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FIXED POINT THEOREM IN PROBABILISTIC G-METRIC SPACES

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Abstract : In this paper we study some fixed point theorems in probabilistic G-metric spaces. We also generalized some previously known results.

Key words : G-meric spaces , Menger spaces, probabilistic G-metric space, t-norm,

1.1 Introduction and preliminaries

In 2006 the concept of generalized metric space was introduced [4]. For more results in these spaces one can see [2] and [3].

On the other hand, in 1942, Menger [1] introduced the notion of probabilistic metric space (briefly PM-space) as a generalization of metric space. Such a probabilistic generalization of metric spaces appears to be well adapted for the investigation of physical quantities and physiological thresholds. The development of fixed point theory in PM-spaces was due to Schweizer and Sklar [5, 6]. Fixed point theory has been always an active area of research since 1922 with the celebrated Banach contraction fixed point theorem.

Let X be a nonempty set, and let $G : X \times X \times X \to R^+$ be a function satisfying the following axioms:

1. $G(x, y, z) = 0$ if $x = y = z$,
2. $G(x, x, y) > 0$, for all $x, y \in X$, with $x \neq y$,
3. $G(x, x, y) \leq G(x, y, z)$, for all $x, y, z \in X$, with $z \neq y$,
4. $G(x, y, z) = G(x, z, y) = G(y, z, x) = G(z, y, x) = \dots$ (symmetry in all three variables),
5. $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$, for all $x, y, z, a \in X$, (rectangle inequality).

Then the function G is called a generalized metric, or, more specifically a G −metric on X, and the pair (X, G) is called a G −metric space. A sequence (x_n) in a G −metric space (X, G) is said to be G −convergent to x if

$\lim_{n,m\to\infty} G(x, x_n, x_m) = 0$; which means that, for any $\varepsilon > 0$, there exists $N \in N$ such that $G(x, x_n, x_m) < \varepsilon$, for all $n, m \geq N$. Also a sequence (xn) is called G −Cauchy if for a given $\varepsilon > 0$, there is $N \in N$ such that $G(x_n, x_m, x_l) < \varepsilon$, for all $n, m, l \geq N$; that is if $G(x_n, x_m, x_l) \to 0$ as $n, m, l \to \infty$.

We may construct G −metrics using an ordinary metric. Indeed if (X, D) is a metric space, then define

$$G_s(x,y,z) = d(x,y) + d(z,y) + d(x,z).$$
$$G_m(x,y,z) = \max\{d(x,y), d(y,z), d(x,z)\},$$

for all $x,y,z \in X$. One can verify that G_s and G_m are $G-$metric.

A distribution function is a function $F: [-\infty,\infty] = R \to [0,1]$ that is nondecreasing and left continuous on R; moreover, $F(-\infty) = 0$ and $F(\infty) = 1$.

The set of all the distribution functions is denoted by Δ and the set of those distribution functions such that $F(0) = 0$ is

denoted by Δ^+.

A natural ordering in Δ is defined by $F \le G$ whenever $F(x) \le G(x)$, for every $\in R$. The maximal element in this order for Δ^+ is ε_0, where for $-\infty \le a \le \infty$ the distribution function ε_a is defined by

$$\varepsilon_a(x) = \begin{cases} 0 & \text{if } -\infty \le x \le a \\ 1 & \text{if } a < x \le \infty \end{cases}$$

A binary operation on Δ^+ which is commutative, associative, nondecreasing in each place, and has ε_0 as identity, is said to be triangle function.

Also a probabilistic metric space (abbreviated, PM-space) is an ordered triple (S,F,τ) where S is a nonempty set, τ is a triangle function and $F: S \times S \to \Delta^+$ $\left(F(p,q) \text{ is denoted by } F_{p,q}\right)$ satisfies the following conditions:

1. $F_{p,p} = \varepsilon_0$,
2. If $p \ne q$, then $F_{p,q} \ne \varepsilon_0$,
3. $F_{p,q} = F_{q,p}$,
4. $F_{p,r} \ge \tau\left(F_{p,q}, F_{q,r}\right)$,

for every $p,q,r \in S$.

