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A STUDY ON THE ZEROS OF POLAR DERIVATIVE OF A POLYNOMIAL

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Abstract: According to Enestrom and Kakeya theorem ``all the zeros of a polynomial $f(z) = \sum_{i=0}^{n} k_i z^i$ with real coefficient lie in $|z| \leq 1$ if $0 \leq k_0 \leq k_1 \leq k_2 \leq \cdots \leq k_{n-1} \leq k_n$" see [5, 11]. This article provides a region for the zeros of polar derivative of f (z) which does not lie in the region must be simple. By imposing some conditions on hypothesis in different ways.

Key words: Enestrom-Kakeya theorem, zeros, polynomial, polar derivative.

Mathematics Subject Classiffcation: 30C10, 30C15

1. INTRODUCTION

Let $D_\alpha f(z) = nf(z) + (\alpha - z)f'(z)$ denote the polar derivative of a polynomial f(z) of degree n with respect to real number α. Regarding the distribution of zeros of f(z), Enestrom and Kakeya [5, 11], given the following result.

Theorem 1.1. Let $f(z) = \sum_{i=0}^{n} k_i z^i$ be the n^{th} degree polynomial with real coefficients such that for some $0 \leq k_0 \leq k_1 \leq k_2 \leq \cdots \leq k_{n-1} \leq k_n$. Then all zeros of f (z) lie in $|z| \leq 1$.

Regarding the multiplicity of zeros of f(z), Aziz and Mohammad in [1] proved the following result **Theorem 1.2.** Let $f(z) = \sum_{i=0}^{n} k_i z^i$ be the n^{th} degree polynomial with real coefficients such that for some $0 \leq k_0 \leq k_1 \leq k_2 \leq \cdots \leq k_{n-1} \leq k_n$. Then all zeros of f(z) of modulus greater than or equal to $\frac{n}{n+1}$ are simple.

Gulzar, Zargar, Akhter in [9] are extended the above results to the polar derivatives, in [2, 3, 4, 6, 10] there exist some generalizations and extensions of Enestrom Kakeya theorems, in this article also f(z) is the polynomial of degree n with real coefficients and b_t denotes the coefficient of differentiation of polar derivative $(t-1)[t\alpha k_t + (n-(t-1))k_{t-1}]$ for $t = 2,3,4,\ldots,n$ and c_t denotes $(t-1)[(n-(t-1))k_{t-1}]$ for $t = 2,3,4,\ldots,n$

2. MAIN RESULTS

Theorem 2.1. Let $f(z)=\sum_{i=0}^{n} k_i z^i$ be the n^{th} degree polynomial, let α be real number, $s\geq 1, \eta \geq 0$ such that for some

$$b_n \leq b_{n-1} \leq \cdots \leq b_{m+1} \leq s b_m \geq b_{m-1} \geq \cdots \geq b_3 \geq b_2 - \eta.$$

Then all zeros of $D_\alpha f(z)$ which does not lie in

$$|z| \leq \frac{-b_n + s b_m + 2s(|b_m| - b_m) - b_2 + |b_2| + 2\eta}{|b_n|}$$

are simple. Where $b_t = (t-1)\left[t\alpha k_t + \left(n-(t-1)\right)k_{t-1}\right]$ for $t = 2,3,4,\dots,n$

Corollary 2.1. Let $f(z)=\sum_{i=0}^{n} k_i z^i$ be the n^{th} degree polynomial, let α be real number, $s\geq 1, \eta \geq 0$ such that for some

$$0 < b_n \leq b_{n-1} \leq \cdots \leq b_{m+1} \leq s b_m \geq b_{m-1} \geq \cdots \geq b_3 \geq b_2 - \eta > 0.$$

Then all zeros of $D_\alpha f(z)$ which does not lie in

$$|z| \leq \frac{-b_n + s b_m + 2\eta}{|b_n|}$$

are simple. Where $b_t = (t-1)[t\alpha k_t + (n-(t-1))k_{t-1}]$ for $t = 2,3,4,\dots,n$

Corollary 2.2. Let $f(z)=\sum_{i=0}^{n} k_i z^i$ be the n^{th} degree polynomial, let α be real number, such that for some

$$b_n \leq b_{n-1} \leq \cdots \leq b_{m+1} \leq b_m \geq b_{m-1} \geq \cdots \geq b_3 \geq b_2.$$

Then all zeros of $D_\alpha f(z)$ which does not lie in

$$|z| \leq \frac{-b_n + 2|b_m| - b_m + |b_2| - b_2}{|b_n|}$$

are simple. Where $b_t = (t-1)\left[t\alpha k_t + \left(n-(t-1)\right)k_{t-1}\right]$ for $t = 2,3,4,\dots,n$

Corollary 2.3. Let $f(z)=\sum_{i=0}^{n} k_i z^i$ be the n^{th} degree polynomial, let α be real number, such that for some

$$0 < b_n \leq b_{n-1} \leq \cdots \leq b_{m+1} \leq b_m \geq b_{m-1} \geq \cdots \geq b_3 \geq b_2 > 0.$$

Then all zeros of $D_\alpha f(z)$ which does not lie in

$$|z| \leq \frac{-b_n + b_m}{|b_n|}$$

are simple. Where $b_t = (t-1)\left[t\alpha k_t + \left(n-(t-1)\right)k_{t-1}\right]$ for $t = 2,3,4,\dots,n$

Corollary 2.4. Let $f(z)=\sum_{i=0}^{n} k_i z^i$ be the n^{th} degree polynomial, let α be real number ,$s\geq 1, \eta \geq 0$ such that for some

$$c_n \leq c_{n-1} \leq \cdots \leq c_{m+1} \leq s c_m \geq c_{m-1} \geq \cdots \geq c_3 \geq c_2 - \eta.$$

Then all zeros of $D_\alpha f(z)$ which does not lie in

$$|z| \leq \frac{-c_n + 2s(c_m + |c_m|) - 2c_m + |c_2| - c_2 + 2\eta}{|c_n|}$$

are simple. Where $c_t = (t-1)\left[\left(n-(t-1)\right)k_{t-1}\right]$ for $t = 2,3,4,\dots,n$

