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Potential and Nonlinear Potential Gradient Estimates

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Abstract

In this paper we extend the proof of potential and nonlinear potential pointwise gradient estimates for solutions to measure data problems whose prototype is given by $-\Delta_{\epsilon+2}u_r = \mu_r$. We show that in the case $\epsilon \leq 0$ a pointwise gradient estimate is possible using standard, linear Riesz potentials. The proof is based on the identification of a natural quantity that on one hand respects the natural scaling of the problem, and on the other allows to encode the weaker coercivity properties of the operaotrs considered, in the case $\epsilon \leq 0$. In the case $\epsilon > 0$ we prove a new gradient estimate employing nonlinear potentials of Wolff type.

Keywords: Gradient estimates; nonlinear potential theory; ϵ-Laplacian; regularity thoery

1. Introduction and Results

We consider possibly degenerate quasilinear equations with $(2+\epsilon)$-growth of the type

$$-\nabla a(x_r, Du_r) = \mu_r \tag{1.1}$$

in a bounded domain $\Omega_r \subset \mathbb{R}^n$ with $n \geq 2$, where μ_r is a Radon measure defined on Ω_r with finite total mass. Eventually letting $\mu_r(\mathbb{R}^n \backslash \Omega_r) = 0$ we shall assume that μ_r is defined on the whole $\mathbb{R}^n$. The continuous vector field $a: \Omega_r \times \mathbb{R}^n \to \mathbb{R}^n$ is assumed to be C^1-regular in the gradient variable z_r, with the partial derivative with respect to the gradient variable $a_{z_r}(\cdot)$ being itself continuous, and satisfying the following growth, ellipticity and continuity assumptions:

$$\begin{cases} |a(x_r, z_r)| + |a_z(x_r, z_r)|(|z_r|^2 + (1+\epsilon)^2)^{\frac{1}{2}} \leq (1+2\epsilon)(|z_r|^2 + (1+\epsilon)^2)^{\frac{1+\epsilon}{2}}, \\ (1+\epsilon)^{-1}(|z_r|^2 + (1+\epsilon)^2)^{\frac{\epsilon}{2}} |\lambda_r|^2 \leq \langle a_z(x_r, z_r)\lambda_r | \lambda_r \rangle, \\ |a(x_r, z_r) - a(x_0, z_r)| \leq (1+\epsilon)\omega(|x_r - x_0|)(|z_r|^2 + (1+\epsilon)^2)^{\frac{1+\epsilon}{2}} \end{cases} \tag{1.2}$$

where $x_r, x_0 \in \Omega_r$ and $z_r, \lambda_r \in \mathbb{R}^n$. Assume that $\epsilon \geq 0$ such that, the function $\omega: [0, \infty) \to [0,1]$ is a modulus of continuity i.e., a non-decreasing subadditive function such that

$$\omega(0) = 0 = \lim_{\rho \downarrow 0} \omega(\rho)$$

and $\omega(\cdot) \leq 1$. We assume

$$\int_0^R \omega(\rho) \frac{d\rho}{\rho} := d(R) < \infty, \tag{1.3}$$

whenever $R < \infty$. The prototype of (1.1) is, choosing $\epsilon = -1$, clearly given by the $(2+\epsilon)$-Laplacian equation with coefficients

$$-\nabla(\gamma(x_r)|Du_r|^{\epsilon} Du_r) = \mu_r. \tag{1.4}$$

In this case $\omega(\cdot)$ represents the modulus of continuity of the function $\gamma(\cdot)$, which is in fact assumed to be Dini continuous and satisfying the "ellipticity" condition $0 \leq \gamma(x_r) \leq \epsilon$.

By a weak (distributional) solution to equation (1.1) we mean a function $u_r \in W^{1,1+\epsilon}_{\mathrm{loc}}(\Omega_r)$ such that the distributional relation

$$\int_{\Omega_r} \sum_r \langle a(x_r, Du_r), D\varphi_r \rangle \, dx_r = \int_{\Omega_r} \sum_r \varphi_r d\mu_r$$

holds whenever $\varphi_r \in C_0^\infty(\Omega_r)$ has compact support in Ω_r.

The authors in [20] were able to prove pointwise estimates for u_r in terms of the (truncated) Wolff potential $\mathbf{W}^{\mu_r}_{\beta,2+\epsilon}(x_r, R)$ defined by

$$\sum_r \mathbf{W}^{\mu_r}_{\beta,2+\epsilon}(x_r, R) := \sum_r \int_0^R \left(\frac{|\mu_r|(B(x_r,\rho))}{\rho^{n-\beta(2+\epsilon)}} \right)^{\frac{1}{1+\epsilon}} \frac{d\rho}{\rho} \qquad \beta \in (0, n/(2+\epsilon)]. \tag{1.5}$$

More precisely, in [20] – and in [35,22], where a different and interesting approach was later developed – we can define the estimate

$$\sum_r |u_r(x_r)| \leq c_n \sum_r \left(\fint_{B(x_r,R)} (|u_r| + R(1+\epsilon))^{\gamma} dx_r \right)^{1/\gamma} + c_n \sum_r \mathbf{W}^{\mu_r}_{1,2+\epsilon}(x_r, R), \;\; \gamma > 1+\epsilon, \tag{1.6}$$

valid whenever $B(x_r, R) \Subset \Omega_r$, with x_r being a Lebesgue point of u_r; the constant depends on n, ϵ. In [8] the authors have proved that the pointwise a priori estimate

$$\sum_r |Du_r(x_r)| \leq c_n \sum_r \fint_{B(x_r,R)} (|Du_r| + 1 + \epsilon) \, dy_r + c_n \sum_r \mathbf{W}^{\mu_r}_{\frac{1}{2+\epsilon},2+\epsilon}(x_r, R)$$

$$= c_n \sum_r \fint_{B(x_r,R)} (|Du_r| + 1 + \epsilon) \, dy_r + \sum_r \int_0^R \left(\frac{|\mu_r|(B(x_r,\rho))}{\rho^{n-1}} \right)^{\frac{1}{1+\epsilon}} \frac{d\rho}{\rho} \tag{1.7}$$

holds at every Lebesgue point x_r of Du_r when $\epsilon \geq 0$. The constant c_n depends this time upon $n, \epsilon, \omega(\cdot)$; an extension of (1.7) to a class of anisotropic operators has been later given in [5]. Estimate (1.7) holds in particular for $W^{1,2+\epsilon}$- solutions to (1.1).

We first give an extension to the priori estimate (1.7) when $\epsilon \leq 0$, and in the case solutions to measure data problems like (1.1) belong to the Sobolev space $W^{1,1}$; this is known to happen in the case

$$\epsilon < \frac{1}{n}. \tag{1.8}$$

The previous bound is optimal to have $W^{1,1}$- solutions, are revealed by the analysis of the so-called nonlinear fundamental solution $G_{2+\epsilon}$ to the problem

$$\begin{cases} -\Delta_{2+\epsilon} G_{2+\epsilon} = \delta & \text{in } B_1, \\ G_{2+\epsilon} = 0 & \text{on } \partial B_1, \end{cases} \tag{1.9}$$

where δ is the Dirac measure charging the origin, and B_1 is the ball centered at the origin with radius equal one. In this case we have

$$G_{2+\epsilon}(x_r) \approx \begin{cases} \left(|x_r|^{\frac{2+\epsilon-n}{1+\epsilon}} - 1\right) & \text{if } -1 < \epsilon \neq n-2, \\ G_{2+\epsilon} = 0 & \text{if } \epsilon = n-2, \end{cases}$$

Let us recall that truncated linear Riesz potentials are defined as

$$\mathbf{I}_\beta^{\mu_r}(x_r, R) := \int_0^R \frac{\mu_r(B(x_r,\rho))}{\rho^{n-\beta}} \frac{d\rho}{\rho}, \qquad \beta \in (0, n]$$

Theorem 1.1 *(Linear potential gradient bound). Let* $u_r \in C^1(\Omega_r)$ *be a weak solution to* (1.1) *with* $\mu_r \in L^1(\Omega_r)$, *under the assumptions* (1.2) *with* $n < 1/\epsilon$. *Then there exist a constant* c_n *such that* $c_n \equiv c_n(n, \epsilon, \omega(\cdot), \operatorname{diam}(\Omega_r)) > 0$ *such that the pointwise estimate*

$$\sum_r |Du_r(x_r)| \le c_n \sum_r \fint_{B(x_r,R)} (|Du_r| + 1 + \epsilon)\, dy_r + c_n \sum_r \left[\mathbf{I}_1^{|\mu_r|}(x_r, R)\right]^{\frac{1}{1+\epsilon}} \tag{1.10}$$

holds whenever $B(x_r, R) \subseteq \Omega_r$.

