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FUZZY (LEFT, RIGHT) SIMPLE RING AND FUZZY BI-IDEAL-FREE RING

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Abstract : In this paper we introduced the notions of fuzzy (left, right) simple ring and fuzzy bi-ideal-free ring and defined, studied some properties, characterised level subset has been studied fuzzy bi-ideal in a ring.

Keywords : fuzzy (left, right) simple ring, fuzzy bi-ideal-free ring, fuzzy subset.

1. Introduction

Zedeh [1] introduced the notion of a fuzzy subset A of a non-empty set X as a function from X into [(0, 1]. Rosenfeld [2] used this concept and developed some results in fuzzy group theory. Since then, the study of fuzzy algebraic structures has been pursued in many directions such as groups, modules, vector space and so on. Kuroki [3] introduced the notion of fuzzy ideals and fuzzy bi-ideals in semi-groups. Liu [4] introduced the notion of fuzzy ideal of a ring. Subsequently, Mukharjee and Sen [4] Swamy and Raju [5], Yue [6], Dixit et.al. [7], Raj Kumar and [8], Zie [9] developed the theory of fuzzy rings. Since then many researchers explored on the generalization of the notions of fuzzy set and its application to many mathematical branches.

In this paper we introduced the notions of fuzzy (left, right) simple ring and fuzzy bi-ideal-free ring and defined, studied some properties, characterised level subset has been studied fuzzy bi-ideal in a ring.

2. Preliminaries

In this section we give some definitions and results that are used in this paper.

Definition 2.1. Let μ_i, $i \in I$ be fuzzy subsets of a ring R. The intersection of the fuzzy sets μ_i is defined as follows:

$$[\bigcap \mu_i](x) = \inf_{i \in I} [\mu_i](x) : i \in I], x \in X.$$

Definition 2.2. Let μ be a fuzzy subset of a ring R. Imμ is defined by Im$\mu = \{t \in [0, 1] : \mu(x) = t$ for some $x \in X\}$. Let $t \in [0, 1]$. The set $\mu_1 = \{x \in R : \mu(x) \geq t)$ is called a level subset of μ. Clearly, $\mu_1 \subseteq \mu_2$ whenever $t \geq s$.

Definition 2.3. A fuzzy subset μ of a ring R is called a fuzzy left (right) ideal of R, if for every $x, y \in R$

(i) μ is a fuzzy subgroup of $(R, +)$, i.e., $\mu(x\text{-}y) \geq \min\{\mu(x), \mu(y)\}$.

(ii) $\mu(xy) \geq \mu(y)$, $[\mu(xy) \geq \mu(x)]$.

If μ is both a fuzzy left ideal and a fuzzy right ideal of R, then it is called a fuzzy ideal of R.

Definition 2.4. A ring R is said to be simple if $R^2 \neq \{0\}$ and R has no ideals other than (0) and R.

3. Fuzzy Bi-ideals

Definition 3.1. A non-empty fuzzy subset μ (i.e. $\mu(x) = 0$ for some $x \in R$ of a ring R is called a fuzzy bi-ideal of R if

(i) $\mu(x\text{-}y) \geq \min(\mu(x), \mu(y)\}$,

(ii) $\mu(xy) \geq \min \mu(x), \mu(y)\}$,

(iii) $\mu(xyz) \geq \min(\mu(x), \mu(z)\}$, for all $x, y, z \in R$.

Example 3.2. Let R be the ring of all 2 × 2 matrices over the ring of integers with respect to the matrix addition and multiplication. Let μ be a fuzzy subset of R defined as follows:

$$\mu\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) \begin{cases} = 1 \text{ if } a = b = c = d = 0 \\ = 1/2 \text{ if a is non-zero'even integer and } b = c = d = 0, \\ = 1/3 \text{ if a is non-zero odd integer and } b = c = d = 0, \\ = 0 \text{ in all other cases.} \end{cases}$$

Then μ is a fuzzy bi-ideal of R.

Proposition 3.3. A subset A of a ring R is a bi-ideal of R iff its characteristic function X is a fuzzy bi-ideal of R.

Proposition 3.4. Every fuzzy left (right, two-sided) ideal of a ring R is a fuzzy bi-ideal of R.

Remark 3.5. The converse of the above proposition is not true. The fuzzy subset μ given in Example 3.2 is a fuzzy bi-ideal of R but it is not a fuzzy left ideal of R for

$$\mu\left(\begin{pmatrix}2&0\\0&0\end{pmatrix}\right)=\frac{1}{2} \text{ and } \mu\left(\begin{pmatrix}2&3\\4&5\end{pmatrix}\right)\left(\begin{pmatrix}2&0\\0&0\end{pmatrix}\right)=\mu\left(\begin{pmatrix}4&0\\8&0\end{pmatrix}\right)=0.$$

$$\text{so } \mu\left(\begin{pmatrix}2&3\\4&5\end{pmatrix}\right)\left(\begin{pmatrix}2&0\\0&0\end{pmatrix}\right)\not\geq\mu\left(\begin{pmatrix}2&0\\0&0\end{pmatrix}\right)$$

Proposition 3.6. The intersection of an arbitrary collection of fuzzy bi-ideals of a ring *R* is a fuzzy bi-ideal of *R* if it is not empty.