Corollary 2.5. Let $f(z)=\sum_{i=0}^{n} k_i z^i$ be the n^{th} degree polynomial, let α be real number ,$s\geq 1, \eta \geq 0$ such that for some

$$0 < c_n \leq c_{n-1} \leq \cdots \leq c_{m+1} \leq s c_m \geq c_{m-1} \geq \cdots \geq c_3 \geq c_2 - \eta > 0.$$

Then all zeros of $D_\alpha f(z)$ which does not lie in

$$|z| \leq \frac{-c_n + 2|c_m| - c_m + |c_2| - c_2}{|c_n|}$$

are simple. Where $c_t = (t-1)\left[\left(n-(t-1)\right)k_{t-1}\right]$ for $t = 2,3,4,\dots,n$

Corollary 2.6. Let $f(z)=\sum_{i=0}^{n} k_i z^i$ be the n^{th} degree polynomial, let α be real number, $s\geq 1, \eta \geq 0$ such that for some

$$c_n \leq c_{n-1} \leq \cdots \leq c_{m+1} \leq c_m \geq c_{m-1} \geq \cdots \geq c_3 \geq c_2.$$

Then all zeros of $D_\alpha f(z)$ which does not lie in

$$|z| \leq \frac{-c_n + 2|c_m| - c_m + |c_2| - c_2}{|c_n|}$$

are simple. Where $c_t = (t-1)\left[\left(n-(t-1)\right)k_{t-1}\right]$ for $t = 2,3,4,\dots,n$

Corollary 2.7. Let $f(z)=\sum_{i=0}^{n}k_iz^i$ be the n^{th} degree polynomial, let α be real number, $s\geq 1, \eta \geq 0$ such that for some

$$0<c_n\leq c_{n-1}\leq\cdots\leq c_{m+1}\leq c_m\geq c_{m-1}\geq\cdots\geq c_3\geq c_2>0.$$

Then all zeros of $D_\alpha f(z)$ which does not lie in

$$|z|\leq\frac{-c_n+c_m}{|c_n|}$$

are simple. Where $c_t=(t-1)[(n-(t-1))k_{t-1}]$ for $t=2,3,4,\ldots,n$

Remark 2.1.

(1) Theorem 2.1 reduces to Corollary 2.1 if $b_j\geq 0$
(2) Theorem 2.1 reduces to Corollary 2.2 if $s=1,\eta=0$
(3) Theorem 2.1 reduces to Corollary 2.3 if $b_j\geq 0$ and $s=1,\eta=0$
(4) Theorem 2.1 reduces to Corollary 2.4 if $\alpha=0$
(5) Theorem 2.1 reduces to Corollary 2.5 if $c_j\geq 0$ and $\alpha=0$
(6) Theorem 2.1 reduces to Corollary 2.6 if $\alpha=0$ and $s=1,\eta=0$
(7) Theorem 2.1 reduces to Corollary 2.7 if $s=1,\eta=0$, $c_j\geq 0$ and $\alpha=0$

Theorem 2.2. Let $f(z)=\sum_{i=0}^{n}k_iz^i$ be the n^{th} degree polynomial, let α be real number, $0<r\leq 1, \eta\geq 0$ Such that for some

$$rb_n\leq b_{n-1}\leq\cdots\leq b_{m+1}\leq b_m+\eta\geq b_{m-1}\geq\cdots\geq b_3\geq b_2.$$

Then all zeros of $D_\alpha f(z)$ which does not lie in

$$|z|\leq\frac{|b_n|+2b_m+|b_2|-r(b_n+|b_n|)-b_2+4\eta}{|b_n|}$$

are simple. Where $b_t=(t-1)[t\alpha k_t+(n-(t-1))k_{t-1}]$ for $t=2,3,4,\ldots,n$

Corollary 2.8. Let $f(z)=\sum_{i=0}^{n}k_iz^i$ be the n^{th} degree polynomial, let α be real number, $0<r\leq 1, \eta\geq 0$ Such that for some

$$0<rb_n\leq b_{n-1}\leq\cdots\leq b_{m+1}\leq b_m+\eta\geq b_{m-1}\geq\cdots\geq b_3\geq b_2>0.$$

Then all zeros of $D_\alpha f(z)$ which does not lie in

$$|z|\leq\frac{(1-2r)b_n+2b_m+4\eta}{|b_n|}$$

are simple. Where $b_t=(t-1)[t\alpha k_t+(n-(t-1))k_{t-1}]$ for $t=2,3,4,\ldots,n$

Corollary 2.9. Let $f(z)=\sum_{i=0}^{n}k_iz^i$ be the n^{th} degree polynomial, let α be real number, Such that for some

$$b_n\leq b_{n-1}\leq\cdots\leq b_{m+1}\leq b_m\geq b_{m-1}\geq\cdots\geq b_3\geq b_2.$$

Then all zeros of $D_\alpha f(z)$ which does not lie in

$$|z|\leq\frac{2b_m+|b_2|-b_n-b_2}{|b_n|}$$

are simple. Where $b_t=(t-1)[t\alpha k_t+(n-(t-1))k_{t-1}]$ for $t=2,3,4,\ldots,n$

Corollary 2.10. Let $f(z)=\sum_{i=0}^{n}k_iz^i$ be the n^{th} degree polynomial, let α be real number, Such that for some

$$0<b_n\leq b_{n-1}\leq\cdots\leq b_{m+1}\leq b_m\geq b_{m-1}\geq\cdots\geq b_3\geq b_2>0.$$

Then all zeros of $D_\alpha f(z)$ which does not lie in

$$|z|\leq\frac{2b_m-b_n}{|b_n|}$$

are simple. Where $b_t=(t-1)[t\alpha k_t+(n-(t-1))k_{t-1}]$ for $t=2,3,4,\ldots,n$

Corollary 2.11. Let $f(z)=\sum_{i=0}^{n}k_iz^i$ be the n^{th} degree polynomial, let α be real number, $0<r\leq 1, \eta\geq 0$ Such that for some

$$rc_n\leq c_{n-1}\leq\cdots\leq c_{m+1}\leq c_m+\eta\geq c_{m-1}\geq\cdots\geq c_3\geq c_2.$$

