IJCRT.ORG ISSN : 2320-2882

INTERNATIONAL JOURNAL OF CREATIVE RESEARCH THOUGHTS (IJCRT)

An International Open Access, Peer-reviewed, Refereed Journal

Some Common Fixed point Results in Cone Metric Space

***Preeti Mehta*[1], *Badrilal Bhati*[2]**

[1] Supervisor, Head of Department
Department of Mathematics & Statistics,
Bhupal Nobles' University, Udaipur
Rajasthan
[2]Research Scholar, Department of Mathematics & Statistics,
Bhupal Nobles' University, Udaipur
Rajasthan

***Abstract:* -** *In this present paper, we prove some common fixed point theorems in TVS –valued cone metric space. The extension of coincidence points and common fixed points for four mappings satisfying generalized condition without exploiting the notation of continuity of any map involved therein, in a TVS – valued cone metric space is proved. These results extend, unify and generalize several well known comparable results in the existing literature.*

Mathematics subject classification (2010):- *47H10, 54H25, 54C60, 46B40.*

Key words:- *Coincidence points, Common fixed points, TVS- valued cone metric space.*

Introduction and preliminaries

Huang and Zhang [9] generalized the concept of a metric space, replacing the set of real numbers by an ordered Banach space, and obtained some fixed point theorems for non linear mappings satisfying different contractive conditions. Subsequently, Abbas and Jungck [2] and Abbas and Rhoades [1] studies common fixed point theorems in cone metric space. Recently, Beg et al. [5] studied common fixed point of a pair of maps on topological vector space (TVS) valued cone metric space which is the larger class than that of introduced by Hang and Zhang [9]. Jungck [13] defined the pair of self mappings to be weakly compatible if they commute at their coincidence points. In this paper, common fixed point theorems for two pairs of weakly compatible maps, which are more general than $R -$ weakly commuting and compatible mappings, are obtained in the setting of cone metric spaces, without exploiting the notion of continuity. It is worth mentioning that our results do not require the assumption that the cone is normal. Our results extend and unify various comparable results in the literature ([2], [4], [7]).

The following definitions and results will be needed in the sequel.

Let E be always a topological vector space$(in\ shortly, TVS)$*. A subset P of E called a cone if and only if,* *(i)* *P is closed, non- empty and* $P \neq \{0\}$;

(ii) *If* $a, b \in R$ *with* $a, b \geq 0$ *and* $x, y \in P$, *then* $ax + by \in P$.

(iii) $P \cap (-P) = \{0\}$.

For a given cone $P \subset E$, *we define a partial ordering* $\leq$ *with respect to by* $x \leq y$ *if and only if* $y - x \in P$, *where* $x \ll y$ *means that* $y - x \in int\ P$ *(the interior of P). A cone P is said to be normal if there is a number* $K > 0$ *such that*

$$0 \leq x \leq y \ \Rightarrow \| x\| \leq \ K \|y\|, \ \forall\, x, y \in E.$$

The least positive number satisfying the above inequality is called the normal constant of P.

Recently, Rezapour and Hamlbarani [16] proved that there is no normal cone with normal constant $K < 1$ *and for all* $k > 1$, *there are cones with normal constants* $K > k$.

Definition 1.1 *Let X is a non empty set. Suppose that the mapping* $d : X \times X \to E$ *satisfies*

(a) $0 \leq d(x, y)$ *for all* $x, y \in X$ *and* $d(x, y) = 0$ *if and only if* $x = y$;

(b) $d(x, y) = d(y, x)$ *for all* $x, y \in X$;

(c) $d(x, y) \leq d(x, z) + d(z, y)$ *for all* $x, y \in X$.

Then d is called a TVS – valued cone metric on X and (X, d) *is called a TVS-valued cone metric space.*

Definition 1.2 *Let* (X, d) *be a TVS-valued cone metric space. Let* $\{x_n\}$ *be a sequence in X and* $c \in E$ *with* $0 \ll c$.

(a) *The sequence* $\{x_n\}$ *is called a Cauchy sequence if there is an N such that* $d(x_n, x_m) \ll c$ *for all* $m, n \in N$.

(b) *The sequence* $\{x_n\}$ *is said to be convergent if there exist a positive integer N and* $x \in X$ *such that* $d(x_n, x) \ll c$ *for all* $n > N$.

