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New results on Oscillation for Delay Differential Equations with Piecewise Constant Argument

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Abstract: The purpose of this paper is to use some new techniques to obtain new oscillation conditions for equation

$$y'(x)+p(x)y(x)+q(x)y([x-k])=0, x\geq 0 \qquad \textbf{(1)}$$

where $p(x)$ **and** $q(x)$ **are right continuous functions on** $[-k,\infty), k$ **is a positive integer, and** $[.]$**denotes the greatest integer function. Our results improve and generalize the known results in the literature.**

Key words: Differential equation, Piecewise constant argument, Oscillation.

1. INTRODUCTION

In this chapter we consider the delay differential equation with piecewise constant argument of the form

$$y'(x)+p(x)y(x)+q(x)y([x-k])=0, x\geq 0 \qquad (1)$$

where $p(x)$ and $q(x)$ are right continuous functions on $[-k,\infty), k$ is a positive integer, and $[.]$ denotes the greatest integer function. When $p(x)\equiv 0$ (1.1) reduces to

$$y'(x)+q(x)y([x-k])=0, x\geq 0 \qquad (2)$$

Delay differential equations with piecewise constant arguments represent continuous and discrete dynamical systems and combine the properties of both differential and difference equations. They have particular importance in control theory and certain biomedical problems. There have been many papers concerning these equations; see also [1-16]. Most of these papers deal with the constant coefficients case (autonomous equations) or the case when

$q(x) \geq 0$. In particular, there is little in the way of results for the general case where the coefficient b(t) is oscillatory.

By a solution of (1) we mean a function $y(x)$ which is defined on the set $\{-k, -k+1, \ldots, -1, 0\} \cup (0, \infty)$ and which satisfies the conditions:

(i) $y(x)$ is continuous on $[0, \infty)$;

(ii) The derivative $y'(x)$ exists at each point $x \in [0, \infty)$, with the possible exception of the points $[x] \in [0, \infty)$, where one side derivatives exist;

(iii) Equation (1.1) is satisfied on each interval $[n, n+1)$ for $n \in N(0)$, where $N(n_0) := \{n_0, n_0+1, \ldots\}$, and n_0 is any integer.

A nontrivial solution of (1) is said to be oscillatory if it has arbitrarily large zeros. otherwise, it is said to be nonoscillatory.

The purpose of this paper is to use some new techniques to obtain new oscillation conditions for equations (1) and (2) with oscillating coefficients. Our results improve and generalize the known results in the literature. Some examples are also given to demonstrate the advantage of our results.

2. MAIN RESULT

THEOREM 2.1. Assume that

(i) There exist a sequence of intervals $\{[l_n, m_n]\}_{n=1}^{\infty}$ such that l_n, m_n are integers, $m_n \leq l_{n+1}$ and $m_n - l_n \geq 2k$ for $n = 1, 2, 3 \ldots,$ and that

$$q(x) \geq 0, \text{ for } x \in \cup_{n=1}^{\infty} [l_n, m_n]; \tag{3}$$

(ii)
$$\int_0^{\infty} \bar{q}(x) \ln\left[e \int_x^{x+k} \bar{q}(t)dt + 1 - \operatorname{sgn}\left(\int_x^{x+k} \bar{q}(t)dt \right) \right] dx = \infty \tag{4}$$

where

$$\bar{q}(x) = \begin{cases} q(x), x \in \bigcup_{n=1}^{\infty} [l_n + k, m_n); \\ 0, x \in [0, l_1 + k) \cup \left(\bigcup_{n=1}^{\infty} [m_n, l_{n+1} + k,) \right). \end{cases}$$

then all solutions of the differential equations (2) Oscillates.

PROOF. Assume , for the sake of contradiction, that equation (2) has an eventually positive solution $y(x)$ Then there exists an integer $j \geq 1$ such that

$$y(x - 2k) > 0, \; for x \geq l_j. \tag{5}$$

First, we claim that

$$\int_x^{x+k} \bar{q}(t)dt \leq 1, \; for x \geq l_j. \tag{6}$$

In fact by the definition of $\bar{q}(x)$, we have

$$q(x) \geq \bar{q}(x), \text{ for } x \in \bigcup_{n=j}^{\infty} [l_n, m_n] \tag{7}$$

$$y'(x)+\bar{q}(x)y([x-k])\le 0, x\in\bigcup_{n=j}^{\infty}[l_n,m_n] \tag{8}$$

which implies that $y(x)$ is nonincreasing on $[l_n,m_n]$ for $n\ge j$. We consider four possible cases