Then all zeros of $D_\alpha f(z)$ which does not lie in

$$|z|\leq\frac{|c_n|+2c_m+|c_2|-r(c_n+|c_n|)-c_2+4\eta}{|c_n|}$$

are simple. Where $c_t=(t-1)[(n-(t-1))k_{t-1}]$ for $t=2,3,4,\ldots,n$

Corollary 2.12. Let $f(z)=\sum_{i=0}^{n}k_iz^i$ be the n^{th} degree polynomial, let α be real number, $0<r\leq 1,\eta\geq 0$ Such that for some

$$0<rc_n\leq c_{n-1}\leq\cdots\leq c_{m+1}\leq c_m+\eta\geq c_{m-1}\geq\cdots\geq c_3\geq c_2>0.$$

Then all zeros of $D_\alpha f(z)$ which does not lie in

$$|z|\leq\frac{(1-2r)c_n+2c_m+4\eta}{|c_n|}$$

are simple. Where $c_t=(t-1)\big[\big(n-(t-1)\big)k_{t-1}\big]$ for $t=2,3,4,\ldots,n$

Corollary 2.13. Let $f(z)=\sum_{i=0}^{n}k_iz^i$ be the n^{th} degree polynomial, let α be real number, Such that for some

$$c_n\leq c_{n-1}\leq\cdots\leq c_{m+1}\leq c_m\geq c_{m-1}\geq\cdots\geq c_3\geq c_2.$$

Then all zeros of $D_\alpha f(z)$ which does not lie in

$$|z|\leq\frac{2c_m+|c_2|-c_n-c_2}{|c_n|}$$

are simple. Where $c_t=(t-1)\big[\big(n-(t-1)\big)k_{t-1}\big]$ for $t=2,3,4,\ldots,n$

Corollary 2.14. Let $f(z)=\sum_{i=0}^{n}k_iz^i$ be the n^{th} degree polynomial, let α be real number, Such that for some

$$0<c_n\leq c_{n-1}\leq\cdots\leq c_{m+1}\leq c_m\geq c_{m-1}\geq\cdots\geq c_3\geq c_2>0.$$

Then all zeros of $D_\alpha f(z)$ which does not lie in

$$|z|\leq\frac{2c_m-c_n}{|c_n|}$$

are simple. Where $c_t=(t-1)\big[\big(n-(t-1)\big)k_{t-1}\big]$ for $t=2,3,4,\ldots,n$

Remark 2.2.

(1) Theorem 2.2 reduces to Corollary 2.8 if $b_j\geq 0$
(2) Theorem 2.2 reduces to Corollary 2.9 if $r=1,\eta=0$
(3) Theorem 2.2 reduces to Corollary 2.10 if $b_j\geq 0$ and $r=1,\eta=0$
(4) Theorem 2.2 reduces to Corollary 2.11 if $\alpha=0$
(5) Theorem 2.2 reduces to Corollary 2.12 if $c_j\geq 0$ and $\alpha=0$
(6) Theorem 2.2 reduces to Corollary 2.13 if $\alpha=0$ and $r=1,\eta=0$
(7) Theorem 2.2 reduces to Corollary 2.14 if $r=1,\eta=0$, $c_j\geq 0$ and $\alpha=0$

Theorem 2.3. Let $f(z)=\sum_{i=0}^{n}k_iz^i$ be the n^{th} degree polynomial, let α be real number, $r\geq 1,0<\eta\leq 1$ Such that for some

$$b_n\geq b_{n-1}\geq\cdots\geq b_{m+1}\geq rb_m\leq b_{m-1}\leq\cdots\leq b_3\leq b_2+\eta.$$

Then all zeros of $D_\alpha f(z)$ which does not lie in

$$|z|\leq\frac{b_n+2|b_m|-2r(b_m+|b_m|)+|b_2|+b_2+2\eta}{|b_n|}$$

are simple. Where $b_t=(t-1)\big[t\alpha k_t+\big(n-(t-1)\big)k_{t-1}\big]$ for $t=2,3,4,\ldots,n$

Corollary 2.15. Let $f(z)=\sum_{i=0}^{n}k_iz^i$ be the n^{th} degree polynomial, let α be real number, $r\geq 1,0<\eta\leq 1$ Such that for some

$$0<b_n\geq b_{n-1}\geq\cdots\geq b_{m+1}\geq rb_m\leq b_{m-1}\leq\cdots\leq b_3\leq b_2+\eta>0.$$

Then all zeros of $D_\alpha f(z)$ which does not lie in

$$|z|\leq\frac{b_n+2(1-2r)b_m+2b_2+2\eta}{|b_n|}$$

are simple. Where $b_t=(t-1)\big[t\alpha k_t+\big(n-(t-1)\big)k_{t-1}\big]$ for $t=2,3,4,\ldots,n$

Corollary 2.16. Let $f(z)=\sum_{i=0}^{n}k_iz^i$ be the n^{th} degree polynomial, let α be real number, Such that for some

$$b_n\geq b_{n-1}\geq\cdots\geq b_{m+1}\geq b_m\leq b_{m-1}\leq\cdots\leq b_3\leq b_2.$$

Then all zeros of $D_\alpha f(z)$ which does not lie in

$$|z|\leq\frac{b_n-2b_m+|b_2|+b_2}{|b_n|}$$

are simple. Where $b_t=(t-1)\big[t\alpha k_t+\big(n-(t-1)\big)k_{t-1}\big]$ for $t=2,3,4,\ldots,n$

Corollary 2.17. Let $f(z)=\sum_{i=0}^{n}k_iz^i$ be the n^{th} degree e polynomial, let α be real number, Such that for some

$$0 < b_n \geq b_{n-1} \geq \cdots \geq b_{m+1} \geq b_m \leq b_{m-1} \leq \cdots \leq b_3 \leq b_2 > 0.$$

Then all zeros of $D_\alpha f(z)$ which does not lie in

$$|z| \leq \frac{b_n - 2b_m + 2b_2}{|b_n|}$$

are simple. Where $b_t = (t-1)[t\alpha k_t + (n-(t-1))k_{t-1}]$ for $t = 2,3,4,\ldots,n$

Corollary 2.18. Let $f(z) = \sum_{i=0}^{n} k_i z^i$ be the n^{th} degree polynomial, let α be real number, $r \geq 1, 0 < \eta \leq 1$ Such that for some

$$c_n \geq c_{n-1} \geq \cdots \geq c_{m+1} \geq rc_m \leq c_{m-1} \leq \cdots \leq c_3 \leq c_2 + \eta.$$