Theorem 1.1 says that, at the least for the considered range of $2+\epsilon$, the gradient of solutions can be pointwise estimated by Riesz potentials exactly as in the case of the standard Poisson equation, provided of course the scaling of the equation is taken into account, i.e. one takes $\left[\mathbf{I}_1^{|\mu_r|}(x_r, R)\right]^{\frac{1}{1+\epsilon}}$ rather than $\mathbf{I}_1^{|\mu_r|}(x_r, R)$. We observe that the operator

$$\mu_r \mapsto \left[\mathbf{I}_1^{|\mu_r|}(\cdot, R)\right]^{\frac{1}{1+\epsilon}} \tag{1.11}$$

defines a new nonlinear potential which has the same scaling and homogeneity properties of $\mathbf{W}^{\mu_r}_{\frac{1}{2+\epsilon}, 2+\epsilon}(\cdot, R)$. The appearance of the Riesz based potential (1.11), see [26], when proving a sort of "level set version'' of (1.10); see also [29]. Also, in Theorem 1.1 we have that the Dini modulus of continuity assumed on the coefficients in (1.3), and known to be sharp for linear elliptic equations – see [14] for counter examples – is now found to apply to the nonlinear case too. Finally, we remark that the constant c_n involved in estimate (1.10) is stable when $2+\epsilon$ approaches 2 i.e. letting $(2+\epsilon) \nearrow 2$ in (1.10) we recover the usual $\boldsymbol{I}_1$ estimate valid for the case of the Poisson equation $-\Delta u_r = u_r$ that is

$$\sum_r |Du_r(x_r)| \le c_n \sum_r \fint_{B(x_r,R)} (|Du_r| + 1 + \epsilon)\, dy_r + c_n \sum_r \mathbf{I}_1^{|\mu_r|}(x_r, R) \tag{1.12}$$

Estimate (1.12) has been proved in [27,8] for general nonlinear equations. We can directly take $\epsilon = 0$ in Theorem 1.1 in order to obtain (1.12).

We present a refinement of the main result of [8] in the case $\epsilon > 0$, that is we replace the Wolff potential in the right-hand side of (1.7) with another, slightly smaller nonlinear potential of Wolff type, namely we employ $\mathbf{W}^{\mu_r}_{\frac{2+\epsilon}{4+3\epsilon},\frac{4+3\epsilon}{2+\epsilon}}$ instead of $\mathbf{W}^{\mu_r}_{\frac{1}{2+\epsilon},2+\epsilon}$.We observe that the two potentials still coincide with the Riesz potential $\mathbf{I}_1^{|\mu_r|}$ when $\epsilon = 0$.We shall assume that there exists a positive number ϵ satisfying

$$0 < 1+\epsilon < \min\{1,4/(n-2),\epsilon\} \tag{1.13}$$

such that the renormalized Hölder continuity property

$$\left|a_{z_r}(x_r,z_2) - a_{z_r}(x_r,z_1)\right| \leq (1+2\epsilon)(|z_1|^2+|z_2|^2+(1+\epsilon)^2)^{\frac{-1}{2}}|z_2-z_1|^{1+\epsilon} \tag{1.14}$$

holds whenever $z_1, z_2 \in \mathbb{R}^n$ and $x_r \in \Omega_r$.

Theorem 1.2 *(Nonlinear potential gradient bound). Let $u_r \in C^1(\Omega_r)$be a weak solution to* (1.1) *with $\mu_r \in L^1(\Omega_r)$, under the assumptions* (1.2) *and* (1.14), *with $\epsilon > 0$. Then there exists a constant $c_n \equiv c_n(n,\epsilon,\omega(\cdot)) > 0$ such that the pointwise estimate*

$$\sum_r |Du_r(x_r)| \leq c_n \sum_r \left(\fint_{B(x_r,R)} (|Du_r|+1+\epsilon)^{\frac{2+\epsilon}{2}} dy_r \right)^{\frac{2+\epsilon}{2}}$$
$$+c_n \sum_r \left(\int_0^R \left(\frac{|\mu_r|(B(x_r,\rho))}{\rho^{n-1}} \right)^{\frac{2+\epsilon}{2(\epsilon+1)}} \frac{d\rho}{\rho} \right)^{\frac{2+\epsilon}{2}} \tag{1.15}$$

holds whenever $B(x_r,R) \subseteq \Omega_r$.

The previous theorem refines the main result of [8] – that is (1.7) – in two respects. First we observe that when formulating condition (1.3) in [8] we replaced $\omega(\cdot)$ by $[\omega(\cdot)]^{2/(2+\epsilon)}$, thereby considering a slightly stronger continuity condition, still of Dini type. As already mentioned in the case $\epsilon < 0$, we find that the same optimal conditions valid for linear equation actually works in the general degenerate case $\epsilon \neq 0$; see again [14]. The second and more substantial improvement has already been anticipated above, and concerns the right-hand side nonlinear potential employed in the pointwise estimate (1.15), in the sense that the following inequality holds true:

$$\sum_r \left[\mathbf{W}^{\mu_r}_{\frac{2+\epsilon}{4+3\epsilon},\frac{4+3\epsilon}{2+\epsilon}}\right]^{\frac{2}{2+\epsilon}} = \sum_r \left(\int_0^R \left[\frac{|\mu_r|(B(x_r,\rho))}{\rho^{n-1}} \right]^{\frac{2+\epsilon}{2(\epsilon+1)}} \frac{d\rho}{\rho} \right)^{\frac{2}{2+\epsilon}}$$
$$\leq \int_0^{2R} \sum_r \left[\frac{|\mu_r|(B(x_r,\rho))}{\rho^{n-1}} \right]^{\frac{1}{\epsilon+1}} \frac{d\rho}{\rho} = \sum_r \mathbf{W}^{\mu_r}_{\frac{1}{2+\epsilon},2+\epsilon}(x_r,2R) \tag{1.16}$$

The previous estimate is indeed a consequence of the elementary inequality

$$\sum_{k=0}^{\infty} a_k^{1+\epsilon} \leq \left(\sum_{k=0}^{\infty} a_k \right)^{1+\epsilon}, \qquad \epsilon \geq 0, \qquad a_k \geq 0, \quad \forall k \in \mathbb{N} \tag{1.17}$$

applied with $\epsilon = \epsilon/2$ to perform the following standard computation:

$$\int_0^R \sum_r \left[\frac{|\mu_r|(B(x_r,\rho))}{\rho^{n-1}} \right]^{\frac{2+\epsilon}{2(\epsilon+1)}} \frac{d\rho}{\rho} \leq \sum_r \sum_{k=0}^{\infty} \int_{R/2^{k+1}}^{R/2^k} \left(\frac{|\mu_r|(B(x_r,\rho))}{\rho^{n-1}} \right)^{\frac{2+\epsilon}{2(\epsilon+1)}} \frac{d\rho}{\rho}$$

$$\leq \sum_r \sum_{k=0}^{\infty} \left(\frac{|\mu_r| \left(B\left(x_r, \frac{R}{2^k} \right) \right)}{\left(\frac{R}{2^{k+1}} \right)^{n-1}} \right)^{\frac{2+\epsilon}{2(\epsilon+1)}}$$

$$\leq \sum_r \left(\sum_{k=0}^{\infty} \left(\frac{|\mu_r| \left(B(x_r, R/2^k) \right)}{(R/2^{k+1})^{n-1}} \right)^{\frac{1}{\epsilon+1}} \right)^{\frac{2+\epsilon}{2}}$$

$$\leq \sum_r \left(\int_0^{2R} \left(\frac{|\mu_r| (B(x_r, \rho))}{\rho^{n-1}} \right)^{\frac{1}{\epsilon+1}} \frac{d\rho}{\rho} \right)^{\frac{2+\epsilon}{2}}$$

Both estimates (1.7) and (1.15) allow getting the sharp integrability result

$$\mu_r \in L(1+\epsilon, \gamma) \Longrightarrow Du_r \in L\left(\frac{(1+2\epsilon)(1+\epsilon)}{\epsilon}, \gamma \right) \qquad \text{for } \epsilon > 0 \text{ and } 0 < \gamma < \infty. \tag{1.18}$$

When instead asking for the limiting case $Du_r \in L^{\infty}$, in improvement in terms of the second index in the Lorentz scale is allowed by (1.15) with respect to (1.17). Indeed, while (1.15) allows to include that $\mu_r \in L(n, (2+\epsilon)/(2+2\epsilon))$ implies the local boundedness of Du_r, inequality (1.7) requires that $\mu_r \in L(n, 1/(1+\epsilon))$, which is a stronger condition for $\epsilon > 0$. Turning our attention to the case when μ_r is genuinely a measure, we have that the potentials in the two sides of (1.16) become essentially equivalent when for instance the measure uniformly concentrates on a set with dimension that can be described via ordinary Hausdorff measues. This is for instance the case when the measure concentrates uniformly on a σ-Alfhors regular set S_r

$$\mu_r = \mathcal{H}^{\sigma} \llcorner S_r, \qquad \mu_r(B_R) \approx R^{\sigma}, \quad \sigma \in [0, n],$$

which holds whenever B_R is centered on S_r. Here $\mathcal{H}^{\sigma}$ denotes the σ-dimensional Hausdorff measure. Relevant examples are given by surface measures related to manifolds, where the quantities appearing in the two sides of (1.16) still become equivalent. A strict inequality occurs for instance in the case of those measures uniformly concentrated on sets whose Hausdorff dimension can be described only using in terms of a Gauge function $\gamma(\cdot)$ of non-power type: $\mu_r = \mathcal{H}^{\gamma(\cdot)} \llcorner S_r$ where $\mu_r(B_R) \approx \gamma(R)$. For this we refer for instance to [33].