If 1), 3), 4) and are satisfied, then (S,F,τ) is called a probabilistic pseudo-metric space.

In section 3.2, we introduce the notion generalized probabilistic metric space. Then Some examples and elementary properties of these spaces are discussed. In section 3.3, generalized Menger probabilistic $G-$metric space is studied. Finally in section 3.4, some fixed point theorem in generalized Menger probabilistic metric spaces are investigated.

2.1 Probabilistic G-Metric Space

Definition 2.1. Suppose X is a nonempty set, τ is a triangle function and $G: X \times X \times X \to \Delta^+$, is a mapping satisfying

G_1 $G(p,p,p) = \varepsilon_0$,

G_2 if $p \ne q$, then $G(p,p,q) \ne \varepsilon_0$,

G_3 if $q \ne r$, then $G(p,p,q) \ge G(p,q,r)$,

G_4 $G(p,q,r) = G(p,r,q) = G(q,r,p) = \dots$,

G_5 $G(p,q,r) \ge \tau\left(G(p,s,s), G(s,q,r)\right)$,

for all $p,q,r,s \in X$. Then (X,G,τ) is called a generalized probabilistic metric space (or briefly, probabilistic G-metric

space). (X,G,τ) is called a probabilistic pseudo $G-$metric space if G_1, G_3, G_4 and G_5 are satisfied.

A probabilistic $G-$metric space (X,G,τ) is said to be symmetric if for every $x,y \in X$,

$$G(x,y,y) = G(y,x,x).$$

A probabilistic $G-$metric space (X,G,τ) is called proper if $\tau(\varepsilon_a,\varepsilon_b) \ge \varepsilon_{a+b}$, for all $a,b \in [0,\infty)$.

In the following two examples, we construct two probabilistic G-metric space using a PM$-$space and a $G-$metric

space, respectively.

Example 2.2. With $\tau(F,G) = \min\{F,G\}$, let (X,F,τ) be a probabilistic metric space. If $G_M: X^3 \to \Delta^+$ is defined by

$$G_m(p,q,r) = \min\{F_{p,q}, F_{p,r}, F_{q,r}\},$$

then (X, G_M, τ) is a probabilistic $G-$metric space.

Indeed if $p = q = r$ then

$$G_m(p,q,r) = \min\{F_{p,q}, F_{p,r}, F_{q,r}\} = \min\{\varepsilon_0, \varepsilon_0, \varepsilon_0\} = \varepsilon_0.$$

Also for $p \neq q$ by definition of probabilistic metric, $F_{p,q} \neq \varepsilon_0$, so

$$G_m(p,p,q) = \min\{F_{p,q}, F_{p,p}\} = \min\{F_{p,q}, \varepsilon_0\} = F_{p,q} \geq \varepsilon_0.$$

Now if $q \neq r$ then

$$G_m(p,p,q) = \min\{F_{p,q}, \varepsilon_0\} = F_{p,q}$$
$$\geq \min\{F_{p,q}, F_{p,r}, F_{q,r}\} = G_m(p,q,r).$$

Commutativity of G_m is trivial by commutativity of F. For proving G_5, let $p,q,r,s \in X$. We have

$$\min\{G_m(p,s,s), G_m(s,q,r)\} = \min\{F_{p,s}, F_{s,s}, F_{s,q}, F_{s,r}, F_{q,r}\}.$$

Thus

$$\begin{aligned} G_m(p,q,r) &= \min\{F_{p,q}, F_{p,r}, F_{q,r}\} \\ &\geq \min\{\min\{F_{p,s}, F_{s,q}\}, F_{q,r}, \min\{F_{p,s}, F_{s,r}\}\} \\ &= \min\{\min\{F_{p,s}, F_{s,s}\}, \min\{F_{s,q}, F_{q,r}, F_{s,r}\}\} \\ &= \tau\left(G_m(p,s,s), G_m(s,q,r)\right). \end{aligned}$$