Then all zeros of $D_\alpha f(z)$ which does not lie in

$$|z| \leq \frac{c_n + 2|c_m| - 2r(c_m + |c_m|) + |c_2| + c_2 + 2\eta}{|c_n|}$$

are simple. Where $c_t = (t-1)[(n-(t-1))k_{t-1}]$ for $t = 2,3,4,\ldots,n$

Corollary 2.19. Let $f(z) = \sum_{i=0}^{n} k_i z^i$ be the n^{th} degree polynomial, let α be real number, $r \geq 1, 0 < \eta \leq 1$ Such that for some

$$0 < c_n \geq c_{n-1} \geq \cdots \geq c_{m+1} \geq rc_m \leq c_{m-1} \leq \cdots \leq c_3 \leq c_2 + \eta > 0.$$

Then all zeros of $D_\alpha f(z)$ which does not lie in

$$|z| \leq \frac{c_n + 2(1-2r)c_m + 2c_2 + 2\eta}{|c_n|}$$

are simple. Where $c_t = (t-1)[(n-(t-1))k_{t-1}]$ for $t = 2,3,4,\ldots,n$

Corollary 2.20. Let $f(z) = \sum_{i=0}^{n} k_i z^i$ be the n^{th} degree polynomial, let α be real number, Such that for some

$$c_n \geq c_{n-1} \geq \cdots \geq c_{m+1} \geq c_m \leq c_{m-1} \leq \cdots \leq c_3 \leq c_2.$$

Then all zeros of $D_\alpha f(z)$ which does not lie in

$$|z| \leq \frac{c_n - 2c_m + |c_2| + c_2}{|c_n|}$$

are simple. Where $c_t = (t-1)[(n-(t-1))k_{t-1}]$ for $t = 2,3,4,\ldots,n$

Corollary 2.21. Let $f(z) = \sum_{i=0}^{n} k_i z^i$ be the n^{th} degree polynomial, let α be real number, Such that for some

$$0 < c_n \geq c_{n-1} \geq \cdots \geq c_{m+1} \geq c_m \leq c_{m-1} \leq \cdots \leq c_3 \leq c_2 > 0.$$

Then all zeros of $D_\alpha f(z)$ which does not lie in

$$|z| \leq \frac{c_n - 2c_m + 2c_2}{|c_n|}$$

are simple. Where $c_t = (t-1)[(n-(t-1))k_{t-1}]$ for $t = 2,3,4,\ldots,n$

Remark 2.3.

(1) Theorem 2.3 reduces to Corollary 2.15 if $b_j \geq 0$

(2) Theorem 2.3 reduces to Corollary 2.16 if $r = 1, \eta = 0$

(3) Theorem 2.3 reduces to Corollary 2.17 if $b_j \geq 0$ and $r = 1, \eta = 0$

(4) Theorem 2.3 reduces to Corollary 2.18 if $\alpha = 0$

(5) Theorem 2.3 reduces to Corollary 2.19 if $c_j \geq 0$ and $\alpha = 0$

(6) Theorem 2.3 reduces to Corollary 2.20 if $\alpha = 0$ and $r = 1, \eta = 0$

(7) Theorem 2.3 reduces to Corollary 2.21 if $r = 1, \eta = 0$, $c_j \geq 0$ and $\alpha = 0$

Theorem 2.3. Let $f(z) = \sum_{i=0}^{n} k_i z^i$ be the n^{th} degree polynomial, let α be real number, $s \geq 1, \eta > 1$ Such that for some

$$sb_n \geq b_{n-1} \geq \cdots \geq b_{m+1} \geq b_m - \eta \leq b_{m-1} \leq \cdots \leq b_3 \leq b_2.$$

Then all zeros of $D_\alpha f(z)$ which does not lie in

$$|z| \leq \frac{s(b_n + |b_n|) - |b_n| - 2b_m + |b_2| + b_2 + 4\eta}{|b_n|}$$

are simple. Where $b_t = (t-1)[t\alpha k_t + (n-(t-1))k_{t-1}]$ for $t = 2,3,4,\ldots,n$

Corollary 2.22. Let $f(z) = \sum_{i=0}^{n} k_i z^i$ be the n^{th} degree polynomial, let α be real number, $s \geq 1, \eta > 1$ Such that for some

$$0 < sb_n \geq b_{n-1} \geq \cdots \geq b_{m+1} \geq b_m - \eta \leq b_{m-1} \leq \cdots \leq b_3 \leq b_2 > 0.$$

Then all zeros of $D_\alpha f(z)$ which does not lie in

$$|z| \leq \frac{(2s-1)b_n - 2b_m + 2b_2 + 4\eta}{|b_n|}$$

are simple. Where $b_t = (t-1)\left[t\alpha k_t + \left(n-(t-1)\right)k_{t-1}\right]$ for $t = 2,3,4,\ldots,n$

Corollary 2.23. Let $f(z) = \sum_{i=0}^{n} k_i z^i$ be the n^{th} degree polynomial, let α be real number, $s \geq 1, \eta > 1$ Such that for some

$$b_n \geq b_{n-1} \geq \cdots \geq b_{m+1} \geq b_m \leq b_{m-1} \leq \cdots \leq b_3 \leq b_2.$$

Then all zeros of $D_\alpha f(z)$ which does not lie in

$$|z| \leq \frac{b_n - 2b_m + |b_2| + b_2}{|b_n|}$$

are simple. Where $b_t = (t-1)\left[t\alpha k_t + \left(n-(t-1)\right)k_{t-1}\right]$ for $t = 2,3,4,\ldots,n$

Corollary 2.24. Let $f(z) = \sum_{i=0}^{n} k_i z^i$ be the n^{th} degree polynomial, let α be real number, $s \geq 1, \eta > 1$ Such that for some

$$0 < b_n \geq b_{n-1} \geq \cdots \geq b_{m+1} \geq b_m \leq b_{m-1} \leq \cdots \leq b_3 \leq b_2 > 0.$$

Then all zeros of $D_\alpha f(z)$ which does not lie in

$$|z| \leq \frac{b_n - 2b_m + 2b_2}{|b_n|}$$

are simple. Where $b_t = (t-1)\left[t\alpha k_t + \left(n-(t-1)\right)k_{t-1}\right]$ for $t = 2,3,4,\ldots,n$

Corollary 2.25. Let $f(z) = \sum_{i=0}^{n} k_i z^i$ be the n^{th} degree polynomial, let α be real number, $s \geq 1, \eta > 1$ Such that for some

$$sc_n \geq c_{n-1} \geq \cdots \geq c_{m+1} \geq c_m - \eta \leq c_{m-1} \leq \cdots \leq c_3 \leq c_2.$$

