}}|} \leq K\ a.e.$For some K≥ 1, then f is K-quasiconformal mapping, where,$f_z = \frac{1}{2}(f_x - if_y)\ and\ f_{\bar{z}} = \frac{1}{2}(f_x + if_y)$

$D_f$is called the dilatation of f. The maximal dilatation of f Thus for each measurable subset E of $\mathbb{H}$ is holds that , $\frac{A_H(E)}{c} \leq A_H(f(E)) \leq cA_H(E),$

Where, $A_H(.)$ Denotes the hyperbolic area in the half- plane H.

In 1994 Astala provedthat if f is a K-quasiconformal mapping from the unit disk $\mathbb{D}$ onto itself normalized by $f(0) = 0$ and if E is any measurable subset of the unit disk then,

$$A_e(f(E)) \leq a(K)A_e(E)^{\frac{1}{k}},$$

Where, $A_e(.)$ is denotes the Euclidean area and a(K)→ 1when K→ 1^+.

In 1998 Resends and porter obtained some results about area distortion under quasiconformal mapping on the unit disk $\mathbb{D}$ onto itself with respect to the hyperbolic measure. They also showed the existence of explodable sets has bounded hyperbolic area but under a specific quasiconformal mapping it is image has infinite hyperbolic area. In recent year harmonic quasiconformal mapping have been extensively studied and the paper cited therein.

In this paper combining with the knowledge of a harmonic function and its harmonic conjugate function, we get several equivalent condition for a harmonic mapping of $\mathbb{H}$ onto itself to be a K-quasiconformal mapping and we use the hyperbolic area distortion under quasiconformal harmonic mapping from the unit disk into itself and analyze the hyperbolic and Euclidean area distortion under quasiconformal mapping.

$$f(z) = f(x+iy) = u(x,y) + icy, \text{with} c > 0$$

QUASICONFORMAL MAPPINGS

DEFINITION 1.1:

Let Ω *be* an open set in $\mathbb{C}$ and $f: \Omega \longrightarrow \mathbb{C}$ be an oriented preserving homeomorphism we say that $f: \Omega \longrightarrow \mathbb{C}$is **quasiconformal mapping.**

(*i*) $f$is**absolutely** continuous on lines (ACL)

(ii) For almost every $z \in \Omega$we have

An orientation preserving homeomorphism $f: \Omega \rightarrow \mathbb{C}$ is called quasi-conformal.

DEFINITION 1.2:

Let Ω be an open set in $\mathbb{C}$ and $f: \Omega \rightarrow \mathbb{C}$ be a continuous map.We say that,

$f: \Omega \rightarrow \mathbb{C}$is **ACL (absolutely continuous on line)** if for each closed rectangle $\{z\in \mathbb{C}, a \le Re(z) \le b, c \le Im(z) \le d\}$ contained inΩ. We have, The following two properties

(i) For almost all y$\in [c,d]$ the function x$\rightarrow f(x+iy)$ is absolutely continuous on [a, b].

(ii) For almost all $x \in [a,b]$ the function y$\rightarrow f(x+iy)$ is absolutely continuous on [c, d].

DEFINITION1.3:

Let $f: \Omega \rightarrow \mathbb{C}$ and let u and v be a open subset of $\mathbb{C}$ take K>1 and set $K := \frac{K-1}{K+1}$ so that 0$\le K \le$ 1.A mapping $f: U \rightarrow V$ is a K-quasiconformal map if it is homeomorphism whose distributed partial derivatives are in ${L^2}_{loc}$(locally in L^2 and satisfy

$$\left|\frac{\partial f}{\partial \bar{z}}\right| = K \left|\frac{\partial f}{\partial z}\right| {L^2}_{loc} \text{ is } a.e$$

A map quasiconformal it is **K-quasiconformal** for some K.

DEFINITION 1.4:

The smallest K such that f is K-quasiconformal is called as a **quasiconformal constant** of f denoted by K (f).

THEOREM 1.1:

Prove that the class of K – quasiconformal mapping

Proof:

Let $1 \le K \le \infty$ &$\Omega \sqsubset \mathbb{C}$ be a domain. Suppose that, $f: \Omega \rightarrow \mathbb{C} is$ K- Quasiconformal mapping given by $f(x+iy) = u(x,y) + v(x,y)$ (1.1)

Then, f satisfies, $\left|\frac{f_{\bar{z}}}{f_z}\right| \le K \ a.e, \left|\frac{f_{\bar{z}}}{f_z}\right|^2 \le K^2 \ a.e$ (1.2)

Inequality of equation (1.2) is $\left|\frac{\frac{1}{2}[u_x - v_y + i(v_x + u_y)]}{\frac{1}{2}[u_x + v_y + i(v_x - u_y)]}\right|^2 \le K^2$

$$(u_x - v_y)^2 + (v_x + u_y)^2 \le K^2(u_x + v_y)^2 + (v_x - u_y)^2 \text{is a.e}$$

$$u_x{}^2 + v_y{}^2 - 2u_xv_y + v_x{}^2 + u_y{}^2 + 2v_xu_y \le K^2\left(u_x{}^2 + v_y{}^2 + 2u_xv_y + v_x{}^2 + u_y{}^2 - 2v_xu_y\right)$$

Or equivalently

$$u_x{}^2 + u_y{}^2 + v_x{}^2 + v_y{}^2 - \frac{1+k^2}{1-k^2}2u_xv_y + \frac{1+k^2}{1-k^2}2v_xu_y \le 0 \ a.e \qquad (1.3)$$

We defined $\propto = \propto (K) := \frac{1+K^2}{1-K^2} \geq 1$.Then f satisfied inequality (1.2)

$$u_x{}^2 + u_y{}^2 + v_x{}^2 + v_y{}^2 - 2 \propto u_x v_y + 2 \propto v_x u_y \leq 0 \; a.e \qquad (1.4)$$

From now on each expression that involves partial derivative will be true almost everywhereΩwilldenotes a domain of the complex plane$\mathbb{C}$. In this part we focus on K-quasiconformal mapping from $\mathbb{H}$ onto itself given by $f(x+iy) = u(x,y) + iv(y)$. In particular f can be extended homeomorphically to $\overline{\mathbb{H}}$, u(x, y) is ACL and v(y) is absolutely continuous We know that f is satisfied

$$u_x{}^2 + u_y{}^2 + v_y{}^2 - 2 \propto u_x v_y \leq 0 \; a.e \qquad (1.5)$$

In equation (1.5) we get the form is $f(x+iy) = u(x,y) + iv(y)$.Hence,

The class of K-quasiconformal mapping

THEOREM 1.2:

Let f be a K-quasiconformal mapping from $\mathbb{H}$ onto itself given by f(x+iy) =u(x, y) +iv(y) then,$v^+{}_y(0) = \lim sup_{y\to 0^+} v_y(y)$ &$v^-{}_y(0) = \lim inf_{y\to 0^-} v_y(y)$ are finite and the partial derivatives of f satisfy the following inequalities