Example 2.3. Let (X,F) be a G-metric space. For every $p,q,r \in X$, define

$$G_{p,q,r} = \varepsilon_{F_{p,q,r}}. \qquad 2.1$$

Also let τ is a triangle function for which

$$\tau(\varepsilon_a, \varepsilon_b) \leq \varepsilon_{a+b},$$

for all $a,b \in R^+$. Then it is straightforward to show that (X,G,τ) is a probabilistic $G-$metric space.

Also if a proper probabilistic $G-$metric is of the form 2.1, then $F_{p,q,r}$ is a $G-$metric. Suppose (X,G,τ) is a proper probabilistic $G-$metric space and there exists a function $F: X \times X \times X \to R^+$, such that

$$G_{p,q,r} = \varepsilon_{F_{p,q,r}}$$

then (X,F) is a $G-$metric space.

Indeed in this case

$$\varepsilon_0 = G_{p,p,p} = \varepsilon_{F_{p,p,p}} = \varepsilon_0,$$

so $F_{p,p,p} = 0$. If $p \neq q$ then

$\varepsilon_0 \neq G_{p,p,q} = \varepsilon_{F_{p,p,q}}$,

which implies that $F_{p,p,q} \neq 0$. Also if $q \neq r$ then the fact that $G_{p,p,q} \geq G_{p,q,r}$ implies that $F_{p,p,q} \leq F_{p,q,r}$.

Commutativity of F follows from commutativity of G. For proving

$$F_{p,q,r} \leq F_{p,s,s} + F_{s,q,r},$$

we note that G is proper, so,

$$\varepsilon_{F_{p,q,r}} = G_{p,q,r} \geq \tau\left(\varepsilon_{F_{p,s,s}}, \varepsilon_{F_{s,q,r}}\right) \geq \varepsilon_{F_{p,s,s}} + F_{s,q,r}$$

which implies that (X,G) is a $G-$metric space.

In the following proposition, it is proved that we may construct a probabilistic $G-$metric space using a pseudo probabilistic $G-$metric space. To do this, we introduce the following relation:

Let (X,G,τ) be a probabilistic pseudo $G-$metric space. For $p,q \in X$, we say $p \sim q$ if and only if

$$G(p,p,q) = \varepsilon_0 \text{ and } G(p,q,q) = \varepsilon_0.$$

This relation is an equivalence relation. Indeed if $p \sim q$ and $q \sim r$, then

$$G(p,p,q) = \varepsilon_0, G(p,q,q) = \varepsilon_0 \text{ and } G(q,q,r) = \varepsilon_0, G(r,r,q) = \varepsilon_0$$

But G is a probabilistic pseudo $G-$metric, so

$$G(p,p,r) = G(r,p,p) \geq \tau\left(G(r,q,q), G(q,p,p)\right) = \tau(\varepsilon_0, \varepsilon_0) = \varepsilon_0,$$

which implies that $G(p,p,r) \geq \varepsilon_0$. Now maximality of ε_0 implies that $G(p,p,r) = \varepsilon_0$. Similarly $G(p,r,r) = \varepsilon_0$. This prove that $\sim$ is transitive. The other properties of $\sim$ to be an equivalence relation is trivial.

Proposition 2.4 Let (X, G, τ) be a probabilistic pseudo $G-$metric space, for every $p \in S$, let $p^\star$ denote the equivalence class of p and let $X^\star$ denotes the set of these equivalence classes. Then the expression

$$G^\star(p^\star, q^\star, r^\star) = G(p,q,r), \; p \in p^\star, q \in q^\star, r \in r^\star$$

define a function $G^\star$ from $X^\star \times X^\star \times X^\star$ into Δ^+ and the triple $(X^\star, G^\star, \tau)$ is a probabilistic $G-$metric space, the

quotient space of (X, G, τ).