Although we shall take the strategy adopted in [8] as a guideline, there are here a number of new non-trivial points. We start by the case $\epsilon \leq 0$, that is Theorem 1.1. Usually called the singular case since the modulus of ellipticity tends to infinity when $|Du_r| \to 0$, the case $\epsilon < 0$ is for our ultimate purposes to be considered as a degenerate one. In fact, since estimates of the type (1.7) and (1.10) are estimates *on the size of the gradient*, the difficult case for us is when $|Du_r|$ gets large. In this situation there is a loss of ellipticity in the equation, and estimates become harder to get. Instead a pointwise gradient estimate is in principle easier to get when $2+\epsilon$ gets larger since the coercivity of the operator increases. A manifestation of these difficulties in the case $\epsilon < 0$ is the following major technical difference with respect to the case $\epsilon \geq 0$. In [8] we developed an iteration scheme based on the comparison between the original solution u_r of (1.1) and solutions to homogeneous equations of the type div $a(x_r, Dw_r) = 0$ in a ball B_R with radius R, subject to the boundary condition $u_r \equiv w_r$ on ∂B_R. The outcome was the inequality

$$\sum_r \fint_{B_R} |Du_r - Dw_r| dx_r \le c_n \sum_r \left[\frac{|\mu_r|(B_R)}{R^{n-1}}\right]^{\frac{1}{\epsilon+1}}. \tag{1.19}$$

The quantity on the right-hand side of (1.19) is in turn the density of the potential $\mathbf{W}^{\mu_r}_{\frac{1}{2+\epsilon},2+\epsilon}$, and this is a key point in the proof; indeed, a suitable summation process involving (1.19) finally leads to (1.7). As a consequence of the fact that $\epsilon \le 0$ an estimate of the type (1.19) is no longer possible since the coercivity of the operator is too weak. One of the main challenges here is to find the correct replacement for the quantity appearing in the right-hand side of (1.19) when $\epsilon \le 0$, which allows to rebalance the weak ellipticity. It turns out that the mixed quantity

$$\sum_r \left[\frac{|\mu_r|(B_R)}{R^{n-1}}\right]^{\frac{1}{\epsilon+1}} + \sum_r \left[\frac{|\mu_r|(B_R)}{R^{n-1}}\right] \left(\fint_{B_R} (|Du_r| + 1 + \epsilon)\, dx_r\right)^{-\epsilon} \tag{1.20}$$

depends also on the gradient average, and that for this reason cannot represent the density of a potential, is the right one. In fact, the presence of the gradient in (1.20), coupled with the measure, encodes in an optimal way the weaker ellipticity of the problem.

For the case $\epsilon \ge 0$ the key to the improved nonlinear potential estimate (1.15) is the use of the map $V(Du_r)$ in the estimates, rather than the plain gradient Du_r. Here it is

$$V(Du_r) \equiv V_{1+\epsilon}(Du_r) = (|Du_r|^2 + (1+\epsilon)^2)^{\frac{\epsilon}{4}}\, Du_r. \tag{1.21}$$

The use of $V(Du_r)$ rather than Du_r allows to get better estimates as it allows to incorporate many of the degenerate features of the operator in question in the considered map, allowing for a better potential on the right-hand side. In turn, working with the quantity defined in (1.21) poses additional problems, and in particular a few delicate estimates below the natural growth exponents must be worked out.

2. Notations

We denote by c_n a general constant larger (or equal) than one, possibly varying from line to line; special occurrences will be denoted by c_{n+1} etc; relevant dependences on parameters will be emphasized using parentheses. We also denote by $B(x_0, R) := \{x_r \in \mathbb{R}^n : |x_r - x_0| < R\}$ the open ball with center x_0 and radius $R > 0$; when not important, or clear from the context, we shall omit denoting the center as follows: $B_R \equiv B(x_0, R)$. Unless otherwise stated, different balls in the same context will have the same center. We shall also denote $B \equiv B_1 = B(0,1)$. With A being a measurable subset with positive measure, and with $g^r : A \to \mathbb{R}^k$ being a measurable map, we shall denote

$$\fint_A g^r(x_r)\, dx_r := \frac{1}{|A|} \int_A g^r(x_r) dx_r$$

its integral average. According to what we have stated in the Introduction, when considering a L^1-function μ_r we shall denote

$$|\mu_r|(A) := \int_A |\mu_r(x_r)|dx_r.$$

In other words, we shall deal with L^1-data, but "thinking of the case the datum is a measure". Indeed, when considering equation as (1.1) in order to get the results we are bounded to present, it is sufficient to consider the case $\mu_r \in L^1(\Omega_r)$, the case when μ_r is general Borel measure with finite total mass can be obtained via approximation [8,27].

With $\epsilon \geq 0$, we define

$$V(z_r) \equiv V_{1+\epsilon}(z_r) := ((1+\epsilon)^2 + |z_r|^2)^{\frac{\epsilon}{4}} z_r\,, \quad z_r \in \mathbb{R}^n \tag{2.1}$$

which is easily seen to be a locally bi-Lipschitz bijection of $\mathbb{R}^n$. A basic property of the map $V(\cdot)$, whose proof can be found in [15], is the following: For any $z_1, z_2 \in \mathbb{R}^n$, and any $\epsilon \geq 0$, it holds

$$c_n^{-1}((1+\epsilon)^2 + |z_1|^2 + |z_2|^2)^{\frac{\epsilon}{2}} \leq \frac{|V(z_2) - V(z_1)|^2}{|z_2 - z_1|^2} \leq c_n((1+\epsilon)^2 + |z_1|^2 + |z_2|^2)^{\frac{\epsilon}{2}}\,, \tag{2.2}$$

where $c_n \equiv c_n(n,\epsilon)$. The strict monotonicity properties of the vector field $a(\cdot)$ implied by the left-hand side in $(1.2)_2$ can be recast using the map $V(\cdot)$. Indeed – see also [25] – combining $(1.2)_2$ and (2.2) yields, for $c_n \equiv c_n(n,\epsilon,) > 0$, and whenever $z_1, z_2 \in \mathbb{R}^n$

$$c_n^{-1}|V(z_2) - V(z_1)|^2 \leq \langle a(x_r, z_2) - a(x_r, z_1), z_2 - z_1\rangle. \tag{2.3}$$

Moreover, when $\epsilon \geq 0$, assumption $(1.2)_2$ - via (2.2) – (2.3) - immediately implies

$$c_n^{-1}|z_2 - z_1|^{2+\epsilon} \leq \langle a(x_r, z_2) - a(x_r, z_1), z_2 - z_1\rangle. \tag{2.4}$$

3. Decay Estimates for a_0–Harmonic Functions

The aim of this chapter is to recall a few decay estimates valid for solutions to homogeneous equations of the type

$$\nabla a_0(Dv_r) = 0, \tag{3.1}$$

where the vector field $a_0 : \mathbb{R}^n \to \mathbb{R}^n$ satisfies assumptions $(1.2)_{1,2}$ and (1.14), with the obvious understanding that now no x_r-dependence is involved. Such functions are indeed called a_0-harmonic functions. The peculiar point of the results we are going to present is that a few of the decay estimates presented are not formulated in terms of the gradient Du_r, but rather in terms of the nonlinear quantity $V(Du_r)$. The decay estimates for solutions found here differ from the usual ones in the fact that the exponents involved are smaller than those typically used, and this will require to employ certain rarely used facts from regularity theory of $(2+\epsilon)$-Laplacian type operators.

Theorem 3.1. *Let $v_r \in W^{1,2+\epsilon}(\Omega_r)$ be a weak solution to* (3.1) *under the assumptions* $(1.2)_{1,2}$ *with $\epsilon > 0$ and* (1.14). *Then there exist constants $\beta \in (0,1]$ and $c_n \geq 1$, both depending on n, ϵ such that the following estimate:*

$$\fint_{B_\rho} \sum_r \left|V(Dv_r) - (V(Dv_r))_{B_\rho}\right|^2 dx_r \leq c_n \left(\frac{\rho}{R}\right)^{2\beta} \fint_{B_R} \sum_r \left|V(Dv_r) - (V(Dv_r))_{B_R}\right|^2 dx_r \tag{3.2}$$

holds whenever $B_\rho \subseteq B_R \subseteq \Omega_r$ are concentric balls.

