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The study of commutative rings in commutative algebra

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Abstract:

In ring theory, a branch of abstract algebra, a commutative ring is a ring in which the multiplication operation is commutative. The study of commutative rings is called commutative algebra. Complementarily, noncommutative algebra is the study of noncommutative ring where multiplication is not required to be commutative.

Keywords: Ring theory, Abstract algebra.

Definition

A *ring* is a set R equipped with two binary operations, i.e. operations combining any two elements of the ring to a third. They are called *addition* and *multiplication* and commonly denoted by "+" and "·"; e.g. $a + b$ and $a \cdot b$. To form a ring these two operations have to satisfy a number of properties: the ring has to be an abelian group under addition as well as a monoidunder multiplication, where multiplication distributes over addition; i.e., $a \cdot (b + c) = (a \cdot b) + (a \cdot c)$. The identity elements for addition and multiplication are denoted 0 and 1, respectively.

If the multiplication is commutative, i.e.

$$a \cdot b = b \cdot a,$$

then the ring R is called *commutative*. In the remainder of this article, all rings will be commutative, unless explicitly stated otherwise.

First examples

An important example, and in some sense crucial, is the ring of integers **Z** with the two operations of addition and multiplication. As the multiplication of integers is a commutative operation, this is a commutative ring. It is usually denoted **Z**as an abbreviation of the German word *Zahlen* (numbers).

A field is a commutative ring where and every non-zero element a is invertible; i.e., has a multiplicative inverse $b$such that $a \cdot b = 1$. Therefore, by definition, any field is a commutative ring. The rational, real and complex numbers form fields.

If R is a given commutative ring, then the set of all polynomials in the variable X whose coefficients are in R forms thepolynomial ring, denoted $R[X]$. The same holds true for several variables.

If V is some topological space, for example a subset of some $\mathbf{R}^n$, real- or complex-valued continuous functions on V form a commutative ring. The same is true for differentiable or holomorphic functions, when the two concepts are defined, such as for V a complex manifold.

Divisibility

In contrast to fields, where every nonzero element is multiplicatively invertible, the concept of divisibility for rings is richer. An element a of ring R is called a unit if it possesses a multiplicative inverse. Another particular type of element is the zero divisors, i.e. an element a such that there exists a non-zero element b of the ring such that $ab = 0$. If R possesses no non-zero zero divisors, it is called an integral domain (or domain). An element a satisfying $a^n = 0$ for some positive integer n is called nilpotent.

Localizations

The *localization* of a ring is a process in which some elements are rendered invertible, i.e. multiplicative inverses are added to the ring. Concretely, if S is a multiplicatively closed subset of R (i.e. whenever $s, t \in S$ then so is st) then the *localization* of R at S, or *ring of fractions* with denominators in S, usually denoted $S^{-1}R$ consists of symbols

$$\frac{r}{s} \text{ with } r \in R, s \in S$$

subject to certain rules that mimic the cancellation familiar from rational numbers. Indeed, in this language **Q** is the localization of **Z** at all nonzero integers. This construction works for any integral domain R instead of **Z**. The localization $(R \setminus \{0\})^{-1}R$ is a field, called the quotient field of R.

Ideals and modules

Many of the following notions also exist for not necessarily commutative rings, but the definitions and properties are usually more complicated. For example, all ideals in a commutative ring are automatically two-sided, which simplifies the situation considerably.

Modules and ideals

For a ring R, an *R-module M* is like what a vector space is to a field. That is, elements in a module can be added; they can be multiplied by elements of R subject to the same axioms as for a vector space. The study of modules is significantly more involved than the one of vector spaces in linear algebra, since several features of vector spaces fail for modules in general: modules need not be free, i.e., of the form

Even for free modules, the rank of a free module (i.e. the analog of the dimension of vector spaces) may not be well-defined. Finally, submodules of finitely generated modules need not be finitely generated (unless *R* is Noetherian, seebelow).

Ideals

Ideals of a ring *R* are the submodules of *R*, i.e., the modules contained in *R*. In more detail, an ideal *I* is a non-empty subset of *R* such that for all *r* in *R*, *i* and *j* in *I*, both *ri* and $i + j$ are in *I*. For various applications, understanding the ideals of a ring is of particular importance, but often one proceeds by studying modules in general.