CASE 1. $x\in\bigcup_{n=j}^{\infty}[l_n+k,m_n-k]$. From (8), we have

$$y(x+k)-y(x)+\int_x^{x+k}\bar{q}(t)y([x-k])dt\le 0,$$

which, by using the nonincreasing nature of $y(x)$ on $[l_n,m_n]$ for $n\ge i$, yields

$$y(x)\ge\int_x^{x+k}\bar{q}(t)y([x-k])dt\ge y(x)\int_x^{x+k}\bar{q}(t)dt.$$

and so

$$\int_x^{x+k}\bar{q}(t)dt\le 1 \tag{9}$$

CASE.2. $x\in\bigcup_{n=j}^{\infty}(m_n-k,m_n]$ it follows from (9) that

$$\int_x^{x+k}\bar{q}(t)dt\le\int_{m_n-k}^{x+k}\bar{q}(t)dt=\int_{m_n-k}^{m_n}\bar{q}(t)dt\le 1 \tag{10}$$

CASE.3. $x\in\bigcup_{n=j}^{\infty}[l_n,l_n+k)$. From (9), we have

$$\int_x^{x+k}\bar{q}(t)dt=\int_{l_n+k}^{x+k}\bar{q}(t)dt\le\int_{l_n+k}^{l_n+2k}\bar{q}(t)dt\le 1 \tag{11}$$

CASE.4. $x\in\bigcup_{n=j}^{\infty}[m_n,l_{n+1}]$. By the definition of $\bar{q}(x)$ we have

$$\int_x^{x+k}\bar{q}(t)dt=0. \tag{12}$$

combining cases 1-4, we see that (6) holds.

Second, we claim that there exists a sequence of real numbers $\{\xi_i\}$ such that $\lim_{i\to\infty}\xi_i=\infty$ and that

$$\limsup_{i\to\infty}\frac{y(\xi_i-k)}{y(\xi_i)}<\infty \tag{13}$$

in fact, it follows from (4) that $\limsup_{x\to\infty}\int_x^{x+k}\overline{q}(t)dt>0$, which, together with (10)-(12), yields that there exists a sequence of integers $\{n_i\}(k<n_1<n_2<\ldots)$ and $x_i\in\lfloor l_{n_i}+k, m_{n_i}-k\rfloor$ and $c>0$ such that

$$\int_{x_i}^{x_i+k}\overline{q}(t)dt\ge 2c \quad i=1,2,3\ldots$$

Thus, there exists $\xi_i\in(x_i,x_i+k), i=1,2,\ldots$ such that

$$\int_{x_i}^{\xi_i}\overline{q}(t)dt\ge c \text{ and } \int_{\xi_i}^{x_i+k}\overline{q}(t)dt\ge c \tag{14}$$

From (8), we have

$$y(\xi_i)-y(x_i)+\int_{x_i}^{\xi_i}\overline{q}(t)y([x-k])dt\le 0 \tag{15}$$

and

$$y(x_i+k)-y(\xi_i)+\int_{\xi_i}^{x_i+k}\overline{q}(t)y([x-k])dt\le 0 \tag{16}$$

Using the nonincreasing nature of $y(x)$ in $[l_{n_i}, m_{n_i}]$, from (14)-(16), we get

$-y(x_i)+cy(\xi_i-k)\le 0$ and $-y(\xi_i)+cy(x_i)\le 0$,

or

$$\frac{y(\xi_i-k)}{y(\xi_i)}\le\frac{1}{c^2}, \qquad i=1,2,3,\ldots$$

which implies that the second claim holds.

It is the time to complete the proof of theorem 2.1

Set $\lambda(x)=-\dfrac{y'(x)}{y(x)}$ for $x\ge l_j-k$. Then $\lambda(x)\ge 0$ for $x\in\bigcup_{n=j}^{\infty}[l_n,m_n]$, and

$$\lambda(x)=q(x)\exp\left(\int_{[x-k]}^{x}\lambda(t)dt\right), t\ge l_j, \tag{17}$$

or

$$\lambda(x)\int_{x}^{x+k}\overline{q}(t)dt = q(x)\left(\int_{x}^{x+k}\overline{q}(t)dt\right)\exp\left(\int_{[x-k]}^{x}\lambda(t)dt\right), x \ge l_j \tag{18}$$

it is easy to show that

$$\varphi(u)ue^{y} \ge \varphi(u)y + \varphi(u)\ln\left(eu + 1 - \operatorname{sgn} u\right),\ \text{for } u \ge 0 \text{ and } y \in R \tag{19}$$

where $\varphi(0) = 0$ and $\varphi(u) \ge 0$ for $u > 0$

We consider the two possible cases.