This result is standard in the case of the $(2+\epsilon)$-Laplacian equation – see for instance [7] and references therein – and it is more in general known to hold for minima of certain functionals of the Calculus of

Variations with $(2+\epsilon)$-growth; moreover, it extends to minimizers of the $(2+\epsilon)$-Dirichlet functional (4.31) below in the vectorial case. Here we shall present the necessary modifications to the known proofs in order to prove the result in the context of Theorem 3.1.

We divide the proof in several steps.

Step 1: The degenerate case. Here we see that we may reduce to the nondegenerate case $\epsilon > 0$ via an approximation that the reader may for instance find in [12,10]. Let us fix a family $\{\phi_\epsilon^r\}_{\epsilon>0}$ of standard mollifiers in $\mathbb{R}^n$ and obtained in the following way: $\phi_\epsilon^r(z_r) \coloneqq (2+\epsilon)^{-n}\phi^r(z_r/\epsilon)$ Here $\phi^r \in C^\infty(\mathbb{R}^n)$ and it is such that

$$\operatorname{supp} \phi^r = \overline{B_1} \quad \text{and} \quad \int_{\mathbb{R}^n} \phi^r(z_r)\, dz_r = 1. \tag{3.3}$$

We defined the regularized vector fields

$$a_\epsilon(z_r) \coloneqq (a * \phi_\epsilon^r)(z_r), \qquad \epsilon > 0. \tag{3.4}$$

It obviously follows that $a_\epsilon(\cdot) \in C^\infty(\mathbb{R}^n)$ and moreover, as in [12] – whose arguments apply here since (3.3) is assumed – we have that the assumptions (1.2) and (1.14) are satisfied for new values of ϵ. The approximation scheme in question is completely standard and we omit it here (see [1] and [8]). We then define $v_\epsilon \in v_r + W_0^{1,2+\epsilon}(\Omega'_r)$ as the unique solution to the Dirichlet problem

$$\begin{cases} \operatorname{div} a_\epsilon(Dv_\epsilon) = 0 & \text{in } \Omega'_r, \\ v_\epsilon = v_r & \text{on } \partial\,\Omega'_r, \end{cases}$$

for a Lipschitz-regular subdomain $\Omega'_r \Subset \Omega_r$. The final outcome is that – up to choosing a suitable subsequence $\epsilon \equiv \epsilon_n \to 0$ – we have that $v_\epsilon \to v_r$ strongly in $W^{1,2+\epsilon}(\Omega'_r)$. Needless to say this is sufficient to pass $\epsilon \to 0$ in an estimate like (3.2). Therefore in the rest of the proof we shall with no loss of generality assume that $\epsilon > 0$, catching the case $\epsilon = 0$ by passing to the limit the uniform decay estimates obtained in a standard way. Moreover we shall obviously replace Ω'_r by Ω_r since the result we are going to prove is local in nature.

Step 2: L^∞-estimate. Let us denote

$$\tilde{a}_{i,j}(x_r) \coloneqq (|Dv_r(x_r)|^2 + (1+\epsilon)^2)^{-\epsilon}\partial_{z_{r_j}}(a_0^i)(Dv_r(x_r)),$$

and

$$H^r \equiv H^r(Dv_r) \coloneqq (|Dv_r|^2 + (1+\epsilon)^2)^{(2+\epsilon)/2}.$$

It then follows – see for instance the approach in [10] – that $H^r \in W_{\mathrm{loc}}^{1,2}(\Omega_r) \cap L_{\mathrm{loc}}^\infty(\Omega_r)$ and that H^r is a subsolution of a uniformly elliptic equation with measureable coefficients, that is

$$\int_{\Omega_r} \tilde{a}_{i,j}\, D_i H^r D_j \varphi^r\, dx_r \le 0, \quad \varphi^r \ge 0 \tag{3.5}$$

holds with $\varphi_r \in C_0^\infty(\Omega_r)$. In turns this fact implies that $H^r \in L_{\mathrm{loc}}^\infty(\Omega_r)$ and the quantitative estimate

$$\sup_{B_{R/2}} H^r \le \fint_{B_R} H^r\, dx_r \qquad \forall\, B_R \subseteq \Omega_r, \tag{3.6}$$

holds for a constant depending on n, ϵ.

Step 3: A first oscillation estimate. Denoting

$$\phi^r(R) := \fint_{B_R} \left|V(Dv_r) - (V(Dv_r))_{B_R}\right|^2 dx_r, \qquad M(R) := \sup_{B_R} H^r$$

for a fixed ball $B_R \equiv B(x_0, R)$. As a consequence of the weak Harnack type inequality valid for subsolutions of (3.5) we have - see [15] -

$$\phi^r(R) \leq c_n[M(R) - M(R/2)], \tag{3.7}$$

For $c_n \equiv c_n(n,\epsilon)$. Moreover, a standard difference quotient method asserts that $V(Dv_r) \in W^{1,2}_{\mathrm{loc}}(\Omega_r, \mathbb{R}^n)$, while the next reverse Hölder's inequality is just a consequence of the fact that v_r solves (3.1), together with the higher differentiability of $V(Dv_r)$:

$$\sum_r \left(\fint_{B_{R/2}} \left|V(Dv_r) - (V(Dv_r))_{B_{R/2}}\right|^{2\chi} dx_r\right)^{1/\chi} \leq c_n \sum_r \fint_{B_R} \left|V(Dv_r) - (V(Dv_r))_{B_R}\right|^2 dx_r,$$

where $\chi = n/(n-2)$ when $n > 2$ and χ can be chosen arbitrarily large when $n = 2$.

Step 4: Conclusion. Here we prove

Lemma 3.1. *Assume that*

$$\phi^r(R) \leq c_{n+1}\left(\left|(Dv_r)_{B_R}\right|^2 + (1+\epsilon)^2\right)^{(2+\epsilon)/2}, \tag{3.8}$$

holds for a constant c_{n+1}. Then there exists another constant c_{n+2}, depending on n, ϵ and c_{n+1}, such that

$$\phi^r(\rho) \leq c_{n+2}\left(\frac{\rho}{R}\right)^2 \left\{1 + \left(\frac{R}{\rho}\right)^{n+2}\left[\frac{\phi^r(R)}{\left(\left|(Dv_r)_{B_R}\right|^2 + (1+\epsilon)^2\right)^{(2+\epsilon)/2}}\right]^{1+\epsilon}\right\} \phi^r(R)$$

holds whenever $0 < \rho \leq R$.

Once the previous lemma is proved, the proof follows along the lines of [15] – keep in mind that general differential forms are used there. Indeed, a delicate but by now standard iteration argument allows to deduce (3.2) from Lemma 3.1 and the content of Step 3. It therefore remains to prove Lemma 3.1; to which we dedicate in the rest of the proof. We again follow the lines of [15], but at several stages we shall use a different argument since we are not dealing with minimizers of integral functionals.

Let us set $z_0 := (Dv_r)_{B_R}$. Assumption (3.8) used together with (3.6) yields

$$\sup_{B_{R/2}} |Dv_r - z_0|^{2+\epsilon} \leq c_n(|z_0|^2 + (1+\epsilon)^2)^{(2+\epsilon)/2}, \tag{3.9}$$

where c_n depends on ϵ, c_{n+1}. We now introduce the frozen matrix

$$(A_0)^i_j := \left(a^i_0\right)_{z_{r_j}}(z_0),$$

which is an elliptic matrix with constant coefficients in the sense that it satisfies the following ellipticity and growth conditions

$$c_n^{-1}(|z_0|^2 + (1+\epsilon)^2)^{\frac{\epsilon}{2}}|\lambda_r|^2 \leq \langle A_0\lambda_r, \lambda_r\rangle, \qquad |A_0| \leq c_n(|z_0|^2 + (1+\epsilon)^2)^{\frac{\epsilon}{2}}, \tag{3.10}$$

whenever $\lambda_r \in \mathbb{R}^n$, and with $c_n \equiv c_n(n,\epsilon)$. Accordingly, we define $\tilde{v}_r \in v_r + W^{1,2}_0(B_{R/2})$as the unique solution to the following Dirichlet problem:

$$\begin{cases} \nabla\left(A_0 D\tilde{v}_r\right) = 0 & \text{in } B_{R/2}, \\ \tilde{v}_r = v_r & \text{on } \partial B_{R/2}. \end{cases}$$

