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Practical problems of Max-Plus Algebra: Matrix

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Abstract: The optimization problems, such as scheduling or project management, in which the objective function depends on the operations *maximum* and *plus*, can be naturally formulated and solved in max-plus algebra. A system of discrete events, e.g., activations of processors in parallel computing, or activations of some other cooperating machines, is described by a systems of max-plus linear equations. In particular, if the system is in a steady state, such as a synchronized computer network in data processing, then the state vector is an eigenvector of the system. In reality, the entries of matrices and vectors are considered as intervals. The properties and recognition algorithms for several types of interval eigenvectors are studied in this paper. For a given interval matrix and interval vector, a set of generators is defined. Then, the strong and the strongly universal eigenvectors are studied and described as max-plus linear combinations of generators. Moreover, a polynomial recognition algorithm is suggested and its correctness is proved. Similar results are presented for the weak eigenvectors. The results are illustrated by numerical examples. The results have a general character and can be applied in every max-plus algebra and every instance of the interval eigenproblem.

Keywords: system dynamics; steady state; max-plus algebra; interval matrix; interval vector; strong eigenvector; strongly universal; weak eigenvector

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1. Introduction

In many practical problems, the standard algebraic operations "plus" and "product" are inadequate, e.g., in scheduling problems, synchronization problems, or in project management, and binary operations "maximum" and "plus" seem to be more appropriate. This observation leads to the definition and use of so-called max-plus algebra, which has been used by many authors, see e.g., [1–4].

Max-plus algebras represent a suitable mathematical tool for exploration of systems working in discrete steps called discrete event systems (DES, for short). A DES is determined by the transition matrix, A, and the starting state vector, $x^{(0)}$. The sequence of state vectors in time is then computed by recurrent formula $x^{(k+1)} = A \otimes x^{(k)}$ using the matrix operations derived from "maximum" and "plus"
(for more details, see [2,5,6]).

For convenience of the reader, we present here the basic definitions. A max-plus algebra is a triple $(\mathsf{B}, \oplus, \otimes)$, where B is the set of real numbers with added $-\infty = \varepsilon$, and $\oplus = \max$, $\otimes = +$ are binary operations on B. Clearly, ε is the neutral element with respect to $\oplus$ and the absorbing element with

respect to $\otimes$. $\mathsf{B}(m, n)$ ($\mathsf{B}(n)$) denote the set of all matrices (vectors) of the given dimension over B. The linear ordering on B induces a partial ordering on $\mathsf{B}(m, n)$ and $\mathsf{B}(n)$, with respect to all entries of the matrix (vector).

The work of a DES in time often comes to a steady state. In the formal matrix notation, the steady states correspond to max-plus eigenvectors fulfilling the equation $A \otimes x = \lambda \otimes x$, with $A \in \mathsf{B}(n, n)$, $x \in \mathsf{B}(n)$.

It is assumed that an eigenvector is different from the 'zero' vector with all entries equal to ε. In this notation, the intervals between the beginnings of consecutive cycles on every component of DES are equal to a scheduled value $\lambda \in \mathsf{B}$.

In the real world, the entries of matrices and vectors are usually not strict values and should be considered as intervals. The properties and recognition algorithms for several types of interval eigenvectors are studied in this paper. The strong, strongly universal and weak interval eigenvectors in max-plus algebra are investigated, and polynomial algorithms for the recognition versions of these problems are presented.

The results can be applied in every max-plus algebra and every instance of the interval eigenproblem.

2. Definitions and Basic Properties

Our results will be illustrated by the following simple example describing the main ideas of the investigation, see Figure 1.

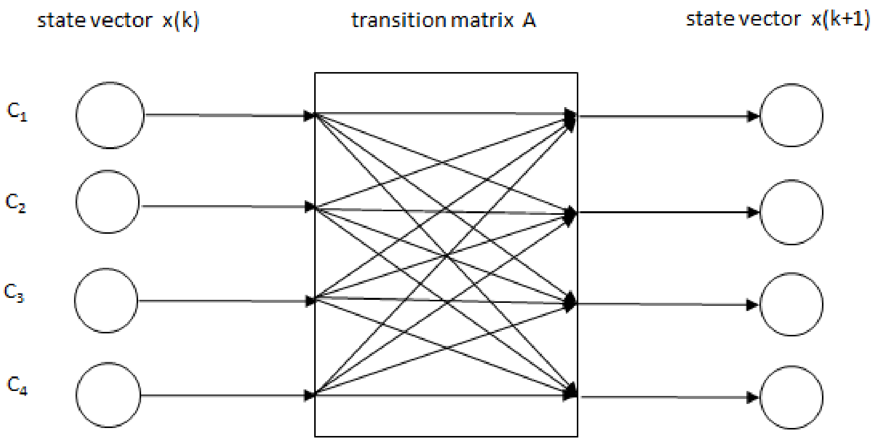

Figure 1. Work of DES in time.

