IJCRT.ORG ISSN : 2320-2882

INTERNATIONAL JOURNAL OF CREATIVE RESEARCH THOUGHTS (IJCRT)

An International Open Access, Peer-reviewed, Refereed Journal

α*-Normal Spaces

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Abstract: In this paper, we established and study a new class of spaces, called α*-normal spaces. The relationships among normal, α-normal and α*-normal spaces are investigated. Moreover, established some functions related with α*-normal spaces and obtain some preservation theorems of α*-normal spaces.

Keywords: α*-open**,** α-open**,** α*-normal, α-normal spaces.

2010 AMS Subject Classification: 54A05, 54C08, 54C10, 54D15.

1. Introduction

In 1937, Stone **[11]** established the concept of regular-open sets. In 2014, Thakur C. K. Raman et al. **[12]** established the concepts of α***-**generalized and α***-**separation axioms in topological spaces and obtained their properties. In 2020, Hamant Kumar **[4]** established the concepts of ii-normal spaces and its properties.

2. Preliminaries

2.1 Definition. A subset A of a topological space X is said to be

1. **regular open [11]** if A = int(cl(A)).

2. **α-open [9]** if cl(int (cl(A))) ⊂ A.

3. **α*-open [12]** if F ⊂ α-int(A) whenever F is αg-closed and F ⊂ A.

The complement α*-closed set is α*-open set.

4. The complement of regular open (resp. **α*-open, α-open**) set is said to be **regular closed** (resp.**α*-closed, α-closed**). The intersection of all α*-closed sets containing A is called the **α*-closure of A** and denoted **α*-cl**(**A**).The union of all α*-open subsets of X which are contained in A is called the **α*-interior of A** and denoted by **α*-int**(**A**).

The family of all regular open (resp. regular closed, α-closed, α*-closed) sets of a space X is denoted by R-O(X) (resp. R-C(X), α-C(X), α*-C(X))).

We have the following implications for the properties of subsets :

open ⇒ α-open ⇒ α*-open

Where none of the implications is reversible as can be seen from the following examples.

2.2. Example. Let X = {a, b, c, d} and τ = {X, ϕ,{a},{b},{a, b},{a, b, c}}. Then

1) closed sets in X are X, ϕ, {d}, {c, d}, {a, c, d}, {b, c, d}.

2) α-open sets in X are ϕ, X, {a}, {b}, {a, b}, {a, b, c} {a, b, d}.

3) α*-open sets in X are ϕ, X, {a}, {b}, {a, b}, {a, d}, {b, c}, {b, d}, {a, b, c}, {a, b, d}. Here we see that {b, c} is α*-open but neither α-open nor open.

2.3. Example. Let X = {a, b, c, d} and τ = {ϕ, {a}, {c}, {a, b}, {a, c},{a, b, c}, X }. Then

1) closed sets in X are X, ϕ, {d}, {b, d}, {c, d}, {a, c, d}, {b, c, d}.

2) α-open sets in X are ϕ, X, {a}, {b}, {a, b}, {a, c}, {a, b, c}, {a, b, d}.

3) α*-open sets in X are X, ϕ, {a}, {b}, {a, b}, {a, c}, {a, d}, {b, c},{b, d}, {a, b, c}, {a, b, d}, {a, c, d}, {b, c, d}.

Here {a, b, d} is α-open as well as α*-open but not open.

3. α*-Normal Spaces

3.1. Definition. A topological space X is said to be **α*-normal** if for every pair of disjoint closed subsets H, K, there exist disjoint α*-open sets U, V of X such that H ⊂ U and K ⊂ V.

3.2. Definition. A topological space X is said to be **α-normal [1]** if for every pair of disjoint closed subsets H, K, there exist disjoint α-open sets U, V of X such that H ⊂ U and K ⊂ V.

3.3. Definition. A topological space X is said to be **quasi α*-normal [7]** if for every pair of disjoint π-closed subsets H, K, there exist disjoint α*-open sets U, V of X such that H ⊂ U and K ⊂ V.

3.4. Definition. A topological space X is said to be **almost α*-normal [6]**if for any two disjoint closed subsets A and B of X, one of which is regular closed there exist disjoint α*-open sets U and V of X such that A ⊂ U and B ⊂ V.

3.5. Definition. A topological space X is said to be **mildly α*-normal [8]** if for any two disjoint regular closed subsets A and B of X, there exist disjoint α*-open sets U and V of X such that A ⊂ U and B ⊂ V.

3.6. Example. Let X = {a, b, c, d} and τ = {ϕ, {a}, {c}, {a, c},{ b, d}, {a, b, d}, {b, c, d}, X }. The pair of disjoint closed subsets of X are A = ϕ and B = {c}. Also U = {a} and V = {b, c, d} are disjoint α*-open sets such that A ⊂ U and B ⊂ V. Hence X is normal as well as α*-normal because every open set is α*-open set.