Then all zeros of $D_\alpha f(z)$ which does not lie in

$$|z| \leq \frac{s(c_n + |c_n|) - |c_n| - 2c_m + |c_2| + c_2 + 4\eta}{|c_n|}$$

are simple. Where $c_t = (t-1)\left[\left(n-(t-1)\right)k_{t-1}\right]$ for $t = 2,3,4,\ldots,n$

Corollary 2.26. Let $f(z) = \sum_{i=0}^{n} k_i z^i$ be the n^{th} degree polynomial, let α be real number, $s \geq 1, \eta > 1$ Such that for some

$$0 < sc_n \geq c_{n-1} \geq \cdots \geq c_{m+1} \geq c_m - \eta \leq c_{m-1} \leq \cdots \leq c_3 \leq c_2 > 0.$$

Then all zeros of $D_\alpha f(z)$ which does not lie in

$$|z| \leq \frac{(2s-1)c_n - 2c_m + 2c_2 + 4\eta}{|c_n|}$$

are simple. Where $c_t = (t-1)\left[\left(n-(t-1)\right)k_{t-1}\right]$ for $t = 2,3,4,\ldots,n$

Corollary 2.27. Let $f(z) = \sum_{i=0}^{n} k_i z^i$ be the n^{th} degree polynomial, let α be real number, $s \geq 1, \eta > 1$ Such that for some

$$c_n \geq c_{n-1} \geq \cdots \geq c_{m+1} \geq c_m \leq c_{m-1} \leq \cdots \leq c_3 \leq c_2.$$

Then all zeros of $D_\alpha f(z)$ which does not lie in

$$|z| \leq \frac{c_n - 2c_m + |c_2| + c_2}{|c_n|}$$

are simple. Where $c_t = (t-1)\left[\left(n-(t-1)\right)k_{t-1}\right]$ for $t = 2,3,4,\ldots,n$

Corollary 2.28. Let $f(z) = \sum_{i=0}^{n} k_i z^i$ be the n^{th} degree polynomial, let α be real number, $s \geq 1, \eta > 1$ Such that for some

$$0 < c_n \geq c_{n-1} \geq \cdots \geq c_{m+1} \geq c_m \leq c_{m-1} \leq \cdots \leq c_3 \leq c_2 > 0.$$

Then all zeros of $D_\alpha f(z)$ which does not lie in

$$|z| \leq \frac{c_n - 2c_m + 2c_2}{|c_n|}$$

are simple. Where $c_t = (t-1)\left[\left(n-(t-1)\right)k_{t-1}\right]$ for $t = 2,3,4,\ldots,n$

Remark 2.1.

(1) Theorem 2.4 reduces to Corollary 2.22 if $b_j \geq 0$
(2) Theorem 2.4 reduces to Corollary 2.23 if $s = 1, \eta = 0$
(3) Theorem 2.4 reduces to Corollary 2.24 if $b_j \geq 0$ and $s = 1, \eta = 0$
(4) Theorem 2.4 reduces to Corollary 2.25 if $\alpha = 0$
(5) Theorem 2.4 reduces to Corollary 2.26 if $c_j \geq 0$ and $\alpha = 0$
(6) Theorem 2.4 reduces to Corollary 2.27 if $\alpha = 0$ and $s = 1, \eta = 0$
(7) Theorem 2.4 reduces to Corollary 2.28 if $s = 1, \eta = 0$, $c_j \geq 0$ and $\alpha = 0$

3. Proofs of the Theorems

Proof of the Theorem 2.1.

Let $f(z) = k_0 + k_1 z + k_2 z^2 + \cdots + k_n z^n$ be the n^{th} degree polynomial with real coefficients. By definition of polar derivative, we have $D_\alpha f(z) = nf(z) + (\alpha - z)f'(z)$
Therefore $D_\alpha f(z) = nf(z) + \alpha f'(z) - zf'(z)$

$$D_\alpha f(z) = n(k_0 + k_1 z + k_2 z^2 + \cdots + k_n z^n) + \alpha(k_0 + k_1 z + k_2 z^2 + \cdots + k_n z^n)' - z(k_0 + k_1 z + k_2 z^2 + \cdots + k_n z^n)'$$

$$D_\alpha f(z) = n(k_0 + k_1 z + k_2 z^2 + \cdots + k_n z^n) + \alpha(k_1 + 2k_2 z + \cdots + z\, k_n z^{n-1}) - z(k_1 + 2k_2 z + \cdots + z\, k_n z^{n-1})$$

$$D_\alpha f(z) = [n\alpha k_n + (n-(n-1))k_{n-1}]z^{n-1} + [(n-1)\alpha k_{n-1} + (n-(n-2))k_{n-2}]z^{n-2} + \cdots + [2\alpha k_2 + (n-1)k_1]z + [\alpha k_1 + nk_0]$$

$$D'_\alpha f(z) = b_n z^{n-2} + b_{n-1} z^{n-3} + b_{n-2} z^{n-4} + \cdots + b_4 z^2 + b_3 z + b_2$$

Where $b_t = (t-1)[t\alpha k_t + (n-(t-1))k_{t-1}]$ for $t = 2,3,4,\ldots,n$
Now consider $g(z) = (1-z)D'_\alpha f(z)$, so that

$$g(z) = (1-z)(b_n z^{n-2} + b_{n-1} z^{n-3} + b_{n-2} z^{n-4} + \cdots + b_4 z^2 + b_3 z + b_2)$$

$$g(z) = -b_n z^{n-1} + (b_n - b_{n-1})z^{n-2} + (b_{n-1} - b_{n-2})z^{n-3} + (b_{n-2} - b_{n-3})z^{n-4} + \cdots + (b_{m+1} - b_m)z^{m-1} + (b_m - b_{m-1})z^{m-2} + \cdots + (b_4 - b_3)z^2 + (b_3 - b_2)z + b_2$$

Then

$$|g(z)| \geq |b_n||z|^{n-2}\left[|z| - \frac{1}{|b_n|}\left\{|b_n - b_{n-1}| + \frac{|b_{n-1} - b_{n-2}|}{|z|} + \frac{|b_{n-2} - b_{n-3}|}{|z|^2} + \cdots + \frac{|b_3 - b_2|}{|z|^{n-3}} + \frac{|b_2|}{|z|^{n-2}}\right\}\right]$$