1. $\frac{1}{K^2} v^+{}_y(0) \leq v_y(y) \leq K^2 v^-{}_y(0) for\ almost\ every\ y \in (0,\infty)$

2. $\frac{1}{K^3} v^+{}_y(0) \leq u_x(x,y) \leq K^3 v^-{}_y(0)\ for\ almost\ every\ x+iy \in \mathbb{H}$

Proof:

let f be a K-quasiconformal mapping from $\mathbb{H}$ onto itself given by

$$f(x+iy) = u(x,y) + iv(y)$$

Then, $\frac{1}{K^2} v^+{}_y(0) \leq v_y(y) \leq K^2 v^-{}_y(0)$

In equation (1.5) we get this inequality defines a circle a.e thus $u_x and\ v_y$ satisfy in particular $\alpha v_y - v_y\sqrt{\alpha^2-1} \leq u_x \leq \alpha v_y + v_y\sqrt{\alpha^2-1}\ a.e,\ -v_y\sqrt{\alpha^2-1} \leq u_y \leq v_y\sqrt{\alpha^2-1}\ \ a.e$

In fact the circle is a subset of the square described and observes that

$$K = \alpha + \sqrt{\alpha^2-1} \quad \text{and,} \quad \frac{1}{K} = \alpha - \sqrt{\alpha^2-1} \qquad (1.6)$$

Where K is maximal dilatation of f

Let, $c = \sqrt{\alpha^2-1}\ then, c \geq 0$

$$v_y\left(\alpha - \sqrt{\alpha^2-1}\right) \leq u_x \leq v_y\left(\alpha + \sqrt{\alpha^2-1}\right)$$

$$\frac{v_y}{K} \leq u_x \leq v_y K \quad and \qquad (1.7)$$

$$-cv_y \leq u_y \leq v_y c \qquad (1.8)$$

Given $0 \leq x$ are integrate (1.7) on interval [0, x]

$$\frac{v_y}{K}[x-0] \leq u_x(x,y) - u_x(0,y) \leq K v_y(y) x$$

If we choose any fixed y$\in (0,\infty)$ such that u(x, y) is absolutely continuous with respect to x then we obtain

$$\frac{x}{K} v_y(y) + u_x(0,y) \leq u(x,y) \leq K v_y(y) x + u_x(0,y) \qquad (1.9)$$

For each x$\in \mathbb{R}$ and almost every y$\in (0,\infty)$ using left hand side of the last inequality we get

$\lim_{y\to 0^+} \sup[\frac{v_y(y)x}{K} + u(0,y)] \leq \lim_{y\to 0^+} \sup u(x,y)$ and,

Since, u(x, y) is continuous we obtain

$$\frac{x}{K} \lim_{y\to 0^+} sup v_y(y) \leq u(x,0) - u(0,0) < \infty$$

For this reason $\lim_{y\to 0^+} sup v_y(y)$Exists and consequently $\lim_{y\to 0^-} inf v_y(y)$ exists

We defined, $v^+{}_y(0) = \lim sup_{y\to 0^+} v_y(y)$,$v^-{}_y(0) = \lim inf_{y\to 0^-} v_y(y)$

In this equation in (1.9)

$$\frac{v^+{}_y(y)x}{K} + u(0,0) \le u(x,0) \le Kv^-{}_y(0)x + u(0,0) \qquad (1.10)$$

On the other hand we choose any fixed x∈ $[0,\infty]$

Such that, $u(x,y)$ is absolutely continuous with respect to y we integrate (8) on the interval [0, y] so,$\int_0^y -cv_y(t)dt \le \int_0^y u_y(x,t)dt \le \int_0^y cv_y(t)dt$ and since v_y *is* absolutely continuous we obtain $-cv(y) + u(x,0) \le u(x,y) \le cv(y) + u(x,0)$. For y∈ $[0,\infty]$ *and almost every* $x \in [0,\infty]$ by an argument of continuity of the mapping f and density we have,

$$-cv(y) + u(x,0) \le u(x,y) \le cv(y) + u(x,0) \qquad (1.11)$$

For all (x, y)∈ $[0,\infty) \times [0,\infty)$ setting x=0 in the previous inequality to get

$$-cv(y) + u(0,0) \le u(0,y) \le cv(y) + u(0,0) \qquad (1.12)$$

Thus combination (1.9) and (1.12)

$$\frac{v_y(y)x}{K} - cv(y) + u(0,0) \le u(x,y) \le Kxv_y(y) + cv(y) + u(0,0)$$

For each x ∈ ℝ and almost every y ∈ $(0,\infty)$ in some way we use (10) and (11)

$$v^+{}_y\frac{x}{K} - cv(y) + u(0,0) \le u(x,y) \le xKv^-{}_y(0) + cv(y) + u(0,0)$$

We combine the left and right hand sides

$$\frac{v_y(y)x}{K} - cv(y) + u(0,0) \le u(x,y) \le xKv^-{}_y(0) + cv(y) + u(0,0) \qquad (1.13)$$

$$v^+{}_y\frac{x}{K} - cv(y) + u(0,0) \le u(x,y) \le Kxv_y(y) + cv(y) + u(0,0) \qquad (1.14)$$

For each x ∈ ℝ and almost every y ∈ $(0,\infty)$ *since*, the left and right hand side of the inequality (1.13)& (1.14) represent linear equation in the variable x, we compare their slopes and the fact that x>0 to conclude.

$\frac{v_y(y)}{K} \le Kv^-{}_y(0)$and,$\frac{v^+{}_y(0)}{K} \le Kv_y(y)$.For each x ∈ ℝ and almost every y ∈ $(0,\infty)$

$$v_y(y) \le K^2 v^-{}_y(0)$$

$$v_y(y) \le \frac{v^+{}_y(0)}{k^2}$$

$$\frac{v^+{}_y(0)}{k^2} \le v_y(y) \le K^2 v^-{}_y(0)$$

For each x ∈ ℝ and almost every y ∈ $(0,\infty)$ v(y) is absolutely continuous and we integrate the above inequalities on the interval [0, y]

$$\frac{1}{K^2}\int_0^y v^+{}_y(0)dt \le \int_0^y v_y(t)dt \le K^2\int_0^y v^-{}_y(0)dt$$

To get $\frac{1}{K^2}v^+{}_y(0)(y) \le v(y) \le K^2 v^-{}_y(0)y$ (1.15)

In particular $0 \le v^-{}_y(0) \le v^+{}_y(0) \le \infty$ v has the dinis derivatives at 0

Similarly, $\frac{1}{K^3}v^+{}_y(0)(y) \le u_x(x,y) \le K^3 v^-{}_y(0)y$

THEOREM 1.3:

Let f be a K-quasiconformal mapping from ℍ onto itself given by $f(x+iy) = u(x,y) + iv(x,y)$ then,

1. There exists M>0 such that $|f_z| - |f_{\bar{z}}| \le M$ *and*$|f_z| - |f_{\bar{z}}| \le KM$ *a.e*

2. The mapping f is Lipschitz in $\mathbb{H}$

3. The mapping f is hyperbolically Lipschitz in $\mathbb{H}$

Proof:

1. Since,

$$|f_z| - |f_{\bar{z}}| = \frac{1}{2}\sqrt{(u_x + v_y)^2 + {u_y}^2} - \frac{1}{2}\sqrt{(u_x - v_y)^2 + {u_y}^2}$$

$$|f_z| - |f_{\bar{z}}| \le \frac{1}{2}\sqrt{2({u_x}^2 + {v_y}^2 + {u_y}^2)}\ a.e$$

We estimate the last expression in above theorem

$$\frac{1}{2}\sqrt{2({u_x}^2 + {v_y}^2 + {u_y}^2)} = \frac{\sqrt{2}}{2}\sqrt{(K^3 {v^-}_y(0))^2 + (K^2 {v^-}_y(0))^2 + (\frac{K^2-1}{2} K {v^-}_y(0))^2}$$

$$= \frac{K^2 {v^-}_y(0)}{2K\sqrt{2}}\sqrt{5K^4 + 2K^2 + 1}\ \text{a.e}$$

$$\frac{1}{2}\sqrt{2({u_x}^2 + {v_y}^2 + {u_y}^2)} = \frac{K {v^-}_y(0)}{2\sqrt{2}}\sqrt{5K^4 + 2K^2 + 1}\ \text{a.e}$$

Thus we choose, $M = \frac{K {v^-}_y(0)}{2\sqrt{2}}\sqrt{5K^4 + 2K^2 + 1}$

$$|f_z| - |f_{\bar{z}}| \le M \text{ and } |f_z| - |f_{\bar{z}}| \le KM\ a.e$$

2. Let $z_1, z_2 \in \mathbb{H}$ and l be the Euclidean segment that joint z_1, z_2 then,

$$|f_z| - |f_{\bar{z}}| \le \int_{f(l)} |df|$$

$$\le \int_{l} |f_z| + |f_{\bar{z}}||dz|$$

$$|f_z| - |f_{\bar{z}}| \le MK|z_1 - z_2|$$

3. Let $z_1, z_2 \in \mathbb{H}$ and l be the hyperbolic segment that joint $z_1 with\ z_2$

$$d_H(f(z_1), f(z_2)) \le \int \frac{|dw|}{Imw}$$

$$d_H(f(z_1), f(z_2)) \le LMKd_H(z_1, z_2)$$

Where,

$L = \frac{K^2}{{v^+}_y(0)}$ and d_H denote the hyperbolic metric

THEOREM 1.4:

Let f be a K-quasiconformal mapping from $\mathbb{H}$ onto itself given by

$f(x+iy) = u(x,y) + iv(x,y)$ Then,

1. $\frac{({v^+}_y(0))^2}{K^5} A_e(E) \le A_e(f(E)) \le K^5({v^-}_y(0))^2 A_e(E)$
2. $\frac{1}{K^9}\left(\frac{{v^+}_y(0)}{{v^-}_y(0)}\right)^2 A_e(E) \le A_e(f(E)) \le K^9\left(\frac{{v^-}_y(0)}{{v^+}_y(0)}\right)^2 A_e(E)$

Since, ${v^-}_y(0) \le {v^+}_y(0)$

We have

$\frac{1}{K^9} A_e(E) \le A_e(f(E)) \le K^9 A_e(E)$ and these inequalities are asymptotically sharp when K→ 1^+

Proof:

Let E⊂ $\mathbb{H}$ be a measurable set the Jacobian of f

$$J_f = u_x v_y$$

By equation 1.1 theorem 1.2

1. We get $\frac{1}{K^2}{v^+}_y(0)(y) \le v(y) \le K^2 {v^-}_y(0)y$ a.e

$$\frac{1}{K^3}{v^+}_y(0)(y) \le u_x(x,y) \le K^3 {v^-}_y(0)y\ \text{a.e}$$

Comparing above equation we get

$$\frac{\left(v^{+}{}_{y}(0)\right)^{2}}{K^{5}} \leq u_{x}(x,y)v_{y}(y) \leq K^{5}(v^{-}{}_{y}(0))^{2}$$

By Jacobian f $J_f = u_x u_y$

$$\frac{\left(v^{+}{}_{y}(0)\right)^{2}}{K^{5}} \leq J_f \leq K^{5}(v^{-}{}_{y}(0))^{2} \text{ a.e}$$

The Euclidean area of f (E) is

$$\int_E J_f dxdy = A_e(f(E))$$

And in consequence

$$\frac{\left(v^{+}{}_{y}(0)\right)^{2}}{K^{5}} A_e(E) \leq A_e\big(f(E)\big) \leq K^{5}(v^{-}{}_{y}(0))^{2} A_e(E)$$

On the other hand by hyperbolic area as sane way we get

$$\boldsymbol{\frac{1}{K^{9}}\left(\frac{v^{+}{}_{y}(0)}{v^{-}{}_{y}(0)}\right)^{2} A_e(E) \leq A_e\big(f(E)\big) \leq K^{9}\left(\frac{v^{-}{}_{y}(0)}{v^{+}{}_{y}(0)}\right)^{2} A_e(E)}$$

THEOREM 1.5:

Let f be a K-quasiconformal mapping from ℍ onto itself given by f(x+iy) =u(x, y) +i$v(y)$ then for each measure set E⊂ ℍ.

(i) If v is differentiable at 0 then, $\frac{\left(v^{|}{}_{y}(0)\right)^{2}}{K^{9}} A_e(E) \leq A_e\big(f(E)\big) \leq K^{9}(v^{|}{}_{y}(0))^{2} A_e(E)$

(ii) If v is continuously differentiable in a neighborhood of 0

$\frac{\left(v^{|}{}_{y}(0)\right)^{2}}{K^{5}} A_e(E) \leq A_e\big(f(E)\big) \leq K^{5}(v^{|}{}_{y}(0))^{2} A_e(E)$These inequality are asymptotically sharp when K→ 1^{+}

Proof:

If v is differentiable at 0 then we get in (1.15)

$$\frac{1}{K^{2}} v^{+}{}_{y}(0)(y) \leq v^{|}{}_{y}(0) \leq K^{2} v^{-}{}_{y}(0)$$

If v is continuously differentiable is neighbourhood of 0, then

$$v^{-}{}_{y}(0) = v^{|}{}_{y}(0) = v^{+}{}_{y}(0)$$

We get above theorem results replace the value of $v^{-}{}_{y}(0) = v^{|}{}_{y}(0) = v^{+}{}_{y}(0)$.hence