Any ring has two ideals, namely the zero ideal {0} and *R*, the whole ring. These two ideals are the only ones precisely if *R* is a field. Given any subset $F = \{f_j\}_{j \in J}$ of *R* (where *J* is some index set), the ideal *generated by F* is the smallest ideal that contains *F*. Equivalently, it is given by finite linear combinations

$$r_1 f_1 + r_2 f_2 + \ldots + r_n f_n.$$

Principal ideal domains

If *F* consists of a single element *r*, the ideal generated by *F* consists of the multiples of *r*, i.e., the elements of the form *rs* for arbitrary elements *s*. Such an ideal is called a principal ideal. If every ideal is a principal ideal, *R* is called a principal ideal ring; two important cases are **Z** and $k[X]$, the polynomial ring over a field *k*. These two are in addition domains, so they are called principal ideal domains.

Unlike for general rings, for a principal ideal domain, the properties of individual elements are strongly tied to the properties of the ring as a whole. For example, any principal ideal domain *R* is a unique factorization domain (UFD) which means that any element is a product of irreducible elements, in a (up to reordering of factors) unique way. Here, an element *a* in a domain is called irreducible if the only way of expressing it as a product

$$a = bc,$$

is by either *b* or *c* being a unit. An example, important in field theory, are irreducible polynomials, i.e., irreducible elements in$k[X]$, for a field *k*. The fact that **Z** is a UFD can be stated more elementarily by saying that any natural number can be uniquely decomposed as product of powers of prime numbers. It is also known as the fundamental theorem of arithmetic.

An element *a* is a prime element if whenever *a* divides a product *bc*, *a* divides *b* or *c*. In a domain, being prime implies being irreducible. The converse is true in a unique factorization domain, but false in general.

The factor ring

The definition of ideals is such that "dividing" *I* "out" gives another ring, the *factor ring R* / *I*: it is the set of cosets of *I*together with the operations

$$(a + I) + (b + I) = (a + b) + \mathrm{I} \text{ and } (a + I)(b + I) = ab + I.$$

For example, the ring $\mathbf{Z}/n\mathbf{Z}$ (also denoted $\mathbf{Z}_n$), where *n* is an integer, is the ring of integers modulo *n*. It is the basis ofmodular arithmetic.

An ideal is *proper* if it is strictly smaller than the whole ring. An ideal that is not strictly contained in any proper ideal is calledmaximal. An ideal *m* is maximal if and only if *R* / *m* is a field. Except for the zero ring, any ring (with identity) possesses at least one maximal ideal; this follows from Zorn's lemma.

Noetherian rings

A ring is called *Noetherian* (in honor of Emmy Noether, who developed this concept) if every ascending chain of ideals

$$0 \subseteq I_0 \subseteq I_1 \ldots \subseteq I_n \subseteq I_{n+1} \subseteq \ldots$$

becomes stationary, i.e. becomes constant beyond some index n. Equivalently, any ideal is generated by finitely many elements, or, yet equivalent, *submodules of finitely generated modules are finitely generated*.

Being Noetherian is a highly important finiteness condition, and the condition is preserved under many operations that occur frequently in geometry. For example, if R is Noetherian, then so is the polynomial ring $R[X_1, X_2, \ldots, X_n]$ (by Hilbert's basis theorem), any localization $S^{-1}R$, and also any factor ring R / I.

Any non-noetherian ring R is the union of its Noetherian subrings. This fact, known as Noetherian approximation, allows the extension of certain theorems to non-Noetherian rings

Artinian rings

A ring is called Artinian (after Emil Artin), if every descending chain of ideals

$$R \supseteq I_0 \supseteq I_1 \ldots \supseteq I_n \supseteq I_{n+1} \supseteq \ldots$$

becomes stationary eventually. Despite the two conditions appearing symmetric, Noetherian rings are much more general than Artinian rings. For example, **Z** is Noetherian, since every ideal can be generated by one element, but is not Artinian, as the chain

$$\mathbf{Z} \supsetneq 2\mathbf{Z} \supsetneq 4\mathbf{Z} \supsetneq 8\mathbf{Z} \supsetneq \ldots$$

shows. In fact, by the Hopkins–Levitzki theorem, every Artinian ring is Noetherian. More precisely, Artinian rings can be characterized as the Noetherian rings whose Krull dimension is zero.