If $|z| > 1$ then $\frac{1}{|z|} < 1$, *then we have*

$$|g(z)| \geq |b_n||z|^{n-2}\Big[|z| - \frac{1}{|b_n|}\{|b_n - b_{n-1}| + |b_{n-1} - b_{n-2}| + |b_{n-2} - b_{n-3}| + \cdots + |b_{m+2} - b_{m+1}| + |b_{m+1} - sb_m| + |sb_m + b_m| + |sb_m - b_{m-1}| + \cdots + |b_3 - (b_2 - \eta)| + |\eta| + |b_2|\}\Big]$$

$$\geq |b_n||z|^{n-2}\Big[|z| - \frac{1}{|b_n|}\{b_{n-1} - b_n + b_{n-2} - b_{n-1} + \cdots + b_{m+1} - b_{m+2} + sb_m - b_{m+1} + (s-1)|b_m| + (s-1)|b_m| + sb_m - b_{m-1} \ldots + b_3 - (b_2 - \eta) + \eta + |b_2|\}\Big]$$

$$\geq |b_n||z|^{n-2}\left[|z| - \frac{1}{|b_n|}\{-b_n + sb_m + 2s|b_m| - 2sb_m - b_2 + \eta + \eta + |b_2|\}\right]$$

$$\geq |b_n||z|^{n-2}\left[|z| - \frac{1}{|b_n|}\{-b_n + sb_m + 2s(|b_m| - b_m) - b_2 + |b_2| + 2\eta\}\right].$$

Hence g (z) > 0 provided $|z| > \frac{1}{|b_n|}\{-b_n + sb_m + 2s(|b_m| - b_m) - b_2 + |b_2| + 2\eta\}$
This shows that all zeros of g (z) whose modulus is greater than 1 are lie in

$$|z| \leq \frac{-b_n + sb_m + 2s(|b_m| - b_m) - b_2 + |b_2| + 2\eta}{|b_n|}.$$

Since zeros of g(z) whose modulus is less than or equal to 1 are already lie in

$$|z| \leq \frac{-b_n + sb_m + 2s(|b_m| - b_m) - b_2 + |b_2| + 2\eta}{|b_n|}$$

it follows that all zeros of g(z) lie in

$$|z| \leq \frac{-b_n + sb_m + 2s(|b_m| - b_m) - b_2 + |b_2| + 2\eta}{|b_n|}.$$

Since all zeros of g(z) are also the zeros of $D'_\alpha f(z)$. Therefore all zeros of $D'_\alpha f(z)$ lie in

$$|z| \leq \frac{-b_n + sb_m + 2s(|b_m| - b_m) - b_2 + |b_2| + 2\eta}{|b_n|}.$$

In other words all zeros of $D_\alpha f(z)$ which does not lie in

$$|z| \leq \frac{-b_n + sb_m + 2s(|b_m| - b_m) - b_2 + |b_2| + 2\eta}{|b_n|}$$

are simple. Where $b_t = (t-1)[t\alpha k_t + (n-(t-1))k_{t-1}]$ for $t = 2,3,4,\dots,n$

Proof of the Theorem 2.2.

Let $f(z) = k_0 + k_1 z + k_2 z^2 + \cdots + k_n z^n$ be the n^{th} degree polynomial with real coefficients. By definition of polar derivative, we have $D_\alpha f(z) = nf(z) + (\alpha - z)f'(z)$

Therefore $D_\alpha f(z) = nf(z) + \alpha f'(z) - zf'(z)$

$$D_\alpha f(z) = n(k_0 + k_1 z + k_2 z^2 + \cdots + k_n z^n) + \alpha(k_0 + k_1 z + k_2 z^2 + \cdots + k_n z^n)' - z(k_0 + k_1 z + k_2 z^2 + \cdots + k_n z^n)'$$

$$D_\alpha f(z) = n(k_0 + k_1 z + k_2 z^2 + \cdots + k_n z^n) + \alpha(k_1 + 2k_2 z + \cdots + z\, k_n z^{n-1}) - z(k_1 + 2k_2 z + \cdots + z\, k_n z^{n-1})$$

$$D_\alpha f(z) = [n\alpha k_n + (n-(n-1))k_{n-1}]z^{n-1} + [(n-1)\alpha k_{n-1} + (n-(n-2))k_{n-2}]z^{n-2} + \cdots + [2\alpha k_2 + (n-1)k_1]z + [\alpha k_1 + nk_0]$$

$$D'_\alpha f(z) = b_n z^{n-2} + b_{n-1} z^{n-3} + b_{n-2} z^{n-4} + \cdots + b_4 z^2 + b_3 z + b_2$$

Where $b_t = (t-1)[t\alpha k_t + (n-(t-1))k_{t-1}]$ for $t = 2,3,4,\dots,n$

Now consider $g(z) = (1-z)D'_\alpha f(z)$, so that

$$g(z) = (1-z)(b_n z^{n-2} + b_{n-1} z^{n-3} + b_{n-2} z^{n-4} + \cdots + b_4 z^2 + b_3 z + b_2)$$

$$g(z) = -b_n z^{n-1} + (b_n - b_{n-1})z^{n-2} + (b_{n-1} - b_{n-2})z^{n-3} + (b_{n-2} - b_{n-3})z^{n-4} + \cdots + (b_{m+1} - b_m)z^{m-1} + (b_m - b_{m-1})z^{m-2} + \cdots + (b_4 - b_3)z^2 + (b_3 - b_2)z + b_2$$

Then

$$|g(z)| \geq |b_n||z|^{n-2}\Big[\,|z| - \frac{1}{|b_n|}\Big\{|b_n - b_{n-1}| + \frac{|b_{n-1} - b_{n-2}|}{|z|} + \frac{|b_{n-2} - b_{n-3}|}{|z|^2} + \cdots + \frac{|b_3 - b_2|}{|z|^{n-3}} + \frac{|b_2|}{|z|^{n-2}}\Big\}\Big]$$