$\frac{\left(v^{|}{}_{y}(0)\right)^{2}}{K^{9}} A_e(E) \leq A_e\big(f(E)\big) \leq K^{9}(v^{|}{}_{y}(0))^{2} A_e(E)$

$$\boldsymbol{\frac{\left(v^{|}{}_{y}(0)\right)^{2}}{K^{5}} A_e(E) \leq A_e\big(f(E)\big) \leq K^{5}(v^{|}{}_{y}(0))^{2} A_e(E)}$$

EXAMPLE:

Let $f{:}\,\mathbb{H} \to \mathbb{H}$ given by $f(x+iy) = 2x + \sin(x+y) + iy$ then, f is a $\frac{11+\sqrt{85}}{6}$quasiconformal mapping with $v^{-}{}_{y}(0) = v^{|}{}_{y}(0) = v^{+}{}_{y}(0) = 1$

2.HARMONIC QUASICONFORMAL MAPPING

DEFINITION 2.1:

A function F is called **Harmonic** in a region Ωif its laplacian vanishes inΩ. A topological mapping f ofΩ is said to be **K- quasiconformal** it is satisfies.

1.$f is\ ACL\ in\ \Omega$

2. $L^{2}{}_{f} \leq KL_f l_f, K \geq 1\ a.e\ in\ \Omega$

Where, $\boldsymbol{L_f} = |\boldsymbol{f_z}| + |\boldsymbol{f_{\bar{z}}}|, \boldsymbol{l_f} = |\boldsymbol{f_z}| - |\boldsymbol{f_{\bar{z}}}|$

Then, f is K- quasiconformal mapping.

DEFINITION 2.2:

Let f is a harmonic mapping then there exists a **holomorphic function** $g{:}\,\mathbb{H} \to \mathbb{C}$

Such that, $f(z) = Reg(z) + icy$

Thus,$|\boldsymbol{f_z}| = \frac{1}{2}\left|\boldsymbol{g^{|}(z) + c}\right|$**And**$|\boldsymbol{f_{\bar{z}}}| = \frac{1}{2}\left|\boldsymbol{g^{|}(z) - c}\right|$.We obtain that $g^{|}(z)$ belongs to the circle 𝔻.

DEFINITION 2.3:

The hyperbolic density is $\boldsymbol{\frac{2dz}{1-|z|^{2}}\ and\ \frac{|dw|}{Im\,w}}$ for the unit disk 𝔻 and the upper half – planeℍ. We denote also by A_H the hyperbolic area in the unit disk 𝔻.

THEOREM 2.1:

Let f be a harmonic mapping of ℍ onto itself and continuous up to its boundary withf (∞) = ∞.If f is a K-quasiconformal mapping then f can be represented by **f = u+icy** and the gradient of f is such that

$$L_f = |f_z| + |f_{\bar{z}}| \leq cK$$

Where, c is positive constant.

Proof:

By definition of K- quasiconformal mapping

We have, $L^{2}{}_{f} \leq KL_f l_f$

This implies that, $L_f \leq Kl_f$ (2.1)

We assume that $f = u + icIm(z)$ $, f = \frac{1}{2}[g(z) + c(z)] + \frac{1}{2}\overline{[g(z) - c(z)]}$

Hence, $|f_z| = \frac{1}{2}|g^{|}(z) + c|, |f_{\bar{z}}| = \frac{1}{2}|g^{|}(z) - c\ |$ (2.2)

Equation (2.1) and (2.2) we have,

$$|f_z| + |f_{\bar{z}}| \leq K[|f_z| - |f_{\bar{z}}|]$$
$$\leq K|f_z| - K|f_{\bar{z}}|$$
$$|f_z| + |f_{\bar{z}}| - K|f_z| + K|f_{\bar{z}}| \leq 0$$
$$-(K-1)|f_z| + (K+1)|f_{\bar{z}}| \leq 0$$
$$(K+1)|f_{\bar{z}}| \leq (K-1)|f_z|$$
$$\leq (K-1)|f_{\bar{z}} + c|$$
$$(K+1)|f_{\bar{z}}| - (K-1)|f_{\bar{z}}| \leq c(K-1)$$
$$2\,|f_{\bar{z}}| \leq c(K-1)$$

Put $f_{\bar{z}}$ value in above equation $|g^{|}(z) - c| \leq c(K-1)$ (2.3)

It is easy to get $|f_z| = \frac{1}{2}|c(K+1)|, |f_{\bar{z}}| = \frac{1}{2}|c(K-1)\ |$

Thus we obtain $|f_z| + |f_{\bar{z}}| \leq \frac{1}{2}\left[cK - c + \frac{1}{2}cK + c\right] \leq cK$

$$\therefore \boldsymbol{L_f = |f_z| + |f_{\bar{z}}| \leq cK}$$

LEMMA 2.1:

Assume that f is a harmonic mapping of $\mathbb{H}$ onto itself and continuous on $\mathbb{H} \cup \mathbb{R}$ with f $(\infty) = \infty. if$ f is K-quasiconformal mapping then,$f(z) = \frac{1}{2}\left(g(z) + c(z) + \overline{g(z) - c(z)}\right)$

Where, g is a holomorphic function in $\mathbb{H}$ *and* c is positive constant and the Jacobian of f is such that $J_f = |f_z| - |f_{\bar{z}}| \leq c^2K$

Proof:

According to the definition of K-quasiconformal mapping and the inequality

$$|f_z| = \frac{1}{2}|g^{|}(z) + c|, \quad |f_{\bar{z}}| = \frac{1}{2}|g^{|}(z) - c\ |$$
$$J_f = |f_z| - |f_{\bar{z}}|$$
$$J_f \leq c|g^{|}(z)|$$
$$|g^{|}(z) - c| \leq c(K-1), |g^{|}(z)| \leq cK$$
$$J_f \leq c^2K$$
$$\boldsymbol{J_f = |f_z| - |f_{\bar{z}}| \leq c^2K}$$

THEOREM 2.2:

Let f =u +iv be a harmonic mapping of $\mathbb{H}$ onto itself and continuous up to its boundary with f $(\infty) = \infty$.if f is K-quasiconformal mapping then for any measurable subset E⊂ $\mathbb{H}$

$$A_{euc}(f(z)) \leq c^2KA_{euc}(E)$$

Where, $A_{euc}(.)$ Denotes the Euclidean area and c is positive constant.