Proof. First we prove that $G^\star$ is well defined, i.e. if $r, r' \in p^\star, q, q' \in q^\star$ and $p, p' \in p^\star$, then

$$G(p,q,r) = G(p',q',r').$$

Since $q \sim q', p \sim p'$ and $r \sim r'$ and τ is a triangular function, we have

$$\begin{aligned} G(p,q,r) &\geq \tau\left(G(p,p',p'), G(p',q,r)\right) = G(p',q,r) \\ &\geq \tau\left(G(q,q',q'), G(q',p',r)\right) = G(q',p',r) \\ &\geq \tau\left(G(r,r',r'), G(r',p',q'\right) = G(r',p',q') \\ &= G(p',q',r'). \end{aligned}$$

Similarly we get $G(p',q',r') \leq G(p,q,r)$, so $G^\star$ is well defined. Also trivially,

$$G^\star(p^\star, p^\star, p^\star) = G(p,p,p) = \varepsilon_0.$$

and if $p \neq q$, then $p \notin q^\star, q \notin p^\star$.

Hence $p \nsim q$, so $G(p,p,q) \neq \varepsilon_0$. Thus

$$G^\star_{(p^\star, p^\star, q^\star)} = G(p,p,q) \neq \varepsilon_0.$$

By the fact that,

$$G(p,p,q) \geq G(p,q,r)$$

we lead to

$$G^\star(p,p,q) \geq G^\star(p,q,r).$$

It is trivial to verify the other properties of $G^\star$.

3.1 Menger Probabilistic G −Metric Space

In this section we introduce Menger probabilistic $G-$metric spaces. Recall that a mapping $T: [0,1] \times [0,1] \to [0,1]$

is called a triangular norm (abbreviated, t-norm) if the following conditions are satisfied

1. $T(a,1) = a$, for every $a \in [0,1]$,
2. $T(a,b) = T(b,a)$, for every $a, b \in [0,1]$,
3. $T(a,c) \geq T(b,d)$, whenever $a \geq b$ and $c \geq d$, $(a,b,c,d \in [0,1])$,
4. $T(a, T(b,c)) = T(T(a,b), c)$, $(a,b,c \in [0,1])$.

The following are the four basic t-norms:

(a) The minimum $t-$norm, T_M, is defined by $T_M(x,y) = \min\{x,y\}$.

(b) The product $t-$norm, T_P, is defined by $T_P(x,y) = xy$.

(c) The Lukasiewicz $t-$norm, T_L, is defined by

$$T_L(x,y) = \max\{x+y-1, 0\}.$$

(d) The weakest $t-$norm, the drastic product, T_D, is defined by

$$\begin{cases} T_D(x,y) = \min\{x,y\}, & \text{if} \max\{x,y\} = 1 \\ 0, & \text{otherwise} \end{cases}$$

As regards the pointwise ordering, we have the inequalities

$$T_D < T_L < T_P < T_M.$$

Definition 3.1. Suppose S is a nonempty set and T is a t −norm and $G: S^3 \to \Delta^+$ is a function. The triple (S, G, T) is called a Menger probabilistic G-metric space if for every $p, q, r, s \in S$ and $x, y > 0$,

1. $G(p,p,p) = \varepsilon_0$,
2. If $p \neq q$, then $G(p,p,q) \neq \varepsilon_0$,
3. $G(p,p,q) \geq G(p,q,r)$,
4. $G(p,q,r) = G(p,r,q) = G(q,r,p) = \ldots$,
5. $G(p,q,r)\,(x+y) \geq T\big(G(p,s,s)(x), G(s,q,r)(y)\big)$.

