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Proving Gödel's completeness theorem (in restricted cases) using *Sd* - Topology over the theory of a Model

Soham Dasgupta

(Ex. PGT (Math) at Kendriya Vidyalaya Ordnance Factory, Dumdum, Kolkata, W.B , INDIA)

Abstract*: This is a modification as well as a model theoretic application of my previous paper Sd- Topology over the theory of a Model. The main intention of this article is to proof and visualize the celebrated Gödel's completeness theorem in a different and lucid approach.*

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Introduction: This is a continuation and application of Sd-Topology over the theory of a Model (DOI: 10.17148/IARJSET.2020.71201) to provide a lucid and short proof of the famous Gödel's completeness theorem. Here as we are using Sd Topology so the conditions will be restricted as the cardinality of the universe of the working model should be $card\ \mathcal{U} \geq \aleph_0$ and also the cardinality of the working theory should be card(Th $\mathcal{U}$)$\geq$ *card* ($\mathcal{U}$).

DEFINITION: If is a model for the language $\mathcal{L}$, the theory of $\mathcal{U}$ is denoted by **Th**$\mathcal{U}$, is defined to be the set of all sentences of $\mathcal{L}$ (i.e formulas with no free variables) which are true in $\mathcal{U}$.
So **Th** $= \{ \sigma$ of $\mathcal{L} : \mathcal{U} \vDash \sigma \}$
For example if $\mathcal{L} = \{ <,+,\cdot,0,1\}$ then **Th**$\mathbb{R}$ is famous as the name Real Analysis. Similarly over the same language **Th**$\mathbb{Z}$ is known as number theory again for
$\mathcal{L} = \{+,\cdot,0,1\}$ **Th**$\mathbb{C}$ is renowned as complex analysis.

[Note: here we denote the models for a given language as italic notations as the universe of the model i.e $\mathcal{R} = <\mathbb{R}, I>$ where I is the respective interpretation function , also further we indicate the cardinality of a model as the cardinality of its universe i.e card$\mathcal{R}$ means card$\mathbb{R} = \boldsymbol{c}$]

<u>Construction of *Sd*-topology on Th</u> <u>(*When card* $\mathcal{U} \geq \aleph_0$)</u> :

Lets define a mapping k: P(**Th**$\mathcal{U}$) → P(**Th**$\mathcal{U}$) as
k(A)= A if card A < card $\mathcal{U}$ [where A is subset (i.e sub theory) of **Th**$\mathcal{U}$]
k(A)= **Th**$\mathcal{U}$ if card A $\geq$ card $\mathcal{U}$
It's easy to verify that k is a Kuratowski's Closer Operator
since $k(\emptyset) = \emptyset$,A$\subseteq k$(A) & $k(k(A)) = k(A)$; $\forall A \subseteq$ **Th**$\mathcal{U}$ and k(A U B)=k(A) U k(B) ;$\forall$A,B $\subseteq$ **Th**$\mathcal{U}$

LEMMA: Let k be a Kuratowski's Closer operator on a set X. Then there is a unique topology τ on X such that $k(A)=\bar{A}$; in (X, τ) $\forall A \subseteq X$ where $\bar{A}$ is the closer of A.

Proof: Let $\sum = \{ k(A) : A \subseteq X\}$ since $k(k(A)) = k(A), \forall A \subseteq X$
so we can treat $\sum = \{ A \subseteq X : k(A)=A\}$, Obviously $\emptyset \in \sum$ and $X \in \sum$
Again if A, B ∈ ∑ then as $k(A \cup B)=k(A) \cup k(B) = A \cup B$, so $A\cup B \in \sum$ thus ∑ is closed under finite union. Again if $A_i \in \sum$ then as $\cap_i A_i \subseteq A_i$ so by the property of k ,
$k(\cap_i A_i) \subseteq k(A_i) = A_i$ therefore $\cap_i A_i \subseteq k(\cap_i A_i) \subseteq \cap_i A_i$ and $k(\cap_i A_i) = \cap_i A_i$
thus $\cap_i A_i \in \sum$ so it's also closed under arbitrary intersection. Therefore τ is defined as
$\tau = \{ V : X- V \in \sum \}$, surly a topology on X with ∑ is the set of all closed sets in (X, τ).
Now $\bar{A} \subseteq k(A)$ since k(A) is a closed set containing A. Again $\bar{A} \subseteq k(\bar{A}) \subseteq k(k(A)) =k(A)$.
But $k(A) \subseteq k(\bar{A})$ so we get $k(A)= \bar{A}$.
For uniqueness if τ_1 and τ_2 are two topologies on X such that $Cl_{\tau_1}(A) = Cl_{\tau_2}(A) , \forall A \subseteq X$.
Then $\{ Cl_{\tau_1}(A) : A \subseteq X \} =\{ Cl_{\tau_2}(A) : A \subseteq X \}$ so $\tau_1 = \tau_2$ (Where $Cl_{\tau_1}(A) = \bar{A}$ in (X, τ_1))
(Q.E.D)