The spectrum of a commutative ring

Prime ideals

As was mentioned above, **Z** is a unique factorization domain. This is not true for more general rings, as algebraists realized in the 19th century. For example, in

$$Z[\sqrt{-5}\,]$$

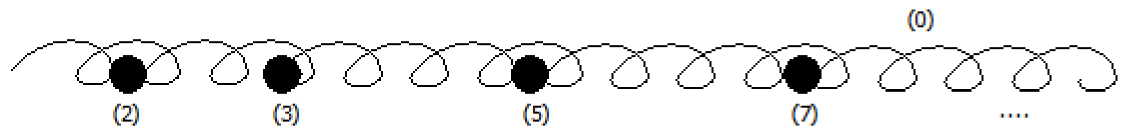
there are two genuinely distinct ways of writing 6 as a product:

$$6 = 2.3 = (1 + \sqrt{-5}\,)(1 - \sqrt{-5}\,)$$

Prime ideals, as opposed to prime elements, provide a way to circumvent this problem. A prime ideal is a proper (i.e., strictly contained in R) ideal p such that, whenever the product ab of any two ring elements a and b is in p, at least one of the two elements is already in p. (The opposite conclusion holds for any ideal, by definition). Thus, if a prime ideal is principal, it is equivalently generated by a prime element. However, in rings such as $(Z\sqrt{-5}\,)$, prime ideals need not be principal. This limits the usage of prime elements in ring theory. A cornerstone of algebraic number theory is, however, the fact that in anyDedekind ring (which includes $(Z\sqrt{-5}\,)$ and more generally the ring of integers in a number field) any ideal (such as the one generated by 6) decomposes uniquely as a product of prime ideals.

Any maximal ideal is a prime ideal or, more briefly, is prime. Moreover, an ideal I is prime if and only if the factor ring R / I is an integral domain. Proving that an ideal is prime, or equivalently that a ring has no zero-divisors can be very difficult. Yet another way of expressing the same is to say that the complement $R \setminus p$ is multiplicatively closed. The localisation $(R \setminus p)^{-1}R$is important enough to have its own notation: R_p. This ring has only one maximal ideal, namely pR_p. Such rings are called local.

The spectrum



Spec (**Z**) contains a point for the zero ideal. The closure of this point is the entire space. The remaining points are the ones corresponding to ideals (p), where p is a prime number. These points are closed.

The *spectrum of a ring R*,[nb 1] denoted by Spec R, is the set of all prime ideals of R. It is equipped with a topology, the Zariski topology, which reflects the algebraic properties of R: a basis of open subsets is given by

$D(f) = \{p \in Spec\ R, f \notin p\}$, where f is any ring element.

Interpreting f as a function that takes the value f mod p (i.e., the image of f in the residue field R/p), this subset is the locus where f is non-zero. The spectrum also makes precise the intuition that localisation and factor rings are complementary: the natural maps $R \to R_f$ and $R \to R / fR$ correspond, after endowing the spectra of the rings in question with their Zariski topology, to complementary open and closed immersions respectively. Even for basic rings, such as illustrated for $R = \mathbf{Z}$ at the right, the Zariski topology is quite different from the one on the set of real numbers.

The spectrum contains the set of maximal ideals, which is occasionally denoted mSpec (R). For an algebraically closed field$k$, mSpec (k[$T_1, ..., T_n$] / ($f_1, ..., f_m$)) is in bijection with the set

$\{x =(x_1, ..., x_n) \in k^n \mid f_1(x) = ... = f_m(x) = 0.\}$

Thus, maximal ideals reflect the geometric properties of solution sets of polynomials, which is an initial motivation for the study of commutative rings. However, the consideration of non-maximal ideals as part of the geometric properties of a ring is useful for several reasons. For example, the minimal prime ideals (i.e., the ones not strictly containing smaller ones) correspond to the irreducible components of Spec R. For a Noetherian ring R, Spec R has only finitely many irreducible components. This is a geometric restatement of primary decomposition, according to which any ideal can be decomposed as a product of finitely many primary ideals. This fact is the ultimate generalization of the decomposition into prime ideals in Dedekind rings.

Affine schemes

The notion of a spectrum is the common basis of commutative algebra and algebraic geometry. Algebraic

geometry proceeds by endowing Spec R with a sheaf (an entity that collects functions defined locally, i.e. on varying open subsets). The datum of the space and the sheaf is called an affine scheme. Given an affine scheme, the underlying ring R can be recovered as the global sections of R^n, Moreover, this one-to-one correspondence between rings and affine schemes is also compatible with ring homomorphisms: any $f : R \to S$ gives rise to a continuous map in the opposite direction

Spec $S \to$ Spec R, $q \mapsto f^{-1}(q)$, i.e. any prime ideal of S is mapped to its preimage under f, which is a prime ideal of R.