If $|z| > 1$ then $\frac{1}{|z|} < 1,$ *then we have*

$$|g(z)| \geq |b_n||z|^{n-2}\Big[\,|z| - \frac{1}{|b_n|}\Big\{|b_n - rb_n| + |rb_n - b_{n-1}| + \cdots + |b_{m+2} - b_{m+1}| + |b_{m+1} - (b_m + \eta)| + |\eta| + |b_m + \eta - b_{m-1}| + \eta \dots + |b_3 - b_2| + |b_2|\Big\}\Big]$$

$$|g(z)| \geq |b_n||z|^{n-2}\Big[\,|z| - \frac{1}{|b_n|}\{|b_n| + 2b_m + |b_2| - r(b_n + |b_n|) - b_2 + 4\eta\}$$

Hence |g(z)| > 0 provided

$$|z| > \frac{|b_n| + 2b_m + |b_2| - r(b_n + |b_n|) - b_2 + 4\eta}{|b_n|}$$

This shows that all zeros of g (z) whose modulus is greater than 1 are lie in

$$|z| \leq \frac{|b_n| + 2b_m + |b_2| - r(b_n + |b_n|) - b_2 + 4\eta}{|b_n|}$$

Since zeros of g(z) whose modulus is less than or equal to 1 are already lie in

$$|z| \leq \frac{|b_n| + 2b_m + |b_2| - r(b_n + |b_n|) - b_2 + 4\eta}{|b_n|}$$

it follows that all zeros of g(z) lie in

$$|z| \leq \frac{|b_n| + 2b_m + |b_2| - r(b_n + |b_n|) - b_2 + 4\eta}{|b_n|}$$

Since all zeros of g(z) are also the zeros of $D'_{\alpha}f(z)$. Therefore all zeros of $D'_{\alpha}f(z)$ lie in

$$|z| \le \frac{|b_n| + 2b_m + |b_2| - r(b_n + |b_n|) - b_2 + 4\eta}{|b_n|}$$

In other words all zeros of $D_{\alpha}f(z)$ which does not lie in

$$|z| \le \frac{|b_n| + 2b_m + |b_2| - r(b_n + |b_n|) - b_2 + 4\eta}{|b_n|}$$

are simple. Where $b_t = (t-1)\left[t\alpha k_t + \left(n-(t-1)\right)k_{t-1}\right]$ for $t = 2,3,4,\dots,n$

Proof of the Theorem 2.3.

Let $f(z) = k_0 + k_1 z + k_2 z^2 + \dots + k_n z^n$ be the n^{th} degree polynomial with real coefficients. By definition of polar derivative, we have $D_{\alpha}f(z) = nf(z) + (\alpha - z)f'(z)$

Therefore $D_{\alpha}f(z) = nf(z) + \alpha f'(z) - zf'(z)$

$$D_{\alpha}f(z) = n(k_0 + k_1 z + k_2 z^2 + \dots + k_n z^n) + \alpha(k_0 + k_1 z + k_2 z^2 + \dots + k_n z^n)' - z(k_0 + k_1 z + k_2 z^2 + \dots + k_n z^n)'$$

$$D_{\alpha}f(z) = n\left(k_0 + k_1 z + k_2 z^2 + \dots + k_n z^n\right) + \alpha(k_1 + 2k_2 z + \dots + z\, k_n z^{n-1}) - z(k_1 + 2k_2 z + \dots + z\, k_n z^{n-1})$$

$$D_{\alpha}f(z) = \left[n\alpha k_n + \left(n-(n-1)\right)k_{n-1}\right]z^{n-1} + \left[(n-1)\alpha k_{n-1} + \left(n-(n-2)\right)k_{n-2}\right]z^{n-2} + \dots + [2\alpha k_2 + (n-1)k_1]z + [\alpha k_1 + nk_0]$$

$$D'_{\alpha}f(z) = b_n z^{n-2} + b_{n-1}z^{n-3} + b_{n-2}z^{n-4} + \dots + b_4 z^2 + b_3 z + b_2$$

Where $b_t = (t-1)\left[t\alpha k_t + \left(n-(t-1)\right)k_{t-1}\right]$ for $t = 2,3,4,\dots,n$

Now consider $g(z) = (1-z)D'_{\alpha}f(z)$, so that

$$g(z) = (1-z)(b_n z^{n-2} + b_{n-1}z^{n-3} + b_{n-2}z^{n-4} + \dots + b_4 z^2 + b_3 z + b_2)$$

$$g(z) = -b_n z^{n-1} + (b_n - b_{n-1})z^{n-2} + (b_{n-1} - b_{n-2})z^{n-3} + (b_{n-2} - b_{n-3})z^{n-4} + \dots + (b_{m+1} - b_m)z^{m-1} + (b_m - b_{m-1})z^{m-2} + \dots + (b_4 - b_3)z^2 + (b_3 - b_2)z + b_2$$

Then

$$|g(z)| \ge |b_n||z|^{n-2}\left[\,|z| - \frac{1}{|b_n|}\left\{|b_n - b_{n-1}| + \frac{|b_{n-1} - b_{n-2}|}{|z|} + \frac{|b_{n-2} - b_{n-3}|}{|z|^2} + \dots + \frac{|b_3 - b_2|}{|z|^{n-3}} + \frac{|b_2|}{|z|^{n-2}}\right\}\right]$$

If $|z| > 1$ then $\frac{1}{|z|} < 1$, *then we have*

$$|g(z)| \ge |b_n||z|^{n-2}\left[\,|z| - \frac{1}{|b_n|}\{|b_n - b_n| + \dots + |b_{m+2} - b_{m+1}| + |b_{m+1} - rb_m| + |rb_m - b_m| + |b_m - rb_m| + |rb_m - b_{m-1}| + \dots + |b_3 - (b_2 + \eta)| + |\eta| + |b_2|\}\right]$$

$$|g(z)| \ge |b_n||z|^{n-2}\left[\,|z| - \frac{1}{|b_n|}\{b_n + 2|b_m| - 2r(b_m + |b_m|) + |b_2| + b_2 + 2\eta\}\right]$$