CASE.1. $x \in \bigcup_{n=j}^{\infty}[l_n, m_n]: \overline{q}(x)$ is right continuous because $q(x)$ is right continuous, and so $\int_{x}^{x+k}\overline{q}(t)dt = 0$ implies that $\overline{q}(x) = 0$. Employing inequality (19) in (18), we get

$$\lambda(x)\int_{x}^{x+k}\overline{q}(t)dt \ge \overline{q}(x)\int_{[x-k]}^{x}\lambda(t)dt + \overline{q}(x)\ln\left(e\int_{x}^{x+k}\overline{q}(t)dt + 1 - \operatorname{sgn}\int_{x}^{x+k}\overline{q}(t)dt\right)$$

CASE.2. $x \in \bigcup_{n=j}^{\infty}[m_n, l_{n+1}).\overline{q}(x) = 0$ and so

$$\lambda(x)\int_{x}^{x+k}\overline{q}(t)dt = \overline{q}(x)\int_{[x-k]}^{x}\lambda(t)dt + \overline{q}(x)\ln\left(e\int_{x}^{x+k}\overline{q}(t)dt + 1 - \operatorname{sgn}\int_{x}^{x+k}\overline{q}(t)dt\right)$$

combining case 1, case 2, we obtain for $x \ge l_j$

$$\lambda(x)\int_{x}^{x+k}\overline{q}(t)dt - \overline{q}(x)\int_{[x-k]}^{x}\lambda(t)dt \ge \overline{q}(x)\ln\left(e\int_{x}^{x+k}\overline{q}(t)dt + 1 - \operatorname{sgn}\int_{x}^{x+k}\overline{q}(t)dt\right)$$

Thus, for $i > 1$

$$\int_{l_{n_i}+k}^{\xi_i}\lambda(x)\int_{x}^{x+k}\overline{q}(t)dtdx - \int_{l_{n_i}+k}^{\xi_i}\overline{q}(x)\int_{[x-k]}^{x}\lambda(t)dtdx \ge \int_{l_{n_i}+k}^{\xi_i}\overline{q}(x)\ln\left(e\int_{x}^{x+k}\overline{q}(t)dt + 1 - \operatorname{sgn}\int_{x}^{x+k}\overline{q}(t)dt\right)dx \tag{20}$$

By interchanging the order of integration, we find that

$$\int_{l_{n_i}+k}^{\xi_i} \bar{q}(x) \int_{[x-k]}^{x} \lambda(t)dtdx \geq \int_{l_{n_i}+k}^{\xi_i} \lambda(x) \int_{x}^{x+k} \bar{q}(t)dtdx$$

substituting it into (20), we have

$$\int_{\xi_i-k}^{\xi_i} \lambda(x) \int_{x}^{x+k} \bar{q}(t)dtdx \geq \int_{l_{n_i}+k}^{\xi_i} \bar{q}(x) \ln\left(e \int_{x}^{x+k} \bar{q}(t)dt + 1 - \operatorname{sgn} \int_{x}^{x+k} \bar{q}(t)dt \right) dx$$

it follows that (6) that

$$\int_{\xi_i-k}^{\xi_i} \lambda(x)dx \geq \int_{l_{n_i}+k}^{\xi_i} \bar{q}(x) \ln\left(e \int_{x}^{x+k} \bar{q}(t)dt + 1 - \operatorname{sgn} \int_{x}^{x+k} \bar{q}(t)dt \right) dx$$

or

$$\ln \frac{y(\xi_i - k)}{y(\xi_i)} \geq \int_{l_{n_i}+k}^{\xi_i} \bar{q}(x) \ln\left(e \int_{x}^{x+k} \bar{q}(t)dt + 1 - \operatorname{sgn} \int_{x}^{x+k} \bar{q}(t)dt \right) dx$$

taking the superior limit as $i \to \infty$, by using (13), we get

$$\int_{l_{n_i}+k}^{\infty} \bar{q}(x) \ln\left(e \int_{x}^{x+k} \bar{q}(t)dt + 1 - \operatorname{sgn} \int_{x}^{x+k} \bar{q}(t)dt \right) dx < \infty$$

This contradicts (4) and completes the proof.