(c) *A cone metric space X is said to be complete if every Cauchy sequence in X is convergent in X.*

It is known that a sequence $\{x_n\}$ *converges to a point* $x \in X$ *if and only if* $d(x_n, x) \to 0$ *as* $n \to \infty$. *A subset A of X is closed if every Cauchy sequence in A has its limit point in A.*

Definition 1.3 *Let f and g be self mappings on a set X. If* $w = fx = gx$ *for some* $x \in X$, *then x is called a coincidence point of f and g, where w is called a point of the coincidence of f and g.*

Definition 1.4 *Let f and g be two self mappings defined on a set X. Then f and g are said to be weakly compatible if they commute at every coincidence point.*

Remarks 1.1 *Let E is a TVS- valued cone metric space with cone P. Then we have the following:*

(1) *If* $a \leq ha$ *for all* $a \in P$ *and* $h \in (0,1)$, *then* $a = 0$.

(2) *If* $0 \leq u \ll c$ *for all* $0 \ll c$, *then* $u = 0$.

(3) *If* $a \leq b + c$ *for all* $0 \ll c$, *then* $a \leq b$.

For more on the properties of the cone, we refer to [10].

Common Fixed Point Results

Lemma 2.1 *Let f, g, S and T be self mappings on a TVS- valued cone metric space X with a cone P having the non empty interior satisfying $f(X) \subset T(X)$ and $g(X) \subset S(X)$. Define the sequences $\{x_n\}$ and $\{y_n\}$ in X by*

$$y_{2n+1} = fx_{2n} = Tx_{2n+1} \quad and \quad y_{2n+2} = gx_{2n+1} = Sx_{2n+2} \;\; \forall\, n \geq 0.$$

Suppose that there exists $\alpha \in [0,1)$ such that

$$d(y_n, y_{n+1}) \leq \alpha d(y_{n-1}, y_n) \quad \forall\, n \geq 1. \qquad 2.1$$

Then either

(1) The pair $\{f,S\},\{g,T\}$ have coincidence points and the sequence $\{y_n\}$ converges to a point X or
(2) $\{y_n\}$ is a Cauchy sequence in X.

Moreover, if X is complete, then the sequence $\{y_n\}$ converges to a point $z \in X$ and

$$d(y_n, z) \leq \frac{\alpha^n}{1-\alpha} d(y_0, y_1), \;\; \forall\, n \geq 1. \qquad 2.2$$

Proof: *To prove of the theorem, suppose that there exists a positive integer n such that $y_{2n} = y_{2n+1}$. Then, from the definition of $\{y_n\}$, $gx_{2n-1} = Sx_{2n} = fx_{2n} = Tx_{2n+1}$ and the mappings f and S have a coincidence point x_{2n}. moreover, from (2.1) yields $y_{2n} = y_m$ for each $m > 2n$ and hence the sequence $\{y_n\}$ converges to a point in X.*

The conclusion holds if $y_{2n+1} = y_{2n+2}$ for some positive integer n.

Assume that $y_{2n} \neq y_{2n+1}$ for all $n \geq 1$. then by (2.1) we have

$$d(y_n, y_{n+1}) \leq \alpha^n\, d(y_0, y_1) \quad \forall\, n \geq 1.$$

for any $m, n \geq 1$ with $m > n$, it follows that

$$d(y_n, y_m) \leq \sum_{i=n}^{m-1} d(y_i, y_{i+1}) \leq \sum_{i=n}^{m-1} \alpha^i\, d(y_0, y_1)$$

$$d(y_n, y_m) \leq \alpha^n\, d(y_0, y_1) \sum_{j=0}^{m-n-1} \alpha^j \qquad 2.3$$

$$d(y_n, y_m) \leq \frac{\alpha^n}{1-\alpha} d(y_0, y_1),$$

Let $0 \ll c$ be given. Choose a symmetric neighborhood V of 0 such that $c + v \subseteq int\, P$. also, choose a positive integer N_1 such that

$$\frac{\alpha^n}{1-\alpha} d(y_0, y_1) \in V, \;\; \forall\, n \geq N_1.$$

Then $\frac{\alpha^n}{1-\alpha} d(y_0, y_1) \ll c$ for all $n \geq N_1$. thus, for all $m, n \geq N_1$,