Proposition 3.7. Let μ be a fuzzy subset of a ring *R*. μ is a fuzzy bi-ideal of *R* iff its level sets μ_1's are bi- ideals of *R* for all $t \in \text{Im}\mu$.

Proposition 3.8. Let *R* be a ring and μ be a fuzzy subset of *R*. μ is a fuzzy right (left) ideals of *R* iff the level sets μ_1's are right (left) ideals of *R* for all $t \in \text{Im}\mu$.

Proposition 3.9. For a ring *R* the following conditions are equivalent:

(i) Every bi-ideal of *R* is a right (left) ideal of *R*.

(ii) Every fuzzy.bi-ideal of *R* is a fuzzy right (left) ideal of *R*.

Proof. (i) implies (ii):

Let μ be a fuzzy bi-ideal of *R*.

$\Rightarrow \mu_1$'s are bi-ideals of *R*.

$\Rightarrow \mu_1$'s are right (left) ideals of *R*.

$\Rightarrow \mu_1$'s is a fuzzy right (left) ideal of *R* by Proposition 3.8.

(ii) implies (i):

Let *A* be a bi-ideal of *R*. Then the characteristic function χ_A of *A* is a fuzzy bi-ideal of *R*. Hence by (ii) χ_A is a fuzzy right (left) of *R*. So *A* is a right (left) ideal of *R*.

4. Fuzzy Simple Rings and Fuzzy (left, right) Simple Ring and Fuzzy Bi-ideal-free Ring

Definition 4.1. A ring *R* with unity is said to be fuzzy left (right) simple if for every fuzzy left (right) ideal δ of *R*, $\delta(x) = \delta(y)$ for every $x(\neq 0), y(\neq 0) \in R$.

Definition 4.2. A ring *R* with unity is said to be fuzzy simple if for every fuzzy ideal δ of *R*, $\delta(x) = \delta(y)$ for every $x(\neq 0), y(\neq 0) \in R$.

Example 4.3. Let *Q* be the field of all rational numbers. Let δ be a fuzzy ideal of *Q*. Let $x(\neq 0) \in Q$. Now $\delta(x) = \delta(x.1) \geq \delta(1)$. Again $\delta(1) = \delta(x.x^{-1}) \geq \delta(x)$. Thus $\delta(x) = \delta(1)$ for every $x(\neq 0) \in Q$. Hence $\delta(x) = \delta(y)$ for every $x(\neq 0), y(\neq 0) \in Q$. Thus *Q* is fuzzy simple.

Proposition 4.4. Let *R* be a unitary ring. Then the following conditions are equivalent: (i) *R* is left (right) simple (ii) *R* is fuzzy left (right) simple

Proof. (i) implies (ii) :

Let be a fuzzy left ideal of *R* and $a(\neq 0), b(\neq 0) \in R$. Now *Ra* is a left ideal of *R*. Since $a = 1_R.a \in Ra$, $Ra \neq \{0\}$. So $Ra = R$. Hence $b = xa$ for some $x \in R$. Similarly $a = yb$ for some $y \in R$. Now $\delta(a) = (yb) \geq \delta(b) = \delta(xa) > \delta(a)$. So $\delta(a) = \delta(b)$ for every $a(\neq 0), b(\neq 0) \in R$. So *R* is fuzzy left simple.

(ii) implies (i):

Let A $(\neq\{0\})$ be a left ideal of *R*. Now χ_A is a fuzzy left ideal of *R*. Since *R* is fuzzy left simple $\chi_A(x) = \chi_A(y)$ for every $x(\neq 0)$ and for every $y(\neq 0) \in R$. Let $x(\neq 0) \in A$. and $y(\neq 0) \in R$. Then $\chi_A(y) = \chi_A(x) = 1$. So $y \in A$. Thus $R \subseteq A$ i.e., $R = A$. So *R* has only two left ideals viz. $\{0\}$ and *R*. Hence *R* is left simple.

Proposition 4.5. Let *R* be a unitary ring. Then the following conditions are equivalent:

(i) *R* is simple

(ii) *R* is fuzzy simple.

Proposition 4.6. Let *R* be a unitary ring. *R* is fuzzy left (right) simple iff *R* is a division ring.

Proof. Since *R* is fuzzy left simple from Proposition 4.4 it follows that *R* is left simple. Consequently it follows that *R* is a division ring.

Converse follows by reversing the above arguments.