THEOREM 2.2. *Assume that* $q(x) \geq 0$ *for* $x > 0$ *and*

$$\int_{0}^{\infty} q(x) \ln\left(e \int_{x}^{x+k} q(t)dt + 1 - \operatorname{sgn} \int_{x}^{x+k} q(t)dt \right) dx = \infty \qquad (21)$$

Then all solutions of the differential equation (2) oscillate.

The conclusion of theorem 2.2 follows from a similar argument to that in Theorem 2.1. We omit the details here.

THEOREM 2.3. *Assume that*

(i) *There exists a sequence of intervals* $\{[l_n, m_n]\}_{n=1}^{\infty}$ *such that* l_n, m_n *are integers,* $m_n \le l_{n+1}$ *and* $m_n - l_n \ge 2k$ *for* $n = 1,2,\ldots$*, and that*

$$q(x) \ge 0 \text{ For } x \in \bigcup_{n=1}^{\infty} [l_n, m_n]$$

(ii) $$\int_0^{\infty} \bar{a}(x) \ln\left(e \int_x^{x+k} \bar{a}(t)dt + 1 - \operatorname{sgn} \int_x^{x+k} \bar{a}(t)dt \right) dx = \infty$$

where

$$\bar{a}(x) = \begin{cases} q(x)\exp\left(\int_{l_n+i}^{x} a(t)dt \right), t \in [l_n + k + i, l_n + k + i + 1), i = 0,1,\ldots j_n - 1 \\ q(x)\exp\left(\int_{l_n+j_n}^{x} a(t)dt \right), t \in [l_n + k + j_n, m_n); \\ 0, t \in [0, l_1 + k) \cup \left(\bigcup_{n=1}^{\infty} m_n, l_{n+1} + k \right) \end{cases}$$

and j_n *is an integer such that* $m_n - 1 \le l_n + k + j_n < m_n$

Then all solutions of equation (1) oscillate.

PROOF. First, we claim that all solutions of differential equation (1) oscillate if and only if all solutions of the differential equation

$$z'(x) + a(x) z([x - k]) = 0, x \ge 0, \tag{22}$$

oscillate, where

$$a(x) = q(x) \exp\left(\int_{n-k}^{x} p(t)dt \right), x \in [n, n+1), n \in N(0) \tag{23}$$

Indeed, let $x(t)$ be a solution of equation (1)then for any $n \in N(0)$, equation (1) can be written as

$$\left(y(x) \exp\left(\int_0^x p(t)dt \right) \right)' + q(x) \exp\left(\int_{n-k}^{x} p(t)dt \right) y([x-k]) \exp\left(\int_0^{n-k} p(t)dt \right) = 0$$

Let

$$a(x) = q(x) \exp\left(\int_{n-k}^{x} p(t)dt \right), z(x) = y(x) \exp\left(\int_0^x p(t)dt \right), x \in [n, n+1). \tag{24}$$

Then, we obtain for $x \in [n, n+1), n \in N(0)$.

$$z'(x) + a(x) z([x-k]) = 0, x \ge 0$$

It follows from (24) that all solutions of equation (1) oscillates if and only if all solutions of (22) oscillate.

The conclusion of the theorem follows from the above claim and Theorem 2.1 immediately from Theorem 2.2 and the proof of Theorem 2.3, we have the following theorem.

THEOREM 2.4. *Assume that* $q(x) \geq 0$ *for* $x \geq 0$ *and*

$$\int_0^\infty a(x) \ln\left(e \int_x^{x+k} a(t)dt + 1 - \operatorname{sgn} \int_x^{x+k} a(t)dt \right) dx = \infty \tag{25}$$

where

$$a(x) = q(x) \exp\left(\int_{n-k}^{x} p(t)dt \right), x \in [n, n+1), n \in N(0) \tag{26}$$

Then the solutions of the differential equation (1) oscillate.

EXAMPLE 2.1. Consider the equation

$$y'(x) + \frac{1}{x+2} y(x) + q(x) y([x-1]) = 0, x \geq 0 \tag{27}$$

where

$$q(x) = \frac{n+1}{x+2} p \sin \frac{\pi}{3} x, x \in [n, n+1), n \in N(0)$$

and $p > 4^{1/3} (2 - \sqrt{3})^{\sqrt{3}/3} \frac{\pi}{3}$.