$$d(y_n, y_m) \leq \frac{\alpha^n}{1-\alpha} d(y_0, y_1) \ll c$$

And so the sequence $\{y_n\}$ is a Cauchy sequence in X. If X is complete, there exists a point $z \in X$ such that $\{y_m\}$ converges to z as $m \to \infty$. Choose a positive integer N_2 such that $d(y_m, z) \ll c$ $for\ all\ m \geq N_2$. thus it follows that,

$$d(y_n, z) \leq d(y_n, y_m) + d(y_m, z)$$

$$d(y_n, z) \leq \frac{\alpha^n}{1-\alpha} d(y_0, y_1) + d(y_m, z)$$

$$d(y_n, z) \leq \frac{\alpha^n}{1-\alpha} d(y_0, y_1) + c$$

Which yields 2.2 by using remark 1 (3). This completes the proof.

Theorem 2.2 *Let f, g, S and T be self mappings of a TVS- valued cone metric space X with a cone P having the non empty interior satisfying $f(X) \subset T(X)$, $g(X) \subset S(X)$ and there exists $\alpha, \beta \in (0,1)$ such that $0 \leq \alpha + \beta < 1$and*

$$d(fx, gy) \leq \alpha \max\{d(Sx, Ty), d(Sx, fx), d(Ty, gy)\}$$

$$+ \beta \max\{d(Sx, gy), d(Ty, fx)\} \qquad 2.4$$

If one of $f(X) \cup g(X)$ and $S(X) \cup T(X)$ is complete, then the pairs $\{f, S\}$ and $\{g, T\}$ have a unique point of coincidence in X. Moreover, if the pairs $\{f, S\}$ and $\{g, T\}$ are weakly compatible, then the mapping $f, g, S,$ and T have a unique common fixed point in X.

Proof:- *For any arbitrary point x_0 in X, construct the sequence $\{x_n\}$ and $\{y_n\}$ in X such that*

$$y_{2n+1} = fx_{2n} = Tx_{2n+1} \quad and \quad y_{2n+2} = gx_{2n+1} = Sx_{2n+2} \ \forall\, n \geq 0.$$

Thus it follows from 2.4

$$d(y_{2n+1}, y_{2n+2}) = d(fx_{2n}, gx_{2n+1})$$

$$d(fx_{2n}, gx_{2n+1}) \leq \alpha \max\begin{Bmatrix} d(Sx_{2n}, Tx_{2n+1}), d(Sx_{2n}, fx_{2n}), \\ d(Tx_{2n+1}, gx_{2n+1}) \end{Bmatrix}$$

$$+ \beta \max\{d(Sx_{2n}, gx_{2n+1}), d(Tx_{2n+1}, fx_{2n})\}$$

$$d(y_{2n+1}, y_{2n+2}) \leq \alpha \max\{d(y_{2n}, y_{2n+1}), d(y_{2n}, y_{2n+1}), d(y_{2n+1}, y_{2n+2})\}$$

$$+ \beta \max\{d(y_{2n+1}, y_{2n+2}), d(y_{2n+1}, y_{2n+1})\}$$

$$d(y_{2n+1}, y_{2n+2}) \leq \alpha d(y_{2n}, y_{2n+1}) + \beta d(y_{2n+1}, y_{2n+2})$$

$$d(y_{2n+1}, y_{2n+2}) \leq \frac{\alpha}{1-\beta} d(y_{2n}, y_{2n+1})$$

suppose that, $q = \frac{\alpha}{1-\beta}$ and processing the same manner, it follows the condition of the lemma 2.1 is satisfied.

Now we show that the pairs $\{f, S\}$ and $\{g, T\}$ have coincidence points in X. In fact, without loss of generality, we may assume that $y_n \neq y_{n+1}$ for any $n \geq 1$. If we have the equality for some n, then from lemma 2.1, the pairs $\{f, S\}$ and $\{g, T\}$ have coincidence points in X. Thus, the sequence is the Cauchy sequence.