In the Menger probabilistic G −metric space (S, G, T) with

$$\sup_{0<t<1} T(t,t) = 1$$

a sequence $\{u_n\}$ in S,

i) is called convergent to $u \in S$ if for every $\varepsilon, \lambda > 0$, there exists $N \in N$ such that, $\forall n \geq N$; $G_{u_n,u,u}(\varepsilon) > 1-\lambda$.

ii) is said to be a Cauchy sequence, if for every $\varepsilon, \lambda > 0$ there exists $N \in N$ such that, $\forall m,n,l \geq N$; $G_{u_m,u_n,u_l}(\varepsilon) > 1-\lambda$.

As usual a Menger probabilistic G −metric space is said to be complete if every Cauchy sequence in S converges to a $u \in S$.

Theorem 3.2. Let (S, G, T_L) be a Menger probabilistic G −metric space and define,

$$G^\star_{p,q,r} = \sup\{t \geq 0 | G_{p,q,r}(t) \leq 1-t\}.$$

Then,

i) $G^\star$ is a G −metric.

ii) S is G −complete if and only if it is $G^\star$ −complete.

Proof. For any $t > 0$, $G_{p,p,p} = \varepsilon_0(t) = 1$, so

$$G^\star_{p,p,p} = \sup\{t \geq 0 | G_{p,p,p}(t) = 1 \leq 1-t\} = 0.$$

Also if $p \neq q$, then $G_{p,p,q} \neq \varepsilon_0$. Hence

There exists $t \in (0,1)$ s.t. $G_{p,p,q}(t) < 1$,

so

$$G^\star_{p,p,q} = \sup\{t \geq 0 | G_{p,p,q}(t) \leq 1-t\} > 0.$$

Now for any $p, q, r \in S$ we know, $G_{p,p,q} \geq G_{p,q,r}$, so

$$\{t | G_{(p,p,q)}(t) \leq 1-t\} \subseteq \{t | G_{p,q,r}(t) \leq 1-t\}.$$

Hence $G^\star_{p,p,q} \leq G^\star_{p,q,r}$.

These prove first, second and the third part of definition of G −metric for $G^\star$. Commutativity of $G^\star$ is trivial.

We are going to prove that,

$$G^\star_{p,q,r} \leq G^\star_{p,s,s} + G^\star_{s,q,r}, \qquad 3.1$$

for all $p, q, r, s \in S$.

To do this, put

$$A = \{t | G_{p,q,r} \leq 1-t\}$$
$$B = \{\lambda | G_{(p,s,s)(\lambda)} \leq 1-\lambda\}$$
$$C = \{\mu | G_{s,q,r} \leq 1-\mu\}.$$

Suppose $t_1 > G^\star(p,s,s)$ and $t_2 > G^\star_{s,q,r}$ are upper bounds for B and C, respectively. Then

$$G(p,s,s)(t_1) > 1-t_1 \text{ and } G_{s,q,r}(t_2) > 1-t_2.$$

Therefore

$$G_{p,q,r}(t_1+t_2) \geq T_L\big(G_{p,s,s}(t_1), G_{(s,q,r)(t_2)}\big)$$
$$\geq G_{p,s,s}(t_1)+G_{s,q,r}(t_2)-1$$
$$> 1-(t_1+t_2).$$

Thus t_1+t_2 is an upper bound for A. Hence G^*satisfies 3.1. Consequently G^* is a $G-$metric.

For proving ii), let (S, G, T_L) be $G-$complete and (u_n) be a Cauchy sequence in the G^*-metric. We prove that (u_n) is Cauchy with the probabilistic $G-$metric G. Let $\varepsilon, \lambda > 0$ be given. If $\varepsilon < \lambda$ then for $\varepsilon, N \in N$ s.t. $\forall m,n,l \geq N$; $G^*_{u_m,u_n,u_l} < \varepsilon$,

since (u_n) is G^*-Cauchy. By definition of G^*, for every $m,n,l \geq N$

$$G_{u_m,u_n,u_l}(\varepsilon) > 1-\varepsilon > 1-\lambda.$$

Now if $\lambda < \varepsilon$ then for , $\exists N \in N, \forall m,n,l \geq N; G^*_{u_m,u_n,u_l} < \lambda$,