This means that the ratio between the highest and the lowest eigenvalue of A_0 is bounded by a constant depending on n, ϵ and therefore classical estimates for solutions to linear elliptic equations apply (see for instance [15]). In particular, for $c_n \equiv c_n(n,\epsilon) \geq 1$ it holds that

$$\sum_r \fint_{B_{R/2}} (|D\tilde{v}_r - z_0|^{3+\epsilon} + |D\tilde{v}_r - z_0|^{2+\epsilon})dx_r \ \leq c_n \sum_r \fint_{B_{R/2}} (|Dv_r - z_0|^{3+\epsilon} + |Dv_r - z_0|^{2+\epsilon})dx_r. \tag{3.11}$$

Again as in [15] we arrive at

$$\sum_r \phi^r(\rho) \leq c_n \left(\frac{\rho}{R}\right)^2 \sum_r \phi^r(R/2) + c_n(|z_0|^2 + (1+\epsilon)^2)^{\frac{\epsilon}{2}} \left(\frac{R}{\rho}\right)^n \sum_r \fint_{B_{R/2}} |Dv_r - D\tilde{v}_r|^2 dx_r, \tag{3.12}$$

for every $\rho \leq R/2$. We have to estimate the last integral in (3.12): denoting $w_r := v_r - \tilde{v}_r$ we have, by mean of the first inequality in (3.10)

$$(|z_0|^2 + (1+\epsilon)^2)^{\frac{\epsilon}{2}} \sum_r \fint_{B_{R/2}} |Dw_r|^2 dx_r \leq c_n \sum_r \fint_{B_{R/2}} \langle A_0 Dw_r, Dw_r \rangle dx_r, \tag{3.13}$$

Now, notice that by writing

$$a_0(z_r) - a_0(z_0) = \int_0^1 (a_0)_{z_r}(tz_r + (1-t)z_0)dt(z_r - z_0),$$

and applying (1.14), we obtain

$$|a_0(z_r) - a_0(z_0) - A_0(z_r - z_0)| \leq c_n(|z_0|^2 + (1+\epsilon)^2)^{\frac{-1}{2}}|z_r - z_0|^{2+\epsilon} + c_n|z_r - z_0|^{1+\epsilon}. \tag{3.14}$$

In turn, using Young's inequality repeatedly, and the fact that both v_r and $\tilde{v}_r$ are solutions, and of course using (3.14), it holds that

$$\begin{aligned}
&\sum_r \fint_{B_{R/2}} \langle A_0 Dw_r, Dw_r \rangle dx_r \\
&= \sum_r \fint_{B_{R/2}} \langle A_0 Dv_r, Dw_r \rangle dx_r \ - \sum_r \fint_{B_{R/2}} \langle a_0(Dv_r - a_0(z_0) - A_0(Dv_r - z_0), Dw_r \rangle dx_r \\
&\leq c_n \sum_r \fint_{B_{R/2}} \left((|z_0|^2 + (1+\epsilon)^2)^{\frac{-1}{2}} |Dv_r - z_0|^{2+\epsilon} + |Dv_r - z_0|^{1+\epsilon} \right) |Dv_r - z_0| dx_r \\
&+ c_n \sum_r \fint_{B_{R/2}} \left((|z_0|^2 + (1+\epsilon)^2)^{\frac{-1}{2}} |Dv_r - z_0|^{2+\epsilon} + |Dv_r - z_0|^{1+\epsilon} \right) |D\tilde{v}_r - z_0| dx_r \\
&\leq c_n \sum_r \fint_{B_{R/2}} (|z_0|^2 + (1+\epsilon)^2)^{\frac{-1}{2}} |Dv_r - z_0|^{3+\epsilon} + |Dv_r - z_0|^{2+\epsilon} dx_r \\
&+ c_n \sum_r \fint_{B_{R/2}} (|z_0|^2 + (1+\epsilon)^2)^{\frac{-1}{2}} |D\tilde{v}_r - z_0|^{3+\epsilon} + |D\tilde{v}_r - z_0|^{2+\epsilon} dx_r \\
&\leq c_n \sum_r \fint_{B_{R/2}} (|z_0|^2 + (1+\epsilon)^2)^{\frac{-1}{2}} |Dv_r - z_0|^{3+\epsilon} + |Dv_r - z_0|^{2+\epsilon} dx_r \\
&\leq c_n \sum_r \fint_{B_{R/2}} (|z_0|^2 + (1+\epsilon)^2)^{\frac{-1}{2}} |Dv_r - z_0|^{3+\epsilon} dx_r
\end{aligned}$$

In the last two lines we used (3.11) and then (3.9). Combining the last inequality with (3.13) yields

$$(|z_0|^2+(1+\epsilon)^2)^{\frac{\epsilon}{2}}\sum_r \fint_{B_{R/2}} |Dv_r - D\tilde{v}_r|^2 dx_r \le c_n \sum_r \fint_{B_{R/2}} (|z_0|^2+(1+\epsilon)^2)^{\frac{-1}{2}} |Dv_r - z_0|^{3+\epsilon} dx_r$$

This last estimate is the analogue of the last inequality at page 38 of [15] and from this point the proof of the Lemma follows as in [15].

We give a version of Theorem 3.1 below the natural growth exponent. Indeed, instead of considering $V(Du_r)$ in L^2, it will be considered in L^1. To begin with we recall a preliminary result on reverse Hölder inequalities.

Lemma 3.2. *Let* $g^r: A \to \mathbb{R}^k$ *be an integrable map such that*

$$\sum_r \left(\fint_{B_R} |g^r|^{\chi_0} dx_r\right)^{1/\chi_0} \le (1+\epsilon)\sum_r \fint_{B_{2R}} |g^r|\, dx_r$$

holds whenever $B_{2R} \subseteq A$, *where* $A \subseteq \mathbb{R}^n$ *is an open subset, and* $\epsilon > 0$. *Then, for every* $t \in (0,1]$ *and* $\chi \in (0, \chi_0]$ *there exists a constant* $c_{n-1} \equiv c_{n-1}(n, \epsilon, t)$ *such that, for every* $B_{2R} \Subset A$ *it holds that*

$$\sum_r \left(\fint_{B_R} |g^r|^{\chi} dx_r\right)^{1/\chi} \le c_{n-1}\sum_r \left(\fint_{B_{2R}} |g^r|^t\, dx_r\right)^{1/t}. \tag{3.15}$$

The proof of the previous result, which is based on a by now standard interpolation/covering argument, can be obtained with minor modifications form [13]. Next, a result which can be inferred from [25]; see also [26].

Lemma 3.3. *Let* $v_r \in W^{1,2+\epsilon}(\Omega_r)$ *be a weak solution to* (3.1) *under the assumptions* (1.2), *and fix* $z_0 \in \mathbb{R}^n$. *For every* $t \in (0,1]$ *there exists* $c_n \equiv c_n(n, \epsilon, t) \ge 1$, *but independent of* $z_0 \in \mathbb{R}^n$ *such that*

$$\sum_r \left(\fint_{B_{2R}} |V(Dv_r) - z_0|^2 dx_r\right)^{1/2} \le c_n \sum_r \left(\fint_{B_R} |V(Dv_r) - z_0|^{2t} dx_r\right)^{1/(2t)}, \tag{3.16}$$

holds whenever $B_R \subseteq \Omega_r$.

We now come to the decay estimate below the natural growth exponent.

Theorem 3.2. *Let* $v_r \in W^{1,2+\epsilon}(\Omega_r)$ *be a weak solution to* (3.1) *under the assumptions* (1.2), *with* $\epsilon > 0$ *and* (1.14). *Then there exist constants* $\beta \in (0,1]$ *and* $c_n \ge 1$, *both depending on* n, ϵ *such that the following estimate:*

$$\fint_{B_\rho} \sum_r \left|V(Dv_r) - (V(Dv_r))_{B_\rho}\right| dx_r \le c_n \left(\frac{\rho}{R}\right)^\beta \fint_{B_R} \sum_r \left|V(Dv_r) - (V(Dv_r))_{B_R}\right| dx_r \tag{3.17}$$

holds whenever $B_\rho \subseteq B_R \subseteq \Omega_r$ *are concentric balls.*

Proof: Using estimate (3.2) and Hölder's inequality we deduce

$$\begin{aligned}\fint_{B_\rho} \sum_r \left|V(Dv_r) - (V(Dv_r))_{B_\rho}\right| dx_r &\le \sum_r \left(\fint_{B_\rho} \left|V(Dv_r) - (V(Dv_r))_{B_\rho}\right|^2 dx_r\right)^{1/2} \\ &\le c_n \left(\frac{\rho}{R}\right)^\beta \sum_r \left[\fint_{B_{R/2}} \left|V(Dv_r) - (V(Dv_r))_{B_{R/2}}\right|^2 dx_r\right]^{1/2} \\ &\le c_n \left(\frac{\rho}{R}\right)^\beta \sum_r \left[\fint_{B_{R/2}} \left|V(Dv_r) - (V(Dv_r))_{B_R}\right|^2 dx_r\right]^{1/2},\end{aligned} \tag{3.18}$$

Whenever $0 < \rho \le R/2$. On the other hand, we may apply estimate (3.16) with $z_0 = (V(Dv_r))_{B_R}$ getting

$$\sum_r \left(\fint_{B_{R/2}} |V(Dv_r) - (V(Dv_r))_{B_R}|^2 dx_r \right)^{1/2} \leq c_n \fint_{B_R} \sum_r |V(Dv_r) - (V(Dv_r))_{B_R}| dx_r$$

Merging this last estimate with (3.18) yields the assertion for case $0 < \rho \leq R/2$; on the other hand estimate (3.17) trivially holds when $R/2 < \rho \leq R$ and the proof is complete.