Suppose that $S(X) \cup T(X)$ *is complete. Then there exists* $u \in S(X) \cup T(X)$ *such that* $y_n \to u$ *as* $n \to \infty$*. Further, the subsequence* $\{Sx_{2n+2}\} = \{gx_{2n+1}\} = \{y_{2n+2}\}$ *and* $\{Tx_{2n+1}\} = \{fx_{2n}\} = \{y_{2n+1}\}$ *of* $\{y_{2n}\}$ *also converges to the point* u*. Now, sinceu* $\in S(X) \cup T(X)$*, we have* $u \in S(X)$ *or* $u \in T(X)$*.*

If $u \in S(X)$*, then we can find* $v \in X$ *such that* $Sv = u$ *and claim that* $fv = u$*. for this, consider*

$$d(fv,u) \le d(fv, gx_{2n+1}) + d(gx_{2n+1}, u)$$

$$d(fv,u) \le \alpha \max\left\{\begin{matrix} d(Sv, Tx_{2n+1}), d(Sv, fv), \\ d(Tx_{2n+1}, gx_{2n+1}) \end{matrix}\right\}$$

$$+ \beta \max\{d(Sv, gx_{2n+1}), d(Tx_{2n+1}, fv)\} + d(gx_{2n+1}, u)$$

as $n \to \infty$ *, we have,*

$$d(fv,u) \le (\alpha + \beta) d(fv,u)$$

Which contradiction, so that $fu = Sv = u$ *and so, since* $u \in f(X) \subset T(X)$*, there exists* $w \in X$ *such that* $Tw = u$*.*

Now, we show that $gw = u$*. In fact consider*

$$d(gw,u) \le d(gw, fx_{2n}) + d(fx_{2n}, u)$$

$$d(gw,u) \le \alpha \max\{d(Sx_{2n}, Tw), d(Sx_{2n}, fx_{2n}), d(Tw, gw)\}$$

$$+ \beta \max\{d(Sx_{2n}, gw), d(Tw, fx_{2n})\} + d(fx_{2n}, u)$$

as $n \to \infty$ *, we have,*

$$d(gw,u) \le (\alpha + \beta) d(gw,u)$$

Which contradiction, so that $gw = u$*. similarly arguments to those given above, we obtain* $gw = Tw = u$*. Thus the pairs* $\{f, S\}$ *and* $\{g, T\}$ *have common fixed point of coincidence in X.*

Now, if the pairs $\{f, S\}$ *and* $\{g, T\}$ *are weakly compatible,* $fu = fSu = Sfu = Su = w_1$ *(say) and* $gu = gTw = Tgw = Tu = w_2$*(say). Now, we have*

$$d(w_1, w_2) = d(fu, gu) \le \alpha \max\{d(Su, Tu), d(Su, fu), d(Tu, gu)\}$$

$$+ \beta \max\{d(Su, gu), d(Tu, fu)\}$$

Which implies, $w_1 = w_2$ *and hence* $fu = gu = Su = Tu$*, i.e the point* u *is coincidence point of the pairs* $\{f, S\}$ *and* $\{g, T\}$*.*

Now, we show that $u = gu$*.*

In fact, we have

$$d(u, gu) = d(fv, gu) \le \alpha \max\{d(Sv, Tu), d(Sv, fv), d(Tu, gu)\}$$

$$+ \beta \max\{d(Sv, gu), d(Tu, fv)\}$$

This implies,

$$d(u, gu) \leq (\alpha + \beta)d(u, gu)$$

Which contradiction, so that $gu = u$ and hence u is a common fixed point of the mappings f, g, S, T.

Finally, for the uniqueness of the point u, suppose that u' is also a common fixed point of f, g, S and T, from 2.4, we have

$$d(fu, gu') \leq \alpha \max\{d(Su, Tu'), d(Su, fu), d(Tu', gu')\}$$

$$+ \beta \max\{d(Su, gu'), d(Tu', fu)\}$$

which implies $u = u'$, and hence u is unique common fixed point of $f, g, S,$ and T, in X.

Suppose that $f(X) \cup g(X)$ is complete and $u \in T(X)$. then similarly we can prove that f, g, S and T, have unique common fixed point in X.

Corollary 2.3 *Let f, g, S and T be self mappings of a TVS- valued cone metric space X with a cone P having the non empty interior satisfying $f(X) \subset T(X)$, $g(X) \subset S(X)$ and there exists $\alpha, \beta \in (0,1)$ such that $0 \leq \alpha + \beta < 1$ and $m, n \geq 1$*

$$d(f^m x, g^n y) \leq \alpha \max\{d(S^m x, T^n y), d(S^m x, f^m x), d(T^n y, g^n y)\}$$

$$+ \beta \max\{d(S^m x, g^n y), d(T^n y, f^m x)\} \qquad 2.4$$

If one of $f(X) \cup g(X)$ and $S(X) \cup T(X)$ is complete, then the pairs $\{f, S\}$ and $\{g, T\}$ have a unique point of coincidence in X. Moreover, if the pairs $\{f, S\}$ and $\{g, T\}$ are weakly compatible, then the mapping $f, g, S,$ and T have a unique common fixed point in X.