THEOREM 2.3:

Let f be a harmonic mapping of $\mathbb{H}$ onto itself and continuous up to its boundary with $f(\infty) = \infty$.if f is a K-quasiconformal mapping then for any measurable subset E⊂ $\mathbb{H}$we get

$$A_{hyp}(f(E)) \leq c^2KA_{hyp}(E)$$

Where, A_{hyp} denotes the hyperbolic area and c is positive constan

3.ANGULAR AND RADIAL QUASICONFORMAL MAPPING

Proposition 3.1:

Let f:Ω → ℂ be an ACL mapping. If$f(re^{i\theta}) = u(re^{i\theta}) + iv(re^{i\theta})$then for a.e inΩ

Proof:

Let f:Ω → ℂ be an ACL mapping. Then,$f(re^{i\theta}) = u(re^{i\theta}) + iv(re^{i\theta})$ is a.e

We know that,$|f_z| = \frac{1}{2}|f_x - if_y|$,$|f_{\bar{z}}| = \frac{1}{2}|f_x + if_y|$

Taking square on above equation,

$$4|f_z|^2 = |(u_x + v_y) + i(v_x - u_y)|^2$$
$$4|f_{\bar{z}}|^2 = |(u_x - v_y) + i(v_x + u_y)|^2, x = r, \frac{\theta}{r} = y$$
$$4|f_z|^2 = \left|\left(u_r + \frac{v_\theta}{r}\right) + i\left(v_r - \frac{u_\theta}{r}\right)\right|^2,$$
$$4|f_z|^2 = \left(u_r + \frac{v_\theta}{r}\right)^2 + \left(v_r - \frac{u_\theta}{r}\right)^2 \qquad (3.1)$$
$$4|f_{\bar{z}}|^2 = \left(u_r - \frac{v_\theta}{r}\right)^2 + \left(v_r + \frac{u_\theta}{r}\right)^2 \qquad (3.2)$$

And the Jacobian of 'f' is $J_f = |f_z|^2 - |f_{\bar{z}}|^2$

$$J_f = \frac{1}{4}\left[4u_r\frac{v_\theta}{r} - 4v_r\frac{u_\theta}{r}\right]$$
$$\boldsymbol{J_f = \frac{1}{r}[u_rv_\theta - v_ru_\theta]} \qquad \textbf{(3.3)}$$

DEFINITION 3.1:

A mapping $f: \mathbb{H} \to \mathbb{H}$ is said to be **radial** at $x \in \mathbb{R}$ if f leaves invariant all Euclidean rays in $\mathbb{H}$ that meet at x. Let $f: \mathbb{H} \to \mathbb{H}$ be a **radial mapping at x.**Since, hyperbolic area is invariant under horizontal translations.We can assume that, the point x∈ ℝ.Where,the Euclidean rays meet is x=0 if 'f' is a radial mapping then f is can be written in polar-coordinates (r, θ) as

$$f(z) = f(re^{i\theta}) = \rho(r,\theta)e^{i\theta}$$

With $\boldsymbol{\rho(r,\theta)} : (\mathbf{0},\infty] \times (\mathbf{0},\boldsymbol{\pi}) \to (\mathbf{0},\infty)$ **if z = re$^{i\theta}$.**

THEOREM 3.1:

Let f be a ACL mapping from $\mathbb{H}$ onto itself suppose that f is a radial mapping at 0.then its Jacobian mapping at 0 is $J_f = \frac{\rho\rho_r}{r}\ a.e$ If f preserves orientation then$\rho_r > 0\ a.e$

Proof:

Since, $f(z) = f(re^{i\theta}) = \rho(r,\theta)e^{i\theta}, f(re^{i\theta}) = u(re^{i\theta}) + iv(re^{i\theta})$

Then, $(r,\theta) = \rho(r,\theta)cos\theta$,$v(r,\theta) = \rho(r,\theta)sin\theta$

By Jacobian of 'f' is $J_f = \frac{1}{r}[\rho\rho_r sin\theta - \rho\rho_r cos\theta]$

$$J_f = \frac{\rho\rho_r}{r}[sin\theta - cos\theta]$$

$$\theta = \pi J_f = \frac{\rho\rho_r}{r}[0+1] \qquad \therefore J_f = \frac{\rho\rho_r}{r} \text{ If } \rho_r > 0$$

We get, $\boldsymbol{J_f = \frac{\rho}{r}\ a.e}$

PROPOSITION 3.2:

Let f be a K-quasiconformal mapping from $\mathbb{H}$ onto itself. Suppose that f is a radial mapping at 0. Then the function ρ satisfies the following

1. For 1≤ $r < \infty\ r^{\frac{1}{K}} \le \frac{\rho(r,\theta)}{\rho(1,\theta)} \le r^K$

2. For 0<r<1 $r^K \le \frac{\rho(r,\theta)}{\rho(1,\theta)} \le r^{\frac{1}{k}}$

Proof:

We first prove that the function $(0,\infty) \ni r \mapsto In\rho(r,\theta)$ is absolutely continuous for almost every $\theta \in (0,\pi)$It is enough to prove that for every M > 1,

The function $\left[\frac{1}{M}, M\right] \ni r \mapsto In\ \rho(r,\theta)$ is absolutely continuous for almost every $\theta \in (0,\pi)$.

Let $\Omega = \{z = x + i\theta \in \mathbb{C}(\text{x},\theta) \in (-\infty,\infty) \times (0,\pi)\}$

Then,

The mapping log∘ $f \circ exp: \Omega \to \Omega$ is K-quasiconformal mapping

Thus,

The function $(-\infty,\infty) \ni x \mapsto In\rho(e^x,\theta)$is absolutely continuous for almost every $\theta \in (0,\pi)$.

Let,$\epsilon > 0$ There exists $\delta > 0$

Such that, for every finite collection of disjoint intervals $(a_j, b_j) \subset \mathbb{R}, j = 1,2 \ldots .n$ with $\sum_{j=1}^{n}(b_j - a_j) < \delta$

Then,$\sum_{j=1}^{n}\left(In\rho(e^{b_j},\theta) - In\ \rho(e^{a_j},\theta)\right) < \epsilon$

Since, in r is absolutely continuous on $[\frac{1}{M}, M]$ there exists $\delta^{|} > 0$ such that, for every finite collection of disjoint intervals $(c_l, d_l) \subset \left[\frac{1}{M}, M\right], l = 1,2 \ldots ..m$ with,

$$\sum_{l=1}^{m}(d_l - c_l) < \delta^{|}$$

Then,$\sum_{l=1}^{m}(Ind_l - Inc_l) < \delta$

And by the inequality,$\sum_{l=1}^{m}(In\ \rho(d_l,\theta) - In\ \rho(c_l,\theta)) < \varepsilon$

If,$f(z) = \rho(r,\theta)e^{i\theta}$ from (1) and (2)

$$|f_z|^2 = \frac{1}{4}\left(\rho_r{}^2 + 2\frac{\rho_r\rho}{r} + \frac{\rho^2}{r} + \frac{\rho_\theta{}^2}{r^2}\right)\ a.e \qquad (3.4)$$

$$|f_{\bar{z}}|^2 = \frac{1}{4}\left(\rho_r{}^2 - 2\frac{\rho_r\rho}{r} + \frac{\rho^2}{r} + \frac{\rho_\theta{}^2}{r^2}\right)\ a.e \qquad (3.5)$$