Now with the help of this lemma and the Kuratowski's Closer Operator *'k'* what we had defined earlier , we can construct a unique topology over **Th$\mathcal{U}$**. Where $k(A) = \bar{A}$, $\forall A \subseteq$ **Th$\mathcal{U}$** . We called that topology is the ***Sd* -topology on Th** . We are denoting that topology further as τ_{Sd} . Another thing to note that if ***card*** $\mathcal{U} < \aleph_0$ or card(Th $\mathcal{U}$)< *card ($\mathcal{U}$)* then *(Th$\mathcal{U}$, τ_{Sd})* will be very boring and elementary so in general we consider *(Th$\mathcal{U}$, τ_{Sd}) with* ***card*** $\mathcal{U} \geq \aleph_0$ and card(Th $\mathcal{U}$)≥ *card ($\mathcal{U}$).*

Some Properties of the topological space (Th , τ_{Sd}):

1) *In (Th, τ_{Sd}) any Sub Theory (i.e non trivial subsets of a theory) containing equal or more sentences than the cardinality of the universe of the respective model , then it's closer is itself the theory. In other words we can say that in (Th, τ_{Sd})if A ⊆* ***Th$\mathcal{U}$*** *and card(A)≥ card ($\mathcal{U}$) then A is dense in (Th$\mathcal{U}$, τ_{Sd}).*

Proof: Its trivial to check that as the operator defined as *k*(A)= **Th$\mathcal{U}$** if card A ≥ card $\mathcal{U}$. Also as in *(Th, τ_{Sd})* , $k(A)= \bar{A}$, $\forall A \subseteq$ **Th$\mathcal{U}$** . So whenever card A ≥ card $\mathcal{U}$ then $\bar{A}$ = ***Th$\mathcal{U}$***. (Q.E.D)

2) *In (Th, τ_{Sd}) any Sub Theory A with card(A) < card ($\mathcal{U}$) is closed . Also vise-versa.*

DEFINITION: A sentence σ∈ *Th$\mathcal{U}$* is said to be a ***Theorem*** iff there exist a convergent sequence of different sentences $\{ \sigma_n \}_{n\in\mathbb{N}}$ (i.e steps) in *Th$\mathcal{U}$, which* converges to σ in *(Th$\mathcal{U}$, τ_{Sd}).*
Then the set $\{\sigma_n : n\in\mathbb{N} \} \cup \{\sigma\}$ is called a ***proof.***

Basically as we know that the last line of a proof is called a Theorem.

<u>Theorem</u>: A sentence is a ***Theorem*** iff it can be proved using finite number of steps.

To proof this theorem using finite steps we have to discuss some lemmas in *(Th$\mathcal{U}$, τ_{Sd})*

LEMMA 1: *In (Th$\mathcal{U}$, τ_{Sd}) if **card** $\mathcal{U} = \aleph_0$ a sequence is convergent iff its semi-constant. (i.e exactly one term of the sequence repeated infinitely many times)*

Proof. In (Th$\mathcal{U}$, τ_{Sd}) if **card** $\mathcal{U} = \aleph_0$ the space is a cofinite space and the above lemma is evident there also that sequence converges to that particular term(point) always.We can find this lemma in almost all general topology book.