Example *An interactive system consisting of n entities (computers, or some other cooperating machines) working in stages can be represented by discrete-event systems. Denote* $C_1, \ldots, C_4$ *the computers in parallel computation sharing partial data to continue the computation in the next stage. Suppose* $x_i(k)$ *stands for the activation time of the k-th stage on* C_i $(i = 1, \ldots, 4)$. *Furthermore, suppose that the entries of a matrix A (called the transition matrix)* a_{ij} *denote the computation time of computer* C_j *while preparing the data for the work of computer* C_i *in the* $(k + 1)$*-st stage* $(i, j = 1, \ldots, 4)$. *The interference of the system can be described by recurrence relations*

$$x_i(k+1) = \max\left\{x_1(k) + a_{i1},\ x_2(k) + a_{i2},\ x_3(k) + a_{i3},\ x_4(k) + a_{i4}\right\},\ i \in \{1,2,3,4\}.$$

The considered system can be written in matrix/vector form as $x(k+1) = A \otimes x(k)$. *Moreover, if we schedule for* λ *the intervals between the beginnings of consecutive cycles on every computer, then we obtain* $x(k+1) = \lambda \otimes x(k)$. *Finally, for steady scheduling of the system, we have to solve the equation*

$$A \otimes x = \lambda \otimes x.$$

Remark 1. *In Example 1, the entries* x_i *are interpreted as activation times of cooperating machines* C_i *in a DES, while* λ *is the length of the interval between the beginnings of consecutive cycles in the steady run of the system. This interpretation implies that* x *cannot contain* ε *entries. Moreover, by a suitable start of the time-measuring, we can achieve* $x_i \geq 0$. *The above interpretation gives* $\lambda > 0$ *as well.*

Remark 2. *On the other hand, the entries* $a_{ij} = \varepsilon$ *can occur in the transition matrix. The interpretation of such situation is: "*C_i *need not wait for* C_j*". Hence, the case* $F = \{(i, j) \in N;\ a_{ij} = \varepsilon\} = N \times N$ *is trivial and is not allowed.*

Remark 3. *In LP computations, the constraints with a single* ε *value on the lower side of the inequality sign are automatically satisfied and may be left out of consideration.*

Let us define, similarly as in [7–10], the interval matrix with bounds $\underline{A}, \overline{A} \in B(n, n)$ and interval vector with bounds $\underline{x}, \overline{x} \in B(n)$,

$$[\underline{A}, \overline{A}] = \left\{A \in B(n,n);\ \underline{A} \leq A \leq \overline{A}\right\}, \quad [\underline{x}, \overline{x}] = \{x \in B(n);\ \underline{x} \leq x \leq \overline{x}\}$$

and suppose that a fixed interval matrix $\mathbf{A} = [\underline{A}, \overline{A}]$ and interval vector $\mathbf{X} = [\underline{x}, \overline{x}]$ are given. The interval eigenproblem for $\mathbf{A}$ and $\mathbf{X}$ consists of recognizing whether $A \otimes x = \lambda \otimes x$ holds true for $A \in \mathbf{A}$, $x \in \mathbf{X}$, $\lambda \in B$, with suitable quantifiers (e.g., for all $A \in \mathbf{A}$, for some $A \in \mathbf{A}$, for all $x \in \mathbf{X}$, for some $x \in \mathbf{X}$) and their various combinations. According to the choice of quantifiers and their order, several different types of interval eigenvectors can be defined (see, e.g., [11–13] for similar classification). The following three types are studied in detail in this paper.