since (u_n) isG^*-Cauchy. By definition of G^*, the fact that G_{u_m,u_n,u_l} is nondecreasing implies that

$$G_{u_m,u_n,u_l}(\varepsilon) \geq G_{u_m,u_n,u_l}(\lambda) > 1-\lambda.$$

Thus (u_n) is $G-$Cauchy. Now by $G-$completeness of S with G, there exists $u \in S$ such that (u_n) is $G-$convergent to u.

So for $\varepsilon > 0$ there exists $N \in N$ such that, for every $m,n \geq N$,

$$G_{u_m,u_n,u}\left(\frac{\varepsilon}{2}\right) > 1-\frac{\varepsilon}{2}.$$

This means that $\frac{\varepsilon}{2}$ is an upper bound for the segment $\{t \,|\, G_{u_m,u_n,u} \leq 1-t\}$. Thus $G^*_{u_m,u_n,u} \leq \frac{\varepsilon}{2}\varepsilon <$, i.e. (u_n) converges to u with $G^\star$ and so S is $G^\star-$complete.

Conversely suppose that S is $G^\star-$complete and (u_n) is a $G-$Cauchy sequence in S. Thus for given $\varepsilon > 0$, there exists

$N \in N$ such that, for all $m,n,l \geq N$;

$$G_{u_m,u_n,u_l}\left(\frac{\varepsilon}{2}\right) > 1-\frac{\varepsilon}{2}.$$

Hence $\forall m,n,l \geq N$;

$$G^\star_{u_m,u_n,u_l} < \frac{\varepsilon}{2} < \varepsilon.$$

This implies that (u_n) is a $G^\star-$Cauchy sequence sequence and so is $G^\star-$convergent to some u in S. Hence for given ε, λ, with $< \lambda$, there exists $N \in N$ such that $\forall m,n \geq N$;

$$G^\star_{u_m,u_n,u} < \varepsilon < \lambda.$$

By definition of $G^\star$, $\forall m,n \geq N$;

$$G_{u_m,u_n,u}(\varepsilon) > 1-\varepsilon > 1-\lambda.$$

Now if $\lambda \leq \varepsilon$ then $\exists N > 0$ s.t. $\forall m,n,l \geq N$;

$$G_{u_m,u_n,u_l}(\varepsilon) > G_{u_m,u_n,u_l}(\lambda) > 1-\lambda \geq 1-\varepsilon,$$

since (u_n) is $G-$Cauchy. By definition of $G^\star$ $\forall m,n,l \geq N$;

$$G^\star_{u_m,u_n,u_l} < \varepsilon$$

But S is $G^\star-$omplete, so there exists $u \in S$ such that (u_n) is $G^\star-$convergent to u. This implies that there exists $N \in N$ such that $\forall m,n \geq N$;

$$G^\star_{u_m,u_n,u} < \lambda < \varepsilon.$$

Finally by definition of $G^\star$ $\forall m,n \geq N$;

$$G_{u_m,u_n,u}(\varepsilon) \geq G_{u_m,u_n,u}(\lambda) > 1-\lambda.$$

Hence (u_n) is $G-$convergent and so S is $G-$complete.

4.1 Fixed Points Of Contractive Maps In Menger Probabilistic G −Metric Space

In this section, first we introduce the concept of G −contractive mapping in Menger probabilistic G −metric space and then its relation with G −contractive map in its dependent G −metric space is studied. This result shows that the existence of a convergent subsequence of an iterate sequence (of a contractive map) implies the existence of a fixed point.

In order to do this, we introduce the following definition;

Definition 4.1. Let (S, G, T) be a Menger probabilistic G −metric space. a mapping $f: S \rightarrow S$ is said to be a G −contraction

if for any $t \in (0,\infty)$,

$$G_{p,q,r}(t) > 1 - t$$

implies that

$$G_{f(p),f(q),f(r)}(kt) > 1 - kt$$

for some fixed $k \in (0,1)$.