LEMMA 2: In (Th$\mathcal{U}$, τ_{Sd}) if **card** $> \aleph_0$ a sequence is convergent iff its eventually constant.
Proof. Let $\{ \sigma_n \}_{n\in\mathbb{N}}$ be a non (eventually) constant sequence in (Th$\mathcal{U}$, τ_{Sd}) converges to $\sigma \in$ Th$\mathcal{U}$.
So $\sigma \notin \{ \sigma_n : \forall n\in\mathbb{N} \}$ but the set $\{ \sigma_n : n\in\mathbb{N} \}$ is countable, so its closed in (Th$\mathcal{U}$, τ_{Sd}).
Then Th$\mathcal{U}$- $\{ \sigma_n : n\in\mathbb{N} \}$ is a neighbourhood of σ which contains no σ_n ,which is a contradiction.
The converse of the lemma is obvious.

Proof of the theorem: If (Th$\mathcal{U}$, τ_{Sd}) is with ***card*** $\mathcal{U} = \aleph_0$ then from the lemma 1 we can see that there any sequence is convergent iff its semi-constant. But as the definition of Theorem its proof cannot repeat any of its sentences. Thus for this case the proof should be finite. The converse is also true as by the lemma1. Now if (Th$\mathcal{U}$, τ_{Sd}) *with **card*** $> \aleph_0$ then by lemma2 the proof of any theorem should be an *eventually constant* sequence of sentences. With the same logic as before the proof should be finite. Again by lemma 2 its converse is also true. (Q.E.D)

Gödel's completeness theorem:

A theory is consistence iff it is satisfiable.

The logical statement of this theorem is $\mathcal{L} \vDash \sigma$ iff $\mathcal{L} \vdash \sigma$.

Stating the theorem in lucid words, if $\sigma \in$ Th$\mathcal{U}$ for every model $\mathcal{U}$ over a language $\mathcal{L}$ then σ is deducible (i.e provable using finite no. steps). Logically speaking a sentence of a theory is a syntactic consequence iff its a semantic consequence.

For example the theorem says that , when a sentence is shown to be provable from the axioms of ring theory by considering an arbitrary ring and showing that the sentence is satisfied by that ring.

Proof: We want to prove the following theorem involving the Sd topology as defined by us and since there is some restrictions as in general we considered *(Th$\mathcal{U}$, τ_{Sd}) with **card*** $\mathcal{U} \geq \aleph_0$ and card(Th $\mathcal{U}$)$\geq$ *card ($\mathcal{U}$).* Thus its evident to say that we will restrict the proof of the theorem over the mentioned conditions.

Well , let for every model $\mathcal{U}$ over a language $\mathcal{L}$ the sentence $\sigma \in$ Th$\mathcal{U}$ with ***card*** $\mathcal{U} \geq \aleph_0$ and card(Th $\mathcal{U}$)$\geq$ *card ($\mathcal{U}$).* Now in the topological space *(Th, τ_{Sd})* any open subset is dense. Let V is such an open subset which does not contain σ . Then as V is dense so there will be a sequence $\{ \sigma_n \}_{n\in\mathbb{N}}$ in V converges to σ. By our previous theorem we had shown that $\{ \sigma_n \}$ is convergent in *(Th$\mathcal{U}$, τ_{Sd})* iff $\{ \sigma_n \}$ is eventually constant. Thus σ is deducible in Th$\mathcal{U}$.

Conversely , Let σ is deducible in Th$\mathcal{U}$. Then there is an eventually constant sequence $\{ \sigma_n \}$ in *(Th, τ_{Sd})* which terminates in σ . So $\{ \sigma_1, \sigma_2, \sigma_3 .... \sigma \}$ is a closed subset of *(Th$\mathcal{U}$, τ_{Sd})* . Thus surely $\sigma \in$ Th$\mathcal{U}$.   [Q.E.D] The converse of the theorem is called Soundness.

Conclusion: The remarkable Gödel's completeness theorem is a milestone in proof theory as well as in the whole field of mathematics, logic and philosophy. But its proof is too much long and difficult to digest by non mathematicians especially the Henkin's proof. Thus here we tried to give an easy and short proof of this theorem by introducing a new topological aspect, however the proof is restricted and conditionally dependent.

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Author's e-mail: *mathsoham@gmail.com*