Proposition 4.7. A commutative unitary ring is a field iff it is fuzzy simple:

Proof. Let *R* be a field. Then *R* is simple. Therefore by Proposition 4.5, *R* is fuzzy simple. Conversely let *R* be fuzzy simple. So by proposition 4.5, *R* is simple. Now since *R* is a commutative unitary ring, *R* is a field.

Definition 4.8. A unitary ring *R* is said to be a bi-ideal-free ring (or free from bi-ideal) if it has only two bi-ideals namely (0) and R itself.

Definition 4.9. A unitary ring *R* is said to be a fuzzy bi-ideal-free ring if for every fuzzy bi-ideal δ of *R*, $\delta(x) = \delta(y)$ for every $x(\neq 0), y(\neq 0) \in R$.

Example 4.10. Let *Q* be the field of all rational numbers. Let δ be a fuzzy bi-ideal of *Q*.

Let $x(\neq 0) \in Q$.

Now $\delta(x) = \delta(1.x.1) \geq \min \delta(1), \delta(1)\} = \delta(1)$.

Again $\delta(1) = S(x.x^{-2}.x) \geq \min \delta(x), \delta(x)\} = \delta(x)$.

Thus $\delta(x) = \delta(1)$ for every $x(\neq 0) \in Q$.

Hence $\delta(x) = \delta(y)$ for every $\delta(x) = \delta(y)$ for every $x(\neq 0), y(\neq 0) \in Q$. Thus *Q* is fuzzy bi-ideal-free.

Proposition 4.11. Let *R* be a unitary ring. Then *R* is a bi-ideal-free ring iff it is fuzzy bi-ideal-free ring:

Proof. First suppose that *R* is a bi-ideal-free ring. So *R* contains no bi-ideal other than $\{0\}$ and *R*. Let $x(\neq 0) \in R$. Now *Rx* is a left ideal of *R* containing *x*. Hence *Rx* is a non-zero bi-ideal of *R*. So $Rx = R$. Let $y(\neq 0) \in R$. Then $y = px$ for some $p \in R$.

Similarly we can show that $xR = R$. So $p = xq$ for some $q \in R$. Let μ be a fuzzy bi-ideal of *R*.

Now $\mu(y) = \mu(px)]\} = \mu(xqx) \geq \min \{\mu(x), \{\mu(x)$.

Similarly we can show that $\mu(x) \geq \mu(y)$.

Hence $\mu(x) \geq \mu(y)$for every $x(\neq 0)$, $y(\neq 0) \in R$. So R is fuzzy bi-ideal free.

Conversely, suppose that R is fuzzy bi-ideal free. Let $A(\neq \{0\})$ be a bi-ideal of R. Then by Proposition 3.3 it follows that xa is a bi-ideal of R. Let $a(\neq 0) \in A$ and $r(\neq 0) \in R$. Then $\chi_A(r) = \chi_A(a) = 1$. So $r \in A$. Then $A = R$. So R contains no non-zero proper bi-ideal of R. Hence R is a bi-ideal-free ring.

Proposition 4.12. R be a unitary ring. If R is fuzzy bi-ideal-free then R is left (right) simple and hence fuzzy left (right) simple.

Proof. Since R is fuzzy bi-ideal-free then by Proposition 4.11 R is bi-ideal free. So R has no proper left ideals i.e., R is left simple. Hence by Proposition 4.4 R is fuzzy left simple.

Proposition 4.13. Let R be a unitary ring. If R is a fuzzy bi-ideal-free ring then R is simple and hence fuzzy simple.

Proposition 4.14. Let R be a unitary ring. If R is fuzzy bi-ideal-free then R is a division ring.

Proof. Since R is fuzzy bi-ideal-free so by Proposition 4.11, R is bi-ideal-free. So R contains no proper left (right) ideal, i.e., R is left (right) simple. Therefore R is a division ring.

Proposition 4.15. A commutative unitary ring is a field iff it is fuzzy bi-ideal-free.

Proof. Let R be a field and $A(\neq \{0\})$ be a bi-ideal of R. Let $a(\neq 0) \in A$. Now since A is a bi-ideal of R, $a.a^{-1}.a \in$ A i.e., $a^{-1} \in A$. Thus $1_R = a.a^{-1} \in A$.

Let $r \in R$. Then $r = I_g\, rI_g \in A$.

Hence $A = R$. So R is bi-ideal-free. Now from Proposition 4.11, it follows that R is fuzzy bi-ideal-free.

Conversely suppose that R is fuzzy bi-ideal-free. Then from Proposition 4.11 it follows that R is bi-ideal-free. Since every ideal is a bi-ideal, R does not contain any proper non-zero ideal. Consequently R is a field.

Corollary 4.16. A commutative unitary ring is fuzzy-bi-ideal-free iff it is simple.

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