The next result has been proved for the case $\epsilon \geq 0$ in [8]; the proof for the case $0 < \epsilon < 1$ can be obtained with minor modifications.

Theorem 3.3. *Let $v_r \in W^{1,1+\epsilon}(\Omega_r)$ be a weak solution to* (3.1) *under the assumptions* $(1.2)_{1,2}$ *with $\epsilon > 0$. Then there exist constants $\beta \in (0,1]$ and $c_n \geq 1$, both depending on n, ϵ such that the estimate*

$$\fint_{B_\rho} \sum_r \left| Dv_r - (Dv_r)_{B_\rho} \right| dx_r \leq c_n \left(\frac{\rho}{R}\right)^\beta \fint_{B_R} \sum_r |Dv_r - (Dv_r)_{B_R}| dx_r \tag{3.19}$$

holds whenever $B_\rho \subseteq B_R \subseteq$ A are concentric balls.

Estimates of this type, with different exponents involved, have been originally developed in [6,21,23].

Remark 3.2 (*Stabilization of the constant I*). A very careful analysis of the estimates involved in the proof of (3.19) reveals a continuous dependence of the constants $\beta > 0$ and $c_n < \infty$ appearing in (3.19). This means whenever $1 + \epsilon$ lies in a compact subset of $(1, \infty)$ then β and c_n vary in a compact subset of $(0,1)$ and $[1, \infty)$, respectively.

4. Decay and Comparison Estimates

We now fix, for the rest of the section, a ball $B(x_0, 2R) \subseteq \Omega_r$ that will be shortly denoted by B_{2R} Unless otherwise stated all the ball considered will concentric to B_{2R}. Moreover, the solution of (1.1) will be always considered under the assumptions of Theorem 1.1, that is of class C^1. In the rest of the sections u_r will be the solution considered in Theorems 1.1 and 1.2.

We derive a few crucial comparison estimates between the original solution of (1.1) and solutions to suitably homogeneous boundary value problems. In the case $\epsilon > 0$ the main point is the use of the function $V(Du_r)$ replacing the gradient Du_r, while in the second case $\epsilon \leq 0$ the main point is that the mixed quantity in (1.20) involving the right-hand side measure μ_r and the gradient average will become into the play.

We start defining $w_r \in u_r + W_0^{1,2+\epsilon}(B_{2R})$ as the unique solution to the homogeneous Dirichlet problem

$$\begin{cases} \nabla a(x_r, Dw_r) = 0 & \text{in } B_{2R}, \\ w_r = u_r & \text{on } \partial B_{2R}. \end{cases} \tag{4.1}$$

Remark 4.1 (*Scaling*). Before going on with the proofs, let us recall a few basic properties of equations of the type (1.1) under the assumptions (1.2), where $\mu_r \in L^1(\Omega_r)$. Let us consider the ball $B_{2R} \equiv B(x_0, 2R) \subset \Omega_r$ and positive number $A > 0$, and let us define the new functions

$$\tilde{u}_r(y) := \frac{u_r(x_0 + 2Ry_r)}{2AR} \quad \text{and} \quad \tilde{\mu}_r(y_r) := \frac{2R\mu_r(x_0 + 2Ry_r)}{A^{1+\epsilon}}, \tag{4.2}$$

and the new vector field

$$\tilde{a}(y_r, z_r) := \frac{a(x_0 + 2Ry_r, Az_r)}{A^{1+\epsilon}},$$

for $y_r \in B_1$ and $z_r \in \mathbb{R}^n$. It is now easy to see that $\tilde{u}_r$ solves the equation

$$-\nabla \tilde{a}(y_r, D\tilde{u}_r) = \tilde{\mu}_r.$$

Moreover the new vector field $\tilde{a}(\cdot)$ satisfies assumptions (1.2) with $1+\epsilon$ replaced by $(1+\epsilon)/A$ (and $\omega(\cdot)$ replaced by $\omega_R(\cdot) \coloneqq \omega(2R\,\cdot)$, but in what follows the properties of $\omega(\cdot)$ will not be important). This observation will be useful in a few lines, when reducing estimates on general balls to the case the ball in equation is B_1.

Lemma 4.1. *Under the assumptions of Theorem* 1.1 *let* $w_r \in u_r + W_0^{1,2+\epsilon}(B_{2R})$ *be as in* (4.1)*; assume that* $\epsilon \geq 0$. *Then the following inequality holds for a constant* $c_n \equiv c_n(n,\epsilon)$:

$$\fint_{B_{2R}} \sum_r |V(Du_r) - V(Dw_r)| dx_r \leq c_n \sum_r \left[\frac{|\mu_r|(B_{2R})}{R^{n-1}}\right]^{\frac{2+\epsilon}{2(1+\epsilon)}}. \tag{4.3}$$

Proof. We start observing that by Remark 4.1, with x_0 being the center of B_{2R}, by taking

$$A \coloneqq \left(\frac{|\mu_r|(B_{2R})}{R^{n-1}}\right)^{\frac{1}{1+\epsilon}} \quad \text{and} \quad \tilde{w}_r(y_r) \coloneqq \frac{2(x_0 + 2Ry_r)}{AR} \tag{4.4}$$

it follows that $\operatorname{div} \tilde{a}(x_r, D\tilde{w}_r) = 0$ and we may reduce ourselves to the case in which the following holds:

$$B_{2R} \equiv B_1 \qquad \text{and in turn} \quad |\mu_r|(B_1) \leq 1, \tag{4.5}$$

thereby proving that

$$\fint_{B_1} \sum_r \left|V_{(1+\epsilon)/A}(D\tilde{u}_r) - V_{(1+\epsilon)/A}(D\tilde{w}_r)\right| dy_r \leq c_n(n,\epsilon). \tag{4.6}$$

Inequality (4.3) then follows by (4.6) scaling back to u_r and w_r and noting that

$$\begin{aligned}\fint_{B_{2R}} \sum_r \left|V_{(1+\epsilon)}(Du_r) - V_{(1+\epsilon)}(Dw_r)\right| dx_r &= A^{(2+\epsilon)/2} \fint_{B_1} \sum_r \left|V_{(1+\epsilon)/A}(D\tilde{u}_r) - V_{(1+\epsilon)/A}(D\tilde{w}_r)\right| dy_r \\ &\leq c_n A^{(2+\epsilon)/2}.\end{aligned}$$

Therefore, from now on we shall argue under the additional assumptions (4.5); it is here needless to remark that we may assume $A > 0$, otherwise the proof trivializes by the strict monotonicity of the vector field $a(\cdot)$. For any integer $k \geq 0$ we define the truncation operators

$$T_k(t) \coloneqq \max\{-k, \min\{k,t\}\}, \qquad \Phi_k(t) \coloneqq T_1(t - T_k(t)), \qquad t \in \mathbb{R}. \tag{4.7}$$

Since both u_r and v_r are solutions agreeing on ∂B_1, we test the weak formulation

$$\int_{B_1} \sum_r \langle a(x_r, Du_r) - a(x_r, Dw_r), D\varphi^r \rangle dx_r = \int_{B_1} \sum_r \varphi^r d\mu_r \tag{4.8}$$

by $\varphi^r \equiv \Phi_k(u_r - w_r)$; using (2.3) – (2.4) and the bound in (4.5), we obtain

$$\sum_r \int_{C_k} |V(Du_r) - V(Dw_r)|^2 + |Du_r - Dw_r|^{2+\epsilon} dx_r \leq c_n \sum_r |\mu_r|(B_1) \leq c_n, \tag{4.9}$$

where

$$C_k \coloneqq \{x_r \in B_1 : k < |u_r(x_r) - w_r(x_r)| \leq k+1\}, \tag{4.10}$$

and $c_n \equiv c_n(n,\epsilon)$. By Hölder's inequality, and the very definition of C_k, for $k > 0$ we find

$$\int_{C_k} \sum_r |V(Du_r) - V(Dw_r)| + |Du_r - Dw_r|^{(2+\epsilon)/2} dx_r$$

$$\leq c_n\, |C_k|^{\frac{1}{2}} \sum_r \left(\int_{C_k} |V(Du_r) - V(Dw_r)|^2 + |Du_r - Dw_r|^{2+\epsilon} dx_r \right)^{1/2}$$

$$\leq^{(4.9)} c_n\, |C_k|^{\frac{1}{2}} \leq \frac{c_n}{k^{\frac{1+\epsilon}{2}}} \sum_r \left(\int_{C_k} |u_r - w_r|^{1+\epsilon} dx_r \right)^{1/2} \tag{4.11}$$

where we choose $1+\epsilon$ in order to satisfy

$$1 < 1+\epsilon < \frac{n(2+\epsilon)}{2(n-1)}. \tag{4.12}$$

Notice that this is possible since $\epsilon \geq 0$. Still, again by Hölder's inequality we have

$$\sum_r \int_{C_0} |V(Du_r) - V(Dw_r)| + |Du_r - Dw_r|^{(2+\epsilon)/2} dx_r \leq c_n(n,\epsilon). \tag{4.13}$$