3.7. Example. Let X = {a, b, c} and τ = {ϕ, {a}, {b, c}, X}. The pair of disjoint closed subsets of X are A = {a} and B = {b, c}. Also U = {a} and V = {b, c} are open sets such that A ⊂ U and B ⊂ V. Therefore X is normal as well as almost normal.

3.8. Example. Let X = {a, b, c} and τ = {ϕ, {a}, {a, b}, {a, c}, X}. Then X is almost normal but it is not normal.

3.9. Example. Let X = {a, b, c, d} and τ = {ϕ, {a}, {c}, {a, c}, {a, d}, {a, b, c}, {a, b, d}, {a, c, d}, X}. The pair of disjoint closed subsets of X are A = {b} and B = {c}. Also U = {a, b} and V = {c} are disjoint α*- open sets such that A ⊂ U and B ⊂ V. Hence X is α*-normal.

3.10. Example. Let X = {a, b, c, d} and τ = {ϕ, {a}, {b}, {c}, {a, b}, {a, c}, {b, c}, {a, b, c}, {a, b, d}, X}. The pair of regularly disjoint closed subsets of X are A = {b, d} and B = {c}. Also U = {a, b, d} and V = {c} are disjoint α*- open sets U and V such that A ⊂ U and B ⊂ V. Hence X is mildly α*-normal.

3.11. Example. Let X = {a, b, c, d} and τ = {ϕ, {a}, {b}, {a, b}, {c, d}, {a, c, d}, {b, c, d}, X}. The pair of disjoint π-closed subsets of X are A = {a} and B = {b}. Also U = {a} and V = {b} are disjoint α*-open sets such that A ⊂ U and B ⊂ V. Hence X is quasi α*-normal.

By the definitions and examples stated above, we have the following diagram:

normal ⇒ α-normal ⇒ α*-normal ⇒ quasi α*-normal ⇒ almost α*-normal ⇒ mildly α*-normal

3.9. Theorem. For a topological space X, the following are equivalent :

(a) X is α*-normal.

(b) For any pair of open sets U and V whose union is X, there exist α*-closed sets A and B such that A ⊂ U and B ⊂ V and A union B = X.

(c) For any closed set F and each open set E, there exists an α*-open set G such that F ⊂ G ⊂ α*-cl(G) ⊂ E.

Proof. (a) ⇒ (b), (b) ⇒ (c), (c) ⇒ (a).

(a) ⇒ (b). Let U and V be any pair of open sets in a α*-normal space X = U union V. Then X – U and X – V are disjoint closed sets of X. By assumption, there exist disjoint α*-open sets U_1, V_1 such that X –U ⊂ U_1 and X – V ⊂ V_1. Let A = X – U_1 and B = X – V_1. Then A and B are α*-closed sets such that A ⊂ U, B ⊂ V and A union B = X.

(b) ⇒ (c). Let F is a closed set and E is an open set containing F. Then X – F and E be open sets whose union is X. Then by (b), there exist α*-closed sets F_1 and F_2 such that F_1 ⊂ X – F and F_2 ⊂ E and F_1 union F_2 = X. Then F ⊂ X – F_1 and X– E ⊂ X – F_2 and (X – F_1) ∩ (X – F_2) = ϕ. Let U = (X – F_1) and V = (X – F_2). Then U and V are disjoint open sets such that F ⊂ U ⊂ X – V ⊂ E. As X – H is α*-closed set, we get α*-cl(U) ⊂ X – H and F ⊂ U ⊂ α*-cl(U) ⊂ E.

(c) ⇒ (a). Let C_1 and C_2 are any two disjoint closed sets of X. Put E = X – C_2 then C_2 ∩ E = ϕ. C_1 ⊂ E, where E is an open set. Then by (c), there exists a α*-open set U of X such that C_1 ⊂ U ⊂ α*-cl(U) ⊂ E. It follows that C_2 ⊂ X – α*-cl(U) = V, say, then V is α*-open and G ∩ H = ϕ. Therefore C_1 and C_2 are separated by α*-open sets U and V. Hence X is α*-normal.

4. Some functions with α*-normal spaces

4.1. Definition. Let (X, τ) be a topological space. A subset L ⊂ X is said to be an α*-neighbourhood (briefly α*-nhd) of a point x belongs to X if there exists an α*-open set M such that x belongs to M ⊂ L.

4.2. Definition. A function f : X → Y is said to be

1) **R-map [3]** if f^{-1}(A) be a regular open in topological space X for each regular open set A of Y.

2) **completely continuous [2]** if f^{-1}(A) be a regular open in X for each open set A of Y.

3) **rc-continuous [5]** if for every regular closed set B in Y, then f^{-1}(B) be a regular closed in X.

4.3. Definition. A function f : X → Y is said to be

1) **softly α*-open** if f(G) belongs to α*-O(Y) for every G belongs to α*-O(X).

2) **softly α*-closed** if f(G) belongs to α*-C(Y) for every G belongs to α*-C(X).