Let $l_n = 6n, m_n = 6n+3$. Then condition (i) in Theorem 2.3 holds, and by simple calculation we obtain

$$\bar{a}(x) = \begin{cases} p \sin \frac{\pi}{3} x, x \in \bigcup_{n=0}^{\infty} [6n+1, 6n+3), \\ 0, x \in [0,1) \cup \left\{ \bigcup_{n=0}^{\infty} [6n+3, 6n+7) \right\} \end{cases}$$

it is not difficult to see that

$$\int_0^6 \bar{a}(x) \ln\left(e \int_x^{x+1} \bar{a}(t)dt + 1 - \operatorname{sgn} \int_x^{x+1} \bar{a}(t)dt \right) dx$$

$$= p \int_1^2 \sin \frac{\pi}{3} x \ln\left[e \int_x^{x+1} p \sin \frac{\pi}{3} t dt \right] dx + p \int_2^3 \sin \frac{\pi}{3} x \ln\left[e \int_x^{3} p \sin \frac{\pi}{3} t dt \right] dx$$

$$= \frac{3p}{2\pi} \ln \frac{27\left(2+\sqrt{3}\right)^{\sqrt{3}} p^3}{4\pi^3} > 0,$$

and so

$$\int_0^\infty \bar{a}(x) \ln\left(e \int_x^{x+1} \bar{a}(t)dt + 1 - \operatorname{sgn} \int_x^{x+1} \bar{a}(t)dt \right) dx = \infty$$

Thus, by theorem 2.3, all solutions of equation (27) oscillate.

REMARK 2.1. It should be noted that $q(x) < 0$ for $3(2k-1) < x < 6k, k = 1,2,\ldots$and so the coefficient $q(x)$ is oscillatory. Thus, the above example does not satisfy the know oscillation conditions in the literature.

EXAMPLE 2.2. Consider the equation

$$y'(x) + \frac{1}{x+2} y(x) + q(x) y([x-1]) = 0, x \geq 0 \tag{28}$$

where

$$q(x) = \begin{cases} \dfrac{(500l + i + 1)p}{x+2}, & x \in [500l + i, 500l + i + 1), i = 0,1,\ldots,29; \\ 0, x \in [500l + 30, 500(l+1)) \end{cases}$$

$l \in N(0)$ and $e^{-29/30} < p < 1$.

It is easy to see that for $l \in N(0)$,

$$\bar{a}(x) = \begin{cases} p, x \in [500l, 500l + 30); \\ 0, x \in [500l + 30, 500(l+1)), \end{cases}$$

and

$$\int_0^{500} \bar{a}(x) \ln\left(e \int_x^{x+1} \bar{a}(t)dt + 1 - \operatorname{sgn} \int_x^{x+1} \bar{a}(t)dt \right) dx = p \ln\left(e^{29} a^{30}\right) > 0$$

It follows that

$$\int_0^{\infty} \bar{a}(x) \ln\left(e \int_x^{x+1} \bar{a}(t)dt + 1 - \operatorname{sgn} \int_x^{x+1} \bar{a}(t)dt \right) dx = \infty .$$

Therefore, by theorem 2.4, all solutions of equation (28) oscillate.

REMARK 2.2. Let

$$a_n = \exp\left(\int_n^{n+1} p(t)dt \right), b_n = \int_n^{n+1} q(x) \exp\left(\int_n^x p(t)dt \right) dx$$

Then it is not difficult to see that

$$\begin{aligned} b_n a_{n-1} &= \int_n^{n+1} q(x) \exp\left(\int_{n-1}^x p(t)dt \right) dx \\ &= \begin{cases} p < 1, n = 500l + i, i = 0,1,\ldots,29; \\ 0, n = 500l + j, j = 30,31,\ldots 499, \end{cases} \end{aligned}$$

where $l \in N(0)$,and so

$$\limsup_{n\to\infty} b_n a_{n-1} = \limsup_{n\to\infty} \int_n^{n+1} q(x) \exp\left(\int_{n-1}^x p(t)dt \right) dx = a < 1,$$

$$\liminf_{n\to\infty} b_n a_{n-1} = \liminf_{n\to\infty} \int_n^{n+1} q(x)\exp\left(\int_{n-1}^{x} p(t)dt\right)dx = 0 < \frac{1}{4},$$

and when $e^{-29/30} < p < \left(3-\sqrt{5}\right)/2$, for any integer $l \geq 0$

$$\limsup_{n\to\infty}\left(b_n a_{n-1} + \sum_{i=0}^{l}\prod_{j=0}^{i} b_{n-j-1}a_{n-j-2}\right) \leq a + a + a^2 + \ldots + a^7 < 1.$$

Therefore, all known results in the literature (see[6,10,11,16]) cannot be applied to (28)

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