Definition *Suppose there is a given interval matrix* $\mathbf{A}$ *and an interval vector* $\mathbf{X}$*. Then,* $\mathbf{X}$ *is called*

- *a strong eigenvector of* $\mathbf{A}$
  if $(\exists\lambda \in \mathsf{B})(\forall A \in \mathbf{A})(\forall x \in \mathbf{X})[\, A \otimes x = \lambda \otimes x]$;
- *a strongly universal eigenvector of* $\mathbf{A}$
  if $(\exists x \in \mathbf{X})(\exists\lambda \in \mathsf{B})(\forall A \in \mathbf{A})[\, A \otimes x = \lambda \otimes x]$;
- *a weak eigenvector of* $\mathbf{A}$
  if $(\exists\lambda \in \mathsf{B})(\exists x \in \mathbf{X})(\exists A \in \mathbf{A})[\, A \otimes x = \lambda \otimes x]$.

Analogously as the above mentioned interval eigenvectors, the corresponding eigenvalues (the strong eigenvalue, the strongly universal eigenvalue, and the weak eigenvalue) of the interval matrix $\mathbf{A}$ *are also defined.*

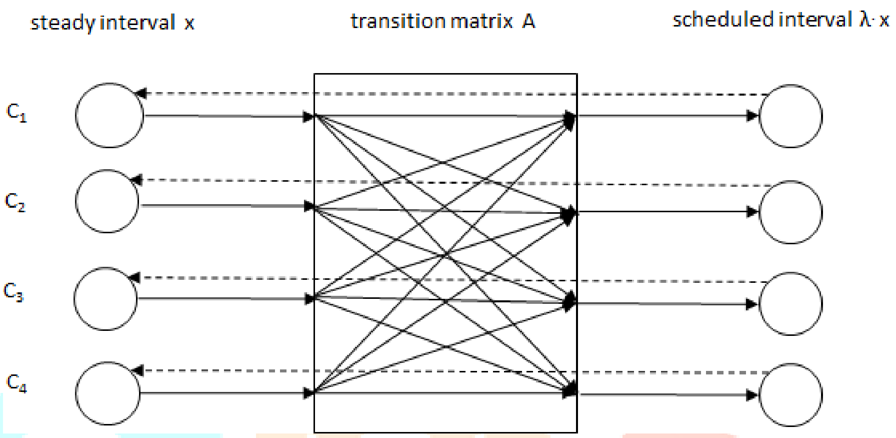

Figure 2. Steady scheduled DES.

Remark 4. *The investigated 'universal' type can be interpreted as follows. In general, the interval vector* $\mathbf{X}$ *is a universal eigenvector of* $\mathbf{A}$ *if there is an eigenvalue* $\lambda \in \mathsf{B}$ *and a vector* $x \in \mathbf{X}$*, which is a steady state of the discrete event system for every transition matrix* $A \in \mathbf{A}$ *(in other words,* x *is a universal eigenvector of* $\mathbf{A}$*, with eigenvalue* λ*).*

It is welcome in scheduling, when there is one common universal eigenvector $x \in \mathbf{X}$ *for all transition matrices* $A \in \mathbf{A}$ *(the interval vector* $\mathbf{X}$ *is then called strongly universal). In such situations, it is possible to choose the activation vector within the given interval* $[\underline{x}, \overline{x}]$*, while every possible transition matrix in the given interval is acceptable.*

Otherwise, the universal eigenvector x *depends on* A*. If* the eigenvalue λ *also depends on* A*, then the interval eigenvector* $\mathbf{X}$ *is called* weakly universal. *The universal and weakly universal interval eigenvectors are not studied in this paper.*

3. Strongly Universal Interval Eigenvectors in a Max-Plus Algebra

In a strongly universal case, there is one (so-called: universal) eigenvector $x \in \mathbf{X}$ which corresponds to all transition matrices in the given interval $\mathbf{A}$. Then, the scheduled run of the DES with the starting state x is satisfied in every stage, if the transition matrix is kept within the $\mathbf{A}$ limits.

Proposition *Let* $\underline{x}, \overline{x} \in \mathsf{B}(n)$*,* $\underline{A}, \overline{A} \in \mathsf{B}(n, n)$*. The interval vector* $\mathbf{X} = [\underline{x}, \overline{x}]$ *is a strongly universal eigenvector of the interval matrix* $\mathbf{A} = [\underline{A}, \overline{A}]$ *if and only if there are* $\lambda \in \mathsf{B}$ *and* $x \in \mathbf{X}$ *such that*

$$\underline{A} \otimes x = \lambda \otimes x, \tag{1}$$
$$\overline{A} \otimes x = \lambda \otimes x. \tag{2}$$

Proof. Assume $\lambda \in \mathsf{B}$, and $x \in \mathbf{X}$ fulfill (1) and (2). Take any $A \in \mathbf{A}$, that is, $\underline{A} \le A \le \overline{A}$. Then, $\underline{A} \otimes x \le A \otimes x \le \overline{A} \otimes x$, by the monotonicity of the operations in a max-plus algebra. This implies $A \otimes x = \lambda \otimes x$, in view of (1) and (2). The converse implication is trivial.