3) **almost α*-irresolute** if for each p belongs to X and every α*-neighbourhood N of f(x), α*-cl(f^{-1}(N)) is a α*-neighbourhood of p.

4.4. Theorem. A function f : X → Y be a softly α*-closed iff for every subset A in Y and for every α*-open set G in X containing f^{-1}(A), there exists an α*-open set H containing A such that $f^{-1}(H) \subset U$.

Proof. Let f is softly α*-closed set. Let A be a subset of Y and G belongs to α*-O(X) containing f^{-1}(A). Put H = Y – f(X – G), then H is a α*-open set of Y such that $A \subset H$ and $f^{-1}(H) \subset U$.

4.5 Lemma. For a function f : X → Y, the following are equivalent:

1) f is almost α*-irresolute.

2) $f^{-1}(V) \subset$ int α*(int-cl(f^{-1}(V))) for each V belong to int-O(Y).

4.6. Theorem. A function f : X → Y is almost int-irresolute iff f(int-cl(U)) ⊂ int-cl(f(U)) for each U belongs to int-O(X).

Proof. Let U belongs to int-O(X). Let y does not belong to int-cl(f(U)). Then there exists V belongs to int-O(Y) such that $V \cap f(U) = \phi$. Therefore $f^{-1}(V) \cap U = \phi$. Since U belongs to int-O(X), we have int-α*(int-cl(f^{-1}(V))) ∩ int-cl(U) = ϕ. Then by **Lemma 4.5** $f^{-1}(V) \cap$ int-cl(U) = ϕ and so V ∩ f(int-cl(U)) = ϕ.This implies that y does not belong to f(int-cl(U)).

If V belongs to int-O(Y), then M = X – int-cl(f^{-1} (V)) belong to int-O(X). By hypothesis, f(int-cl(M)) ⊂ int-cl(f(M)) and so X – int-α*(int-cl(f^{-1}(V))) = int-cl(M) ⊂ f^{-1}(int-cl(f(M))) ⊂ f^{-1}(int-cl(f(X - f^{-1} (V))) ⊂ f^{-1}(int-cl(f(Y – V)) = f^{-1}(Y – V) = X – f^{-1} (V). Hence, $f^{-1}(V) \subset$ int-α*(int-cl(f^{-1}(V))). **By Lemma 4.5**, f is almost int-irresolute.

4.7. Theorem. If f : X → Y is a softly α*-open continuous almost α*-irresolute function from α*-normal space X onto a space Y, then Y is α*-normal.

Proof. Let F_1 be closed subset of Y and F_2 be an open set containing F_1. Then by continuity of f, $f^{-1}(F_1)$ is closed and $f^{-1}(F_2)$ is an open set of X such that $f^{-1}(F_1) \subset f^{-1}(F_2)$. Since X is α*-normal, there exists α*-open set G in X such that $f^{-1}(F_1) \subset U \subset$ α*-cl(G) $\subset f^{-1}(F_2)$ by **theorem 3.9**. Then, $f(f^{-1}(F_1)) \subset f(U) \subset$ f(α*-cl(U)) $\subset f(f^{-1}(F_2))$. Since f is softly α*-open almost α*-irresolute surjection, we obtain $F_1 \subset f(U) \subset \alpha^*\text{-}cl(f(U)) \subset F_2$.Then by **Theorem 3.9**, the space Y is α*-normal.

4.8. Theorem. If $f : X \to Y$ is a softly α*-closed continuous function from α*-normal space X onto a space Y, then Y is α*-normal.

Proof. Let F_1 and F_2 be disjoint closed sets. Then $f^{-1}(F_1)$ and $f^{-1}(F_2)$ are closed sets. Since X is α*-normal, then there exist disjoint α*-open sets G and H such that $f^{-1}(F_1) \subset G$ and $f^{-1}(F_2) \subset H$. By **Theorem 4.4**, there exist α*-open sets A and B such that $F_1 \subset A$ and $F_2 \subset B$, $f^{-1}(A) \subset G$ and $f^{-1}(B) \subset H$. Also, A and B are disjoint. Hence, Y is α*-normal.

4.9. Theorem. A function $f : X \to Y$ is said to be α-closed [**10**] if for every closed set in X, f(G) is α-closed set in Y.

4.10. Theorem. If $f : X \to Y$ is a α*-closed continuous surjection and X is normal, then Y is α*-normal.

Proof. Let F_1 and F_2 be disjoint closed sets of Y. Then $f^{-1}(F_1)$ and $f^{-1}(F_2)$ are closed sets of X by continuity of f. Since X is normal, then there exist disjoint open sets G and H such that $f^{-1}(F_1) \subset G$ and $f^{-1}(F_2) \subset H$. By **Theorem 4.3**, there exist α-open sets U and V in Y such that $F_1 \subset U$ and $F_2 \subset V$. Since each α-open set is α*-open, U and V are disjoint α*-open sets containing F_1 and F_2, respectively. Hence, Y is α*-normal.

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