One can easily see that if $f: S \to S$ is a G −contraction and (u_n) is a convergent sequence to some u in the Menger probabilistic G −metric space S, then $\left(f(u_n)\right)$ converges to $f(u)$.

We recall that a function f on a G −metric space with a G −metric $G^\star$ is called G −contraction if for any $t \in (0,\infty)$,

the relation $G_{p,q,r} < t$ implies that $G_{f(p),f(q),f(r)} < kt$, for some $k \in (0,1)$.

Lemma 4.2. Let (S, G, T_L) be a Menger probabilistic G −metric space and

$$G^\star_{p,q,r} = \sup\left\{t \middle| G_{p,q,r}(t) \le 1 - t\right\}$$

then a function $f: S \to S$ is a G −contraction mapping if and only if it is $G^\star$ −contraction.

Proof. We know that $G^\star$ is a G −metric on S.

Let f be a G −contraction in the Menger probabilistic G −metric space and for $t \in (0,\infty)$

$$G^\star_{p,q,r} < t.$$

By definition of $G^\star$, we get

$$G_{p,q,r}(t) > 1 - t.$$

But f is G −contraction, so

$$G_{f(p),f(q),f(r)}(kt) > 1 - kt,$$

for some fixed $k \in (0,1)$. Now definition of $G^\star$ implies that

$$G^\star_{f(p),f(q),f(r)} < kt,$$

which means that f is $G^\star$ −contraction. The converse of this lemma can be proved similarly.

Theorem 4.3. Let (S, G, T_L) be a Menger probabilistic G −metric space. Suppose A is G −contraction on S and for some u in S, $A^{n_i}(u)$ is a convergent subsequence of $A^n(u)$, then $\xi = A\left(\lim_{i\to\infty} A^{n_i}(u)\right)$ is the unique fixed point of A.

Proof. Let $A(\xi) \neq \xi$, then there exists $t_0 \in (0,\infty)$ such that,

$$G_{A(\xi),\xi,\xi}(t_0) \neq 1.$$

So there exists $\lambda \in (0,1)$, such that

$$1 - \lambda < G_{A(\xi),\xi,\xi}(t_0) < 1.$$

By letting $t = \max\{t_0, \lambda\}$ we get

$$G_{A(\xi),\xi,\xi}(t) \ge G_{A(\xi),\xi,\xi}(t_0) > 1 - \lambda > 1 - t.$$

But A is a G −contraction so for some $k \in (0,1)$,

$$G_{A^2(\xi),A(\xi),A(\xi)}(kt) > 1 - kt.$$

Using induction argument one can see that

$$G_{A^{n+1}(\xi),A^n(\xi),A^n(\xi)}(k^n t) > 1 - k^n t. \qquad 4.1$$

Taking n and n_i large enough such that

$$k^n t < 1 \text{ and } k^{n_i} t < 1$$

and putting $p = A^n(\xi)$, we obtain

$$p = A^n(\xi) = A^n\left(\lim_{i\to\infty} A^{n_i}(u)\right) = \lim_{i\to\infty}\lim_{i\to\infty} A^{n_i}(u) + n(u)$$

Let $s = \max\{k^n t, k^{n_i} t\}$. By 4.1

$$G_{A(p),p,p}(s) > G_{A(p),p,p}(k^n t) > 1 - k^n t > 1 - s.$$

If $G^\star$ is the G −metric introduced in Lemma 4.1, then

$$G^\star_{A(p),p,p} = \sup\{t | G_{A(p),p,p}(t) \le 1 - t\}.$$

So

$$G^\star_{A(p),p,p} < s < 1.$$

By the fact that $A^{n_i}(u) \to \xi$ and $A^{n_i+1}(u) \to A(\xi)$, for every $t, \lambda > 0$, there exists $N \in \mathrm{N}$, such that for every $n_i > N$,

$$G_{A^{n_i}(u),\xi,\xi}(t) > 1 - \lambda, G_{A^{n_i+1}(u),A(\xi),A(\xi)}(t) > 1 - \lambda.$$