Using (4.11), (4.13) and (4.12), and finally Sobolev's embedding theorem, we have

$$\int_{B_1} \sum_r |V(Du_r) - V(Dw_r)| + |Du_r - Dw_r|^{(2+\epsilon)/2} dx_r$$

$$= \int_{C_0} \sum_r |V(Du_r) - V(Dw_r)| + |Du_r - Dw_r|^{(2+\epsilon)/2} dx_r$$

$$+ \sum_{k=1}^{\infty} \int_{C_k} \sum_r |V(Du_r) - V(Dw_r)| + |Du_r - Dw_r|^{(2+\epsilon)/2} dx_r$$

$$\leq c_n + c_n \sum_{k=1}^{\infty} \frac{1}{k^{\frac{1+\epsilon}{2}}} \sum_r \left(\int_{C_k} |u_r - w_r|^{1+\epsilon} dx_r \right)^{\frac{1}{2}}$$

$$\leq c_n + c_n \left[\sum_{k=1}^{\infty} \frac{1}{k^{1+\epsilon}} \right]^{\frac{1}{2}} \sum_r \left(\int_{B_1} |u_r - w_r|^{1+\epsilon} dx_r \right)^{\frac{1}{2}}$$

$$\leq c_n + c_n \left[\sum_{k=1}^{\infty} \frac{1}{k^{1+\epsilon}} \right]^{\frac{1}{2}} \sum_r \left(\int_{B_1} |u_r - w_r|^{(2+\epsilon)/2} dx_r \right)^{\frac{1+\epsilon}{2+\epsilon}}. \tag{4.14}$$

The constant c_n in the last line also depends on $1+\epsilon$. Observe now that by (4.12) it follows $\frac{1+\epsilon}{2+\epsilon} < 1$ and therefore applying Young's inequality in (4.14) yields

$$\sum_r \int_{B_1} |V(Du_r) - V(Dw_r)| + |Du_r - Dw_r|^{(2+\epsilon)/2} dx_r \leq c_n(n,\epsilon).$$

from which (4.6) follows. The proof is complete by making a suitable choice of $1+\epsilon$ in (4.12).

We now switch to the subquadratic case $\epsilon \leq 0$, which involves a more delicate argument, and a scaling procedure with some non-standard quantities reflecting the behavior of $(2+\epsilon)$-Laplacian type operators for $\epsilon \leq 0$.

Lemma 4.2. *Under the assumptions of Theorem* 1.2, *let* $w_r \in u_r + W_0^{1,2+\epsilon}(B_{2R})$ *be as in* (4.1); *assume that* $\epsilon < \frac{1}{n}$. *Then the following inequality holds for a constant* $c_n \equiv c_n(n,\epsilon)$:

$$\sum_r \fint_{B_{2R}} |Du_r - Dw_r| dx_r \le c_n \sum_r \left[\frac{|\mu_r|(B_{2R})}{R^{n-1}}\right]^{\frac{1}{1+\epsilon}} + c_n \sum_r \left[\frac{|\mu_r|(B_{2R})}{R^{n-1}}\right] \left(\fint_{B_{2R}} (|Du_r| + 1 + \epsilon) dx_r\right)^{-\epsilon}. \quad (4.15)$$

Proof. As in the proof of Lemma 4.1 we start by a preliminary reduction appealing to Remark 4.1. In this case we set

$$A := c_n \sum_r \left[\frac{|\mu_r|(B_{2R})}{R^{n-1}}\right]^{\frac{1}{1+\epsilon}} + \sum_r \left[\frac{|\mu_r|(B_{2R})}{R^{n-1}}\right] \left(\fint_{B_{2R}} (|Du_r| + 1 + \epsilon) dx_r\right)^{-\epsilon}.$$

Scaling as in Remark 4.1, and in particular using the notation established in (4.2), we observe that

$$\sum_r [|\tilde{\mu}_r|(B_1)]^{\frac{1}{1+\epsilon}} + \sum_r [|\tilde{\mu}_r|(B_1)] \left(\fint_{B_1} (|D\tilde{u}_r| + (1+\epsilon)/A) dy_r\right)^{-\epsilon}$$

$$= \frac{c_n(n,2+\epsilon)}{A} \sum_r \left[\frac{|\mu_r|(B_{2R})}{R^{n-1}}\right]^{\frac{1}{1+\epsilon}} + \frac{c_n(n,2+\epsilon)}{A} \sum_r \left[\frac{|\mu_r|(B_{2R})}{R^{n-1}}\right] \left(\fint_{B_{2R}} (|Du_r| + 1 + \epsilon) dx_r\right)^{-\epsilon}$$

$$\le c_n(n,\epsilon).$$

Therefore, up to scaling as in Remark 4.1, we may reduce the proof to the case in which $B_{2R} \equiv B_1$ and

$$\sum_r |\mu_r|(B_1) + \sum_r [|\mu_r|(B_1)] \left(\int_{B_1} (|Du_r| + 1 + \epsilon) dy_r\right)^{-\epsilon} \le c_n \quad (4.16)$$

holds for a constant c_n depending on n and ϵ, thereby ultimately reducing ourselves to prove that

$$\sum_r \fint_{B_1} |Du_r - Dw_r| dx_r \le c_n \quad (4.17)$$

in turn holds for a constant c_n depending on n, ϵ.

We start observing that the assumed lower bound $\epsilon < 1/n$ allows to determine $\gamma \in (0,1)$ such that $\epsilon < \gamma/n$ and therefore

$$\frac{n(1+\epsilon)}{n-\gamma} > 1. \quad (4.18)$$

As for the proof of Lemma 4.1, we obtain

$$\int_{C_k} \sum_r |V(Du_r) - V(Dw_r)|^2 dx_r \le c_n(n,\epsilon) \sum_r [|\mu_r|(B_1)], \quad (4.19)$$

where C_k is defined as in (4.10). For every integer $k > 0$ we have

$$\int_{C_k} \sum_r |V(Du_r) - V(Dw_r)|^{2/(2+\epsilon)} dx_r \le c_n |C_k|^{\frac{1+\epsilon}{2+\epsilon}} \left(\int_{C_k} \sum_r |V(Du_r) - V(Dw_r)|^2 dx_r\right)^{\frac{1}{2+\epsilon}}$$

$$\le c_n |C_k|^{\frac{1+\epsilon}{2+\epsilon}} \sum_r [|\mu_r|(B_1)]^{\frac{1}{2+\epsilon}}$$

$$\leq \frac{c_n}{k^{\frac{n(1+\epsilon)}{(2+\epsilon)(n-\gamma)}}} \sum_r \left(\int_{C_k} |u_r - w_r|^{\frac{n}{n-\gamma}} dx_r \right)^{\frac{1+\epsilon}{2+\epsilon}} [|\mu_r|(B_1)]^{\frac{1}{2+\epsilon}} \qquad (4.20)$$

and, again by Hölder's inequality

$$\sum_r \int_{C_0} |V(Du_r) - V(Dw_r)|^{2/(2+\epsilon)} dx_r \leq c_n(n,\epsilon) \sum_r [|\mu_r|(B_1)]^{\frac{1}{2+\epsilon}}.$$