Hence |g(z)| > 0 provided

$$|z| > \frac{b_n + 2|b_m| - 2r(b_m + |b_m|) + |b_2| + b_2 + 2\eta}{|b_n|}$$

This shows that all zeros of g (z) whose modulus is greater than 1 are lie in

$$|z| \le \frac{b_n + 2|b_m| - 2r(b_m + |b_m|) + |b_2| + b_2 + 2\eta}{|b_n|}$$

Since zeros of g(z) whose modulus is less than or equal to 1 are already lie in

$$|z| \le \frac{b_n + 2|b_m| - 2r(b_m + |b_m|) + |b_2| + b_2 + 2\eta}{|b_n|}$$

it follows that all zeros of g(z) lie in

$$|z| \le \frac{b_n + 2|b_m| - 2r(b_m + |b_m|) + |b_2| + b_2 + 2\eta}{|b_n|}$$

Since all zeros of g(z) are also the zeros of $D'_{\alpha}f(z)$. Therefore all zeros of $D'_{\alpha}f(z)$ lie in

$$|z| \le \frac{b_n + 2|b_m| - 2r(b_m + |b_m|) + |b_2| + b_2 + 2\eta}{|b_n|}$$

In other words all zeros of $D_{\alpha}f(z)$ which does not lie in

$$|z| \leq \frac{b_n + 2|b_m| - 2r(b_m + |b_m|) + |b_2| + b_2 + 2\eta}{|b_n|}$$

are simple. Where $b_t = (t-1)\left[t\alpha k_t + \left(n-(t-1)\right)k_{t-1}\right]$ for $t = 2,3,4,\dots,n$

Proof of the Theorem 2.4.

Let $f(z) = k_0 + k_1 z + k_2 z^2 + \cdots + k_n z^n$ be the n^{th} degree polynomial with real coefficients. By definition of polar derivative, we have $D_\alpha f(z) = nf(z) + (\alpha - z)f'(z)$

Therefore $D_\alpha f(z) = nf(z) + \alpha f'(z) - zf'(z)$

$$D_\alpha f(z) = n(k_0 + k_1 z + k_2 z^2 + \cdots + k_n z^n) + \alpha(k_0 + k_1 z + k_2 z^2 + \cdots + k_n z^n)' - z(k_0 + k_1 z + k_2 z^2 + \cdots + k_n z^n)'$$

$$D_\alpha f(z) = n\left(k_0 + k_1 z + k_2 z^2 + \cdots + k_n z^n\right) + \alpha(k_1 + 2k_2 z + \cdots + z\, k_n z^{n-1}) - z(k_1 + 2k_2 z + \cdots + z\, k_n z^{n-1})$$

$$D_\alpha f(z) = \left[n\alpha k_n + \left(n-(n-1)\right)k_{n-1}\right]z^{n-1} + \left[(n-1)\alpha k_{n-1} + \left(n-(n-2)\right)k_{n-2}\right]z^{n-2} + \cdots + [2\alpha k_2 + (n-1)k_1]z + [\alpha k_1 + nk_0]$$

$$D'_\alpha f(z) = b_n z^{n-2} + b_{n-1} z^{n-3} + b_{n-2} z^{n-4} + \cdots + b_4 z^2 + b_3 z + b_2$$

Where $b_t = (t-1)\left[t\alpha k_t + \left(n-(t-1)\right)k_{t-1}\right]$ for $t = 2,3,4,\dots,n$

Now consider $g(z) = (1-z)D'_\alpha f(z)$, so that

$$g(z) = (1-z)(b_n z^{n-2} + b_{n-1} z^{n-3} + b_{n-2} z^{n-4} + \cdots + b_4 z^2 + b_3 z + b_2)$$

$$g(z) = -b_n z^{n-1} + (b_n - b_{n-1})z^{n-2} + (b_{n-1} - b_{n-2})z^{n-3} + (b_{n-2} - b_{n-3})z^{n-4} + \cdots + (b_{m+1} - b_m)z^{m-1} + (b_m - b_{m-1})z^{m-2} + \cdots + (b_4 - b_3)z^2 + (b_3 - b_2)z + b_2$$

Then

$$|g(z)| \geq |b_n||z|^{n-2}\left[\,|z| - \frac{1}{|b_n|}\left\{|b_n - b_{n-1}| + \frac{|b_{n-1} - b_{n-2}|}{|z|} + \frac{|b_{n-2} - b_{n-3}|}{|z|^2} + \cdots + \frac{|b_3 - b_2|}{|z|^{n-3}} + \frac{|b_2|}{|z|^{n-2}}\right\}\right]$$

If $|z| > 1$ then $\frac{1}{|z|} < 1$, *then we have*

$$|g(z)| \geq |b_n||z|^{n-2}\left[\,|z| - \frac{1}{|b_n|}\left\{|b_n - sb_n| + |sb_n - b_{n-1}| + \cdots + |b_{m+1} - (b_m - \eta)| + |\eta| + |b_m - \eta - b_{m-1}| + |\eta| + \cdots + |b_3 - b_2| + |b_2|\right\}\right]$$

$$|g(z)| \geq |b_n||z|^{n-2}\left[\,|z| - \frac{1}{|b_n|}\{s(b_n + |b_n|) - |b_n| - 2b_m + |b_2| + b_2 + 4\eta\}\right]$$

Hence $|g(z)| > 0$ provided

$$|z| > \frac{s(b_n + |b_n|) - |b_n| - 2b_m + |b_2| + b_2 + 4\eta}{|b_n|}$$

This shows that all zeros of $g(z)$ whose modulus is greater than 1 are lie in

$$|z| \leq \frac{s(b_n + |b_n|) - |b_n| - 2b_m + |b_2| + b_2 + 4\eta}{|b_n|}$$

Since zeros of g(z) whose modulus is less than or equal to 1 are already lie in

$$|z| \leq \frac{s(b_n + |b_n|) - |b_n| - 2b_m + |b_2| + b_2 + 4\eta}{|b_n|}$$

it follows that all zeros of g(z) lie in

$$|z| \leq \frac{s(b_n + |b_n|) - |b_n| - 2b_m + |b_2| + b_2 + 4\eta}{|b_n|}$$

Since all zeros of g(z) are also the zeros of $D'_\alpha f(z)$. Therefore all zeros of $D'_\alpha f(z)$ lie in

$$|z| \leq \frac{s(b_n + |b_n|) - |b_n| - 2b_m + |b_2| + b_2 + 4\eta}{|b_n|}$$

In other words all zeros of $D_\alpha f(z)$ which does not lie in

$$|z| \leq \frac{s(b_n + |b_n|) - |b_n| - 2b_m + |b_2| + b_2 + 4\eta}{|b_n|}$$

are simple. Where $b_t = (t-1)\left[t\alpha k_t + \left(n-(t-1)\right)k_{t-1}\right]$ for $t = 2,3,4,\dots,n$

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