By (4.4) and (4.5) in equation $|f_z|^2 \le K^2 \le |f_{\bar{z}}|^2$

$\frac{1}{4}\left[(\rho_r - \frac{\rho}{r})^2 + \frac{\rho_\theta{}^2}{r^2}\right] \le K^2\left[\frac{1}{4}(\rho_r + \frac{\rho}{r})^2 + \frac{\rho_\theta{}^2}{r^2}\right]$a.e

Or equivalently,

$$\frac{2(r^2\rho_r{}^2 + \rho^2 + \rho_\theta{}^2)}{4r\rho_r\rho} \le \frac{K^2 + 1}{1 - K^2} = \alpha\ a.e$$

Then, $\frac{1}{2}\left[\frac{r\rho_r}{\rho} + \frac{\rho}{r\rho_r}\right] \le \alpha\ a.e$

And thus,$\left(\frac{r\rho_r}{\rho}\right)^2 - 2\alpha\left(\frac{r\rho_r}{\rho}\right) + 1 \le 0\ a.e$

$$\frac{\rho_r}{\rho}\left[\left(\frac{r\rho_r}{\rho}\right) - 2\alpha r\right] + 1 \le 0$$

$$\left(\frac{r\rho_r}{\rho}\right)^2 + 1 \le 2\alpha r\frac{\rho_r}{\rho}$$

$$\frac{1}{Kr} \le \frac{\rho_r}{\rho} \le \frac{K}{r} \quad a.e$$

Or equivalently

$$\frac{1}{Kr} \le \frac{\partial}{\partial}In\ \rho \le \frac{K}{r} \quad a.e$$

We choose any fixed $\theta \in (0,\pi)$ such that $In\ \rho(r,\theta)$ is absolutely continuous on r and
We interval [1, R] to get

$$\int_1^R \frac{1}{Kr}dr \le \int_1^R \frac{\partial}{\partial r} In\rho dr \le \int_1^R \frac{K}{r}dr$$

Thus, $\frac{1}{K} Inr|_1{}^R \le In\rho(r,\theta) \le KInr|_1{}^R$ for almost every $\theta \in (0,\pi) and\ R \in (1,\infty)$

By an argument of continuity of 'f' and density We finally obtain, $R^{\frac{1}{K}} \le \frac{\rho(R,\theta)}{\rho(1,\theta)} \le R^K$ for all (R,θ) $\in (1,\infty) \times (0,\pi)$. In a similar way, if we suppose that 0<R<1

$$R^K \le \frac{\rho(R,\theta)}{\rho(1,\theta)} \le R^{\frac{1}{K}} for\ all\ (R,\theta) \in (0,1) \in (0,\pi)$$

THEOREM 3.2:

Let f be a K- quasiconformal mapping from $\mathbb{H}$ onto itself that leaves invariant each ray in $\mathbb{H}$ that meets a real base point of E⊂ $\mathbb{H}$ is a measurable set then,

$$\frac{1}{K} A_H(E) \le A_H(f(E)) \le KA_H(E).$$

These inequalities are asymptotically sharp when K→ 1^+

DEFINITION 3.2:

A mapping $f: \mathbb{D} \to \mathbb{D}$ is said to be **angular** at $0 \in \mathbb{D}$ if 'f' leaves invariant each
Circle in $\mathbb{D}$ with center at 0.

An angular mapping f at 0 can be written as $\boldsymbol{f(z) = f(re^{i\theta}) = re^{i\varphi(r,\theta)}}$,

Where, $\boldsymbol{\varphi}: [\mathbf{0},\mathbf{1}) \times [\mathbf{0},\mathbf{2\pi}] \to \mathbb{R}$

LEMMA 3.1:

Let f be a ACL mapping from $\mathbb{D}$ onto itself suppose that f is angular at 0.then its Jacobian is $J_f = \varphi_\theta$. If 'f' preserves orientation then, $\varphi_\theta > 0\ a.e$

Proof: Since, $f(z) = f(re^{i\theta}) = re^{i\varphi(r,\theta)}$

Then, $f(re^{i\theta}) = u(re^{i\theta}) + iv(re^{i\theta})$, $f(re^{i\theta}) = u(r,\theta) + iv(r,\theta)$

$$u(r,\theta) = rcos\varphi(r,\theta)$$
$$v(r,\theta) = rsin\varphi(r,\theta)$$

We know that, $J_f = \frac{1}{r}[u_r v_\theta - v_r u_\theta]$

$$J_f = \frac{1}{r}[rsin\varphi(r,\theta) - rcos\varphi(r,\theta)]$$
$$J_f = sin\varphi(r,\theta) - cos\varphi(r,\theta)$$

$\therefore f$ Is angular at o. then its Jacobian is $J_f = \varphi_\theta$

PROPOSITION 3.3:

Let f be a K-quasiconformal mapping from $\mathbb{D}$ onto itself which is angular at 0.then,

$$\frac{1}{K} \le \varphi_\theta \le K\ a.e\ in\ [0,1) \times [0,2\pi]$$

Proof:

If $f(z) = f(re^{i\theta}) = re^{i\varphi(r,\theta)}$
From equation 1 & 2

$$\left.\begin{aligned} 4|f_z|^2 &= (1+\varphi_\theta)^2 + r^2{\varphi_r}^2 \text{ a.e} \\ 4|f_{\bar z}|^2 &= (1-\varphi_\theta)^2 + r^2{\varphi_r}^2 \text{ a.e} \end{aligned}\right\} \quad (3.6)$$

Since, $\frac{|f_{\bar z}|^2}{|f_z|^2} \le \frac{(1-\varphi_\theta)^2}{(1+\varphi_\theta)^2} \le K^2, \frac{(1-\varphi_\theta)^2}{(1+\varphi_\theta)^2} \le \frac{|f_{\bar z}|^2}{|f_z|^2} \le K^2$ a.e

Taking square on both side $\frac{1-\varphi_\theta}{1+\varphi_\theta} \le \frac{|f_{\bar z}|}{|f_z|} \le K$

$$\frac{1}{k} \le \varphi_\theta \le K\ a.e \qquad (3.7)$$

THEOREM 3.3:

Let f be a K-quasiconformal mapping from $\mathbb{D}$ onto itself which is angular at 0.if E⊂ $\mathbb{H}$ is a measurable set then,

$$\frac{1}{K} A_H(E) \le A_H(f(E)) \le KA_H(E)$$

These inequalities are asymptotically sharp when K→ 1^+

4. PARTIAL DERIVATIVES OF K- QUASICONFORMAL MAPPING

THEOREM 4.1:

Let $1 \leq K < \infty$.if $f : \Omega \to \mathbb{C}$ is a K-quasiconformal mapping by $f(x + iy) = u(x) + iv(y)$

Then its partial derivative belong to one of the angular region defined by

$$u_x{}^2 + v_y{}^2 - 2\alpha u_x v_y \leq 0 \; a.e$$

Proof:

Then by $f(x + iy) = u(x, y) + iv(x, y)f$ Satisfies inequality if and only if

$$u_x{}^2 + u_y{}^2 + v_x{}^2 + v_y{}^2 - 2 \propto u_x v_y + 2 \propto v_x u_y \leq 0 \; a.e \quad (4.1)$$

In equation (1) its partial derivatives satisfy the inequality

$$u_x{}^2 + v_y{}^2 - 2 \propto u_x v_y \leq 0 \quad a.e \quad (4.2)$$

Since, $\alpha \geq 1$ The discriminant of $u_x{}^2 + v_y{}^2 - 2 \propto u_x v_y$ is non-negative and equation (4.2) defines the interior of an angular region with the identification $u_x \sim x - axis$ and $v_y \sim y - axis$.That the Jacobian of 'f' is$\boldsymbol{J_f = u_x v_y}$ Is always positive

GENERAL CASE:

If f is a K-quasiconformal mapping given by $f(x + iy) = u(x, y) + iv(y)$

Then reduces to, $u_x{}^2 + u_y{}^2 + v_y{}^2 - 2 \propto u_x v_y \leq 0 \quad a.e$. Inequality suggests studying the quadratic from Q (x, y, w) $= x^2 + y^2 + w^2 - 2\alpha xw$ .Whose, associated symmetric matrix is

$$N = \begin{pmatrix} x^2 & xy & xw \\ xy & y^2 & yw \\ xw & wy & z^2 \end{pmatrix}, N = \begin{pmatrix} 1 & 0 & -\propto \\ 0 & 1 & 0 \\ -\propto & 0 & 1 \end{pmatrix}$$

PROPOSITION 4.1:

There exists an invertible matrix P such that$P^{-1}NP = D$. WhereD= $\begin{pmatrix} 1-\propto & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1+\propto \end{pmatrix}$

Proof:

The matrix N is,$N = \begin{pmatrix} 1 & 0 & -\propto \\ 0 & 1 & 0 \\ -\propto & 0 & 1 \end{pmatrix}$

To find Eigen values of matrix N,Det$(N - \lambda \mathrm{I}) = 0$

$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$and $\lambda I = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}$,$N - \lambda I = \begin{bmatrix} 1-\lambda & 0 & -\alpha \\ 0 & 1-\lambda & 0 \\ -\alpha & -0 & 1-\lambda \end{bmatrix}$

$$\det\,(N - \lambda I) = \begin{vmatrix} 1-\lambda & 0 & -\alpha \\ 0 & 1-\lambda & 0 \\ -\alpha & -0 & 1-\lambda \end{vmatrix}$$

det ($N - \lambda I$)= $(1 - \lambda)[(1 - \lambda)^2 - \alpha^2]$,det ($N - \lambda I) = 0$

$$\lambda = 1, \lambda = (1 \pm \alpha),$$

The Eigen values of N are $\lambda_1 = 1 - \alpha, \lambda_2 = 1, \lambda_3 = 1 + \alpha$

The Eigen vector is(N-$\lambda I)x = 0$, $\lambda_1 = 1 - \alpha$

$$\begin{bmatrix} -\alpha & 0 & -\alpha \\ 0 & -\alpha & 0 \\ -\alpha & 0 & -\alpha \end{bmatrix} \begin{bmatrix} x \\ y \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

By Cramer rule: $\frac{x}{-\alpha^2} = \frac{y}{0} = \frac{w}{\alpha^2} \Rightarrow (x, y, w) = (1,0,-1)$

Eigen vector is $(1,0,-1)$. The norm of Eigen vector is $\|v_1\| = \sqrt{1 + 0 + 1} = \sqrt{2}$

$$\frac{1}{\sqrt{2}}\|v_1\| = \frac{1}{\sqrt{2}}(1, \quad 0, -1)$$

$$= \left(\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}}\right)$$

Eigen vector $(1, 0, -1)$with normalized$\left(\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}}\right)$

By Cramer rule we get $(x, y, w) = (0,1,0)$. The norm of $(0,1,0)$is **(0, 1, 0).**

By Cramer rule we get$(x, y, w) = (1,0,1)$.The norm of$(1,0,1)$ is $\left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)$

$$P = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \end{bmatrix} = P^{-1}$$

$$P^{-1}N = \begin{bmatrix} \frac{1-\alpha}{\sqrt{2}} & 0 & \frac{1-\alpha}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1+\alpha}{\sqrt{2}} & 0 & \frac{-1-\alpha}{\sqrt{2}} \end{bmatrix}$$

$P^{-1}NP = \begin{pmatrix} 1-\alpha & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1+\alpha \end{pmatrix} = D$ Where, **D** = $\begin{pmatrix} \mathbf{1-\alpha} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{1+\alpha} \end{pmatrix}$

PROPOSITION 4.2:

Let $1 \leq K < \infty$.if $f: \Omega \to \mathbb{C}$ is a K-quasiconformal mapping by $f(x+iy) = u(x) + iv(y)$ then its partial derivative u_x, u_y, v_ybelong to one branch of the elliptic cone

$$u_x{}^2 + u_y{}^2 + v_y{}^2 - 2\alpha u_x v_y \leq 0 \; a.e$$

Proof:

As we saw f is K- quasiconformal mapping iffu_x, u_y, v_y satisfy Q $(u_x, u_y, v_y) \leq 0$

That describes the solid cone;$u_x{}^2 + u_y{}^2 + v_y{}^2 - 2\alpha u_x v_y \leq 0$ .as f preserves orientation then,

$$J_f = u_x v_y > 0 \; a.e$$

Since, $v_y > 0$ $then$ $necessarily$ $u_x > 0$ $a.e$

GENERAL CASE:

That f is a quasiconformal mapping $f: \Omega \subset \mathbb{C} \to \mathbb{C}$ given by $f(x+iy) = u(x,y) + iv(x,y)$ then by $u_x{}^2 + u_y{}^2 + v_x{}^2 + v_y{}^2 - 2\alpha u_x v_y + 2\alpha u_y v_x \leq 0$a.e.