The resulting equivalence of the two said categories aptly reflects algebraic properties of rings in a geometrical manner.

Similar to the fact that manifolds are locally given by open subsets of $\mathbf{R}^n$, affine schemes are local models for schemes, which are the object of study in algebraic geometry. Therefore, several notions concerning commutative rings stem from geometric intuition.

Dimension

The *Krull dimension* (or dimension) dim R of a ring R measures the "size" of a ring by, roughly speaking, counting independent elements in R. The dimension of algebras over a field k can be axiomatized by four properties:

- The dimension is a local property: $\dim R = \sup_{p \in \operatorname{Spec} R} \dim R_p$.
- The dimension is independent of nilpotent elements: if $I \subseteq R$ is nilpotent then $\dim R = \dim R / I$.
- The dimension remains constant under a finite extension: if S is an R-algebra which is finitely generated as an R-module, then $\dim S = \dim R$.
- The dimension is calibrated by $\dim k[X_1, ..., X_n] = n$. This axiom is motivated by regarding the polynomial ring in $n$variables as an algebraic analogue of n-dimensional space.

The dimension is defined, for any ring R, as the supremum of lengths n of chains of prime ideals

$$p_0 \subsetneq p_1 \subsetneq ... \subsetneq p_n.$$

For example, a field is zero-dimensional, since the only prime ideal is the zero ideal. The integers are one-dimensional, since chains are of the form $(0) \subsetneq (p)$, where p is a prime number. For non-Noetherian rings, and also non-local rings, the dimension may be infinite, but Noetherian local rings have finite dimension. Among the four axioms above, the first two are elementary consequences of the definition, whereas the remaining two hinge on important facts in commutative algebra, thegoing-up theorem and Krull's principal ideal theorem.

Ring homomorphisms

A *ring homomorphism* or, more colloquially, simply a *map*, is a map $f: R \to S$ such that

$$f(a + b) = f(a) + f(b), f(ab) = f(a)f(b) \text{ and } f(1) = 1.$$

These conditions ensure $f(0) = 0$. Similarly as for other algebraic structures, a ring homomorphism is thus a map that is compatible with the structure of the algebraic objects in question. In such a situation S is also called an R-algebra, by understanding that s in S may be multiplied by some r of R, by setting

$$r \cdot s := f(r) \cdot s.$$

The *kernel* and *image* of f are defined by $\ker(f) = \{r \in R, f(r) = 0\}$ and $\operatorname{im}(f) = f(R) = \{f(r), r \in R\}$. The kernel is an ideal of R, and the image is a subring of S.

A ring homomorphism is called an isomorphism if it is bijective. An example of a ring isomorphism, known

as the Chinese remainder theorem,

Commutative rings, together with ring homomorphisms, form a category. The ring $\mathbf{Z}$ is the initial object in this category, which means that for any commutative ring R, there is a unique ring homomorphism $\mathbf{Z} \to R$. By means of this map, an integer $n$can be regarded as an element of R. For example, the binomial formula

$$(x+a)^n = \sum_{k=0}^{n} \binom{n}{k} x^k a^{n-k}$$

which is valid for any two elements *a* and *b* in any commutative ring *R* is understood in this sense by interpreting the binomial coefficients as elements of *R* using this map.

$$\begin{tikzcd}
R \arrow[r] \arrow[d] & S \arrow[d] \arrow[ddr, bend left] & \\
T \arrow[r] \arrow[drr, bend right] & S \otimes_R T \arrow[dr, dotted] & \\
& & W
\end{tikzcd}$$

The universal property of $S \otimes_R T$ states that for any two maps $S \to W$ and $T \to W$ which make the outer quadrangle commute, there is a unique map $S \otimes_R T \to W$ which makes the entire diagram commute.

Given two *R*-algebras *S* and *T*, their tensor product

$S \otimes_R T$

is again a commutative *R*-algebra. In some cases, the tensor product can serve to find a *T*-algebra which relates to *Z* as *S* relates to *R*. For example,

$R[X] \otimes_R T = T[X].$

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