Proof

It follows from Theorem 2.2 that $\{f^m, S^m\}$ and $\{g^n, T^n\}$ have a unique common fixed point $p \in X$. Now, we have

$$f(p) = f(f^m(p)) = f^{m+1}(p) = f^m(f(p))$$

$$S(p) = S(S^m(p)) = S^{m+1}(p) = S^m(S(p))$$

And so $f(p)$ and $S(p)$ are also fixed point for the mappings f^m and S^m. Hence $f(p) = S(p) = p$. by using the same argument in the proof of Theorem 2.2, we obtain $g(p) = T(p) = p$.

This completes the proof.

REFERENCES

1. *M. ABBAS AND B. E. RHOADES, Fixed and periodic point results in cone metric spaces, Appl. Math. Lett.* ***22*** *(2009), 511–515.*
2. *M. ABBAS AND G. JUNGCK, Common Fixed point results of noncommuting Mappings without continuity in cone metric spaces, J. Math. Anal. Appl.* ***341*** *(2008), 418–420.*
3. *M. ABBAS, YEOL JE CHO AND TALAT NAZIR, Common fixed point theorems for four mappings in TVS- Valued cone metric spaces*
4. *M. ARSHAD, A. AZAM AND P. VETRO, Some Common Fixed Point Results in Cone Metric Spaces, Fixed Point Theory and Appl. (2009), Article ID 493965, 11 pages.*

5. I. BEG, A. AZAM AND M. ARSHAD, Common fixed points for maps on topological vector space valued cone metric spaces, Internat. J. Math. Math. Sci. **2009** (2009), 8 pages, Article ID 560264.
6. C. DI BARI AND P. VETRO, ϕ-pairs and common fixed points in cone metric spaces, Rend. Circ. Mat. Palermo **57** (2008), 279–285.
7. B. FISHER, Four mappings with a common fixed point, J. Univ. Kuwait Sci. **8** (1981), 131–139.
8. R. H. HAGHI AND SH. REZAPOUR, Fixed points of multifunctions on regular cone metric spaces,Expo. Math. (2009), dio:10.1016/j.exmath. 2009.04.001.
9. L. G. HUANG AND X. ZHANG, Cone metric spaces and fixed point theorems of Contractive mappings, J. Math. Anal. Appl. **332** (2007), 1467–475.
10. D. ILIC AND V. RAKOCEVIC, Common fixed points for maps on cone metric space, J. Math. Anal. Appl. **341** (2008), 876–882.
11. S. JANKOVI´C , Z. KADELBURG, S. RADENOVI´C AND B. E. RHOADES, Assad-Kirk- Type Fixed Point Theorems for a Pair of Nonself Mappings on Cone Metric Spaces, Fixed Point Theory and Appl. Vol. **2009** , Article ID 761086, 16 pages, doi:10.1155/2009/761086.
12. G. S. JEONG AND B. E. RHOADES, Some remarks for improving fixed point theorems for more than two maps, Indian J. Pure Appl. Math. **28** (1997), 1177–1196.
13. G. JUNGCK, Common xed points for noncontinuous nonself maps on non-metric spaces, Far East J. Math. Sci. **4** (1996), 199–215.
14. S. RADENOVICH AND B. E. RHOADES, Fixed point theorem for two non-self mappings in cone metric spaces, Comput. Math. Appl. **57** (2009), 1701–1707.
15. S. REZAPOUR, A review on topological properties of cone metric spaces, Analysis, Topology and Applications 2008, Vrnjacka Banja, Serbia, from May 30 to June 4, 2008.
16. S. REZAPOUR AND R. HAMLBARANI, Some note on the paper “Cone metric spaces and fixed point theorems of contractive mappings”, J. Math. Anal. Appl. **345** (2008), 719–724.
17. H. H. SCHAEFER, Topological Vector Spaces, Vol. 3 of Graduate Texts in Mathematics, Springer, New York, USA, 1971.