Let $l > j > n + n_i$. We are going to prove that,

$$G^\star_{A^{n_l}(u),A^{n_l+1}(u),A^{n_l+1}(u)} \le k^{l-j} G^\star_{A^{n_j}(u),A^{n_j+1}(u),A^{n_j+1}(u)} . \qquad 4.2$$

If we prove this inequality, then the this together with the facts that $k \in (0,1)$ and

$$G^\star_{A^{n_j}(u),A^{n_j+1}(u),A^{n_j+1}(u)} < 1$$

imply that $\lim_{l\to\infty} A^{n_l}(u) = \lim_{l\to\infty} A^{n_l+1}(u)$ in the generalized metric $G^\star$ and so is valid in the Menger probabilistic G. This leads to the equality $\xi = A(\xi)$ which is a contradiction.

First we prove that,

$$G^\star_{A^{n_l}(u),A^{n_l+1}(u),A^{n_l+1}(u)} \le k G^\star_{A^{n_l-1}(u),A^{n_l}(u),A^{n_l}(u)} . \qquad 4.3$$

To do this, let

$$s \in \{t | G_{A^{n_l}(u),A^{n_l+1}(u),A^{n_l+1}(u)}(t) \le 1 - t\}$$

then

$$G_{A^{n_l}(u),A^{n_l+1}(u),A^{n_l+1}(u)}(s) \le 1 - s.$$

Put $t = \frac{s}{k}$. We find that

$$G_{A^{n_l-1}(u),A^{n_l}(u),A^{n_l}(u)} \le 1 - t,$$

since otherwise by contractivity of A it should be

$$G_{A^{n_l}(u),A^{n_l+1}(u),A^{n_l+1}(u)}(kt) = G_{A^{n_l}(u),A^{n_l+1}(u),A^{n_l+1}(u)}(s)$$
$$> 1 - kt = 1 - s,$$

which is not the case. Therefore

$$t = \frac{s}{k} \in \{t | G_{A^{n_l-1}(u),A^{n_l}(u),A^{n_l}(u)}(t)(t) \le 1 - t\}$$

or equivalently

$$s \in k\{t | G_{A^{n_l-1}(u),A^{n_l}(u),A^{n_l}(u)}(t) \le 1 - t\}.$$

So

$$\sup\{t | G_{A^{n_l}(u),A^{n_l+1}(u),A^{n_l+1}(u)}(t) \le 1 - t\} \le \sup\{t | G_{A^{n_l-1}(u),A^{n_l}(u),A^{n_l}(u)}(t) \le 1 - t\}$$

and consequently 4.3 is valid. Now by induction argument, one leads to 4.3 which completes the proof.

References

1. Menger, K.: Statistical metric, Proc. Nat. Acad. Sci. U.S.A., 28(12) (1942), 535-537.
2. Mustafa Z., Sims B., Fixed point theorems for contractive mappings in complete G-metric spaces, Fixed Point Theory and Application, , Article ID 917175,vol. 2009, (2009)10 pages .
3. Mustafa Z., Shatanawi W. and Bataineh M., Existence of fixed point results in G-metric space, International Journal of Mathematics and Mathematical Sciences, Article ID 283028,vol. 2009, (2009)10 pages.
4. Mustafa Z. and Sims B., A new approach to generalized metric spaces, Journal of Nonlinear and Convex Analysis 7 (2)(2006) 289-297.
5. Schweizer, B, Sklar, A., Probabilistic Metric Spaces, North-Holland, New York (1983)
6. Schweizer B. and Sklar A., Statistical metric spaces, Pacific J. Math, 10 (1960) 313-334.