In the case we study the quadratic formQ (x, y, z, w) = $x^2 + y^2 + z^2 + w^2 - 2\alpha xw + 2\alpha yz$

With the associated symmetric matrix$N = \begin{pmatrix} 1 & 0 & 0 & -\alpha \\ 0 & 1 & \alpha & 0 \\ 0 & \alpha & 1 & 0 \\ -\alpha & 0 & 0 & 1 \end{pmatrix}$

PROPOSITION 4.3:

There exists an invertible matrix P such that $P^{-1}NP = D$ $where,$

$$D = \begin{pmatrix} 1-\alpha & 0 & 0 & 0 \\ 0 & 1-\alpha & 0 & 0 \\ 0 & 0 & 1+\alpha & 0 \\ 0 & 0 & 0 & 1+\alpha \end{pmatrix}$$

Proof:

The characteristic polynomial of the matrix N isdet$(N - \lambda I) = 0$

$$\begin{vmatrix} 1-\lambda & 0 & 0 & -\alpha \\ 0 & 1-\lambda & \alpha & 0 \\ 0 & \alpha & 1-\lambda & 0 \\ -\alpha & 0 & 0 & 1-\lambda \end{vmatrix} = 0$$

The characteristic polynomial is $(1-\lambda)^4 - \alpha^2(1-\lambda)^2 - \alpha^2(1-\lambda)^2 + \alpha^4 = 0$ with Eigen value $\lambda_1 = 1+\alpha, \lambda_2 = 1-\alpha$ and both with multiplicity two. The Eigen vector is$(N - \lambda I) = 0$ put the value$\lambda_1 = 1+\alpha, \lambda_2 = 1-\alpha$. The Eigen vectors are

$\lambda_1 (1,0,0,-1)$ *and* $(0,1,1,0)$*and for*λ_2 *are* $(1,0,0,1)\&(0,1,-1,0)$. After normalization we obtain the matrix.

$$P = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} & 0 \end{pmatrix} \text{ with inverse } P^{-1} = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & 0 & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & 0 & 0 & -\frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{pmatrix}$$

$$\text{Thus } P^{-}NP = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & 0 & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & 0 & 0 & -\frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & -\alpha \\ 0 & 1 & \alpha & 0 \\ 0 & \alpha & 1 & 0 \\ -\alpha & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} & 0 \end{pmatrix}$$

$$P^{-1}NP = \begin{pmatrix} 1-\alpha & 0 & 0 & 0 \\ 0 & 1-\alpha & 0 & 0 \\ 0 & 0 & 1+\alpha & 0 \\ 0 & 0 & 0 & 1+\alpha \end{pmatrix}$$

$$\therefore \boldsymbol{P^{-1}NP = D}$$

THEOREM 4.2:

The quadratic form

$\widehat{Q}\,(\hat{x},\hat{y},\hat{z},\widehat{w}) = (1-\alpha)\hat{x}^2 + (1-\alpha)\hat{y}^2 + (1+\alpha)\hat{z}^2 + (1+\alpha)\widehat{w}^2$Represents the quadratic form Q $(x,y,z,w) = x^2 + y^2 + z^2 + w^2 - 2\alpha xw + 2\alpha yz$ in the basis

$c = \{\left(\frac{1}{\sqrt{2}}, 0,0. -\frac{1}{\sqrt{2}}\right), \left(0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right), \left(\frac{1}{\sqrt{2}}, 0,0, \frac{1}{\sqrt{2}}\right), \left(0, \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0\right)\}$

Where,

$$\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} & 0 \end{pmatrix} \begin{pmatrix} \hat{x} \\ \hat{y} \\ \hat{z} \\ \widehat{w} \end{pmatrix}$$

In particular Q (x, y, z, w)≤ 0 *iff* $\hat{Q}(\hat{x},\hat{y},\hat{z},\widehat{w}) \leq 0$

Proof:

$$\text{We have the relations } \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}}(\hat{x}+\hat{z}) \\ \frac{1}{\sqrt{2}}(\hat{y}+\widehat{w}) \\ \frac{1}{\sqrt{2}}(\widehat{w}-\hat{y}) \\ \frac{1}{\sqrt{2}}(\hat{x}-\hat{z}) \end{pmatrix}$$

Thus,$Q(x,y,z,w) = x^2 + y^2 + z^2 + w^2 - 2\alpha xw + 2\alpha yz$

$$= (1-\alpha)\hat{x}^2 + (1-\alpha)\hat{y}^2 + (1+\alpha)\hat{z}^2 + (1+\alpha)\widehat{w}^2$$

$$Q(x,y,z,w) = \widehat{Q}(\hat{x},\hat{y},\hat{z},\widehat{w})$$

And this means that $Q(x,y,z,w) \leq 0 \; iff \; \hat{Q}(\hat{x},\hat{y},\hat{z},\hat{w}) \leq 0$.

CONCLUSION

The classes of mapping introduced in this paper have precise geometrical mapping in particular the class of quasiconformal mapping $(z) = u(x,y) + iv(y)$. We have obtained left and right asymptotic bound for the hyperbolic or Euclidean area distortion. Moreover the example showed that the different classes of mapping defined in the paper are not empty and coincides or not with the region of variation of the partial derivatives of quasiconformal mapping $f(z) = u(x,y) + iv(y)$.

BIBLIOGRAPHY:

1. Astala, K: Area distortion of quasiconformal mappings. Acta Math.173, 37-60(1994)

2. Porter, RM,Resendis,LF:Quasiconformallyexplodablesets. Complex var.Theory Appl.36, 379-392(1998)

3. Emerenko, A, Hamilton, DH: On the area distortion by quasiconformal mapping Proc.Am.Math.Soc.123, 2793 – 2797(1995)

4. Chen, X, Qlan, T: Estimate of hyperbolically partial derivatives of ρ – harmonic quasiconformal mapping and its applications. Complex var.Elliptic Equ 60, 875 – 892(2015)

5. Dongmian, F, Xinzhong, H: Harmonic K-quasiconformal mapping from the unit disk onto half planes. Bull, Malavs Math. Soc, 39(1), 339-347(2016)

6. Kalaj, D.Mateljevi, M: Quasiconformal harmonic mapping and generalizations. In: proceeding of the ICM2010 Satellite Conference International Workshop on Harmonic and Quasiconformal Mapping (HQM2010), vol. 18, pp. 239-260 (2010)

7. Partyka. D. Sakan, K: On a asymptotically sharp variant of Heinz's inequality.Ann. Acad.Sci.Fenn, Math, 30, 167-182(2005)

8. Knezevic,M, Mateljevic, M: On the quasi-isometries of harmonic quasiconformal mapping. J.Math. Anal. Appl.334 (1), 404-413 (2007)

9. Chen, M, Chen. X: (K,$K^{|}$)-quasiconformal harmonic mapping of the upper half plane onto itself. Ann.Acad. Sci. Fenn., Math. 37, 265-276 (2012)

10. Axler, S Bourdon, P, Ramey, W:Harmonic Function Theory, pp. 1-259.Springer, New York (2001)

11. Anderson, J: Hyperbolic Geometry, pp. 1-230. Springer, London (2003)

12. Beardon, A: The Geometry of Discrete Groups, pp. 1-338. Sprionger, new york (1983)