LP approach. The existence of $\lambda \in \mathsf{B}$, and $x \in \mathbf{X}$ in Proposition satisfying (1) and (2) can be recognized by solving the following linear programming problem: P_{su} with variables $\lambda, x_1, x_2, \ldots, x_n$

$$z = \lambda \to \min \tag{3}$$

subject to

$$\overline{a}_{ij} - x_i + x_j \le \lambda \qquad \text{for every } i, j \in N, \tag{4}$$
$$\underline{x}_j \le x_j \qquad \text{for every } j \in N, \tag{5}$$

$$\underline{x}_j \le x_j \quad \text{for every } j \in N. \tag{6}$$

Theorem *The interval vector* $\mathbf{X} = [\underline{x}, \overline{x}]$, $\underline{x}, \overline{x} \in \overline{B}(n)$, *is a strongly universal eigenvector of the interval matrix* $\mathbf{A} = [\underline{A}, \overline{A}]$, $\underline{A}, \overline{A} \in B(n, n)$, *if and only if the minimization problem* (3)–(6) *has an optimal solution satisfying*

$$\max_{j\in N} (\underline{a}_{ij} - x_i + x_j) = \lambda \quad \text{for every } i \in N, \tag{7}$$

$$\max_{j\in N} (\overline{a}_{ij} - x_i + x_j) = \lambda \quad \text{for every } i \in N. \tag{8}$$

Proof. It is easy to see that, for every pair $(i, j) \in F$, the equality $\overline{a}_{ij} = \overline{\varepsilon}$ implies that is automatically satisfied. That is, this constraint has only to be considered for $(i, j) \in N \times N \setminus F$ (see Remark 3). Analogous limitations are to be applied in (7) and (8).

Now, let us assume that λ^{opt} and x^{opt} are optimal solutions of (3)–(6) with (7) and (8). In view of Proposition , $\mathbf{X}$ is strongly universal. Conversely, assume that $\mathbf{X} = [\underline{x}, \overline{x}]$ is a strongly universal eigenvector of $\mathbf{A} = [\underline{A}, \overline{A}]$. Then, by Proposition, there are $\lambda \in B$ and $x \in \mathbf{X}$ satisfying (1) and (2). By easy equivalent modifications, we see that λ and x satisfy (7) and (8).

Thus, for every $i \in N$, λ is the least upper bound of the set of all $(\underline{a}_{ij} - x_i + x_j)$ with $j \in N$ (and also the least upper bound of the set of all $(\overline{a}_{ij} - x_i + x_j)$ with $j \in N$ and $(i, j) \notin F$). That is, λ, x is an optimal solution of the minimization problem (3)–(6).

Corollary 2. *The problem of recognizing whether a given interval vector* $\mathbf{X}$ *is a strongly universal eigenvector of a given interval matrix* $\mathbf{A}$ *in a max-plus algebra is solvable with the help of an LP minimization problem with* $n + 1$ *variables and* $2n^2 + 2n$ *constraints, and by verifying* $2n$ *max-plus linear equations in* $O(n^3)$*-time.*

4. Conclusions

Three types of interval eigenvectors: the strong, the strongly universal, and the weak interval eigenvector of an interval matrix in max-plus algebra have been studied.

The structure of an eigenvector, and a polynomial algorithm for the corresponding recognition problem have been presented for each of the considered types. Surprisingly, another analogous type of a semi-strongly universal eigenvector turned out to be equivalent to the strongly universal type, in spite of the fact that the first notion is formally weaker than the second one.

The working procedures of the algorithms are illustrated by numerical examples. The examples also show which of the considered types are not equivalent.

The presented results correspond to the authors' systematic effort to solve the recognition problem for various types of interval eigenvectors in max-plus and max-min algebra. Polynomial recognitions of the universal and weakly universal interval eigenvectors remain open for future research.

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