Therefore, keeping (4.18) in mind, we have

$$\sum_r \int_{B_1} |V(Du_r) - V(Dw_r)|^{\frac{2}{2+\epsilon}} dx_r$$

$$= \sum_r \int_{C_0} |V(Du_r) - V(Dw_r)|^{2/(2+\epsilon)} dx_r + \sum_{k=1}^{\infty} \int_{C_k} \sum_r |V(Du_r) - V(Dw_r)|^{2/(2+\epsilon)} dx_r$$

$$\leq c_n [|\mu_r|(B_1)]^{\frac{1}{2+\epsilon}} + c_n \sum_{k=1}^{\infty} \frac{1}{k^{\frac{n(1+\epsilon)}{(2+\epsilon)(n-\gamma)}}} \left[\sum_r \left(\int_{C_k} |u_r - w_r|^{\frac{n}{n-\gamma}} dx_r \right)^{\frac{1+\epsilon}{2+\epsilon}} [|\mu_r|(B_1)]^{\frac{1}{2+\epsilon}} \right]$$

$$\leq c_n \sum_r [|\mu_r|(B_1)]^{\frac{1}{2+\epsilon}} + c_n \left[\sum_{k=1}^{\infty} \frac{1}{k^{\frac{n(1+\epsilon)}{(2+\epsilon)(n-\gamma)}}} \right]^{\frac{1}{2+\epsilon}} \sum_r \left[\left(\sum_{k=1}^{\infty} \int_{C_k} |u_r - w_r|^{\frac{n}{n-\gamma}} dx_r \right)^{\frac{1+\epsilon}{2+\epsilon}} [|\mu_r|(B_1)]^{\frac{1}{2+\epsilon}} \right]$$

$$\leq c_n \sum_r [|\mu_r|(B_1)]^{\frac{1}{2+\epsilon}} + c_n \left[\sum_{k=1}^{\infty} \frac{1}{k^{\frac{n(1+\epsilon)}{(2+\epsilon)(n-\gamma)}}} \right]^{\frac{1}{2+\epsilon}} \sum_r \left[\left(\int_{B_1} |u_r - w_r|^{\frac{n}{n-\gamma}} dx_r \right)^{\frac{1+\epsilon}{2+\epsilon}} [|\mu_r|(B_1)]^{\frac{1}{2+\epsilon}} \right]$$

$$\leq c_n \sum_r [|\mu_r|(B_1)]^{\frac{1}{2+\epsilon}} + c_n \sum_r \left[\left(\int_{B_1} |Du_r - Dw_r| dx_r \right)^{\frac{n(1+\epsilon)}{(2+\epsilon)(n-\gamma)}} [|\mu_r|(B_1)]^{\frac{1}{2+\epsilon}} \right]. \qquad (4.21)$$

In (4.21) the constant obviously depends on γ too. In turn, let us write

$$\sum_r |Du_r - Dw_r| = \sum_r \left[(|Du_r|^2 + |Dw_r|^2 + (1+\epsilon)^2)^{\frac{\epsilon}{2}} |Du_r - Dw_r|^2 \right]^{\frac{1}{2}} \cdot (|Du_r|^2 + |Dw_r|^2 + (1+\epsilon)^2)^{\frac{-\epsilon}{4}}$$

$$\leq c_n \sum_r |V(Du_r) - V(Dw_r)| \cdot (|Du_r|^2 + |Dw_r|^2 + (1+\epsilon)^2)^{\frac{-\epsilon}{4}}$$

$$\leq c_n \sum_r |V(Du_r) - V(Dw_r)| \cdot \left[|Du_r - Dw_r|^{\frac{-\epsilon}{2}} + |Du_r|^{\frac{-\epsilon}{2}} + (1+\epsilon)^{\frac{-\epsilon}{2}} \right], \qquad (4.22)$$

where in the second-last line we used (2.2). Therefore, when $\epsilon \neq 0$, using Young's in the form

$$ab^{\frac{-\epsilon}{2}} \leq \frac{(2+\epsilon)\, \epsilon^{\epsilon/(2+\epsilon)}\, a^{2/(2+\epsilon)}}{2} + \frac{-\epsilon^2 b}{2}, \qquad \epsilon \in (0,1)$$

we again

$$\sum_r |Du_r - Dw_r| \leq c_n \sum_r |V(Du_r) - V(Dw_r)|^{\frac{2}{2+\epsilon}} + \left(\frac{1}{2}\right) \sum_r |Du_r - Dw_r|$$

$$+c_n\sum_r |V(Du_r)-V(Dw_r)|(|Du_r|+1+\epsilon)^{\frac{-\epsilon}{2}} \qquad (4.23)$$

and therefore

$$\sum_r |Du_r-Dw_r| \le c_n\sum_r |V(Du_r)-V(Dw_r)|^{\frac{2}{2+\epsilon}}+c_n\sum_r |V(Du_r)-V(Dw_r)|(|Du_r|+1+\epsilon)^{\frac{-\epsilon}{2}}$$

By using this last estimate together with Hölder's inequality we get

$$\int_{B_1}\sum_r |Du_r-Dw_r|\,dx_r \le c_n\int_{B_1}\sum_r |V(Du_r)-V(Dw_r)|^{2/(2+\epsilon)}\,dx_r$$

$$+c_n\sum_r\left(\int_{B_1}|V(Du_r)-V(Dw_r)|^{\frac{2}{2+\epsilon}}dx_r\right)^{(2+\epsilon)/2}\left(\int_{B_1}(|Du_r|+1+\epsilon)dx_r\right)^{-\epsilon/2}. \qquad (4.24)$$

In turn, combining (4.24) with (4.21) yields

$$\int_{B_1}\sum_r |Du_r-Dw_r|\,dx_r \le c_n\sum_r [|\mu_r|(B_1)]^{\frac{1}{2+\epsilon}}+c_n\sum_r\left(\int_{\boldsymbol{B_1}}|Du_r-Dw_r|dx_r\right)^{\frac{n(1+\epsilon)}{(2+\epsilon)(n-\gamma)}}[|\mu_r|(B_1)]^{\frac{1}{2+\epsilon}}$$

$$+c_n\left\{\sum_r [|\mu_r|(B_1)]\left(\int_{B_1}(|Du_r|+1+\epsilon)dx_r\right)^{\frac{-\epsilon}{2}}\right\}^{\frac{1}{2}}$$

$$+c_n\sum_r\left\{[|\mu_r|(B_1)]\left(\int_{B_1}(|Du_r|+1+\epsilon)dx_r\right)^{\frac{-\epsilon}{2}}\right\}^{1/2}\left(\int_{\boldsymbol{B_1}}|Du_r-Dw_r|dx_r\right)^{\frac{n(1+\epsilon)}{2(n-\gamma)}}$$

$$\le c_n+c_n\sum_r\left(\int_{\boldsymbol{B_1}}|Du_r-Dw_r|dx_r\right)^{\frac{n(1+\epsilon)}{(2+\epsilon)(n-\gamma)}}\left(\int_{\boldsymbol{B_1}}|Du_r-Dw_r|dx_r\right)^{\frac{n(1+\epsilon)}{2(n-\gamma)}} \qquad (4.25)$$

and, keeping in mind (4.16) and the fact that $\epsilon \le 0$, ultimately

$$\sum_r\int_{B_1}|Du_r-Dw_r|dx_r \le c_n+c_n\sum_r\left(\int_{\boldsymbol{B_1}}|Du_r-Dw_r|dx_r\right)^{\frac{n(1+\epsilon)}{(2+\epsilon)(n-\gamma)}}. \qquad (4.26)$$

Now observe that since $\epsilon \ge 2-n$ we have

$$\frac{n(1+\epsilon)}{(2+\epsilon)(n-\gamma)} < \frac{n(1+\epsilon)}{(2+\epsilon)(n-1)} \le 1 \qquad (4.27)$$

so that (4.17) follows from (4.26) applying Young's inequality. The proof is complete.

Remark 4.2 (*Stabilization of the consants II*). The dependence of the constants in (4.2) is stable in the sense that letting $\epsilon \nearrow 0$ in (4.2) we obtain the estimate

$$⨍_{B_{2R}}|Du_r-Dw_r|dx_r \le \frac{c_n|\mu_r|(B_{2R})}{R^{n-1}},$$

and in fact the proof is such that the previous estimate can be obtained taking $\epsilon = 0$. The stability of the constant follows in particular by the use of Young's inequality in (4.23) and (4.26).

We prove a decay estimate for solutions to (1.1) which is obtained using the comparison estimates of the previous section.

With w_r been defined in (4.1) – and keeping the ball $B_{2R} \subset \Omega_r$ fixed as specified at the beginning of the section – we define $v_r \in w_r + W_0^{1,2+\epsilon}(B_R)$as the unique solution to the homogeneous Dirichlet problem

$$\begin{cases} \nabla a(x_0, Dv_r) = 0 & \text{in } B_R, \\ v_r = w_r & \text{on } \partial B_R, \end{cases} \tag{4.28}$$

And prove yet another comparison estimate. We remark that B_R is concentric to B_{2R}. This time we start by the case $\epsilon < 0$.

Lemma 4.3. *Under the assumptions of Theorem* 1.1, *with w_r as in* (4.1) and v_r as in (4.28), *there exists a constant $c_n \equiv c_n(n, \epsilon)$ such that the following inequality holds*:

$$\sum_r \fint_{B_R} |Du_r - Dv_r| dx_r \le c_n \sum_r \left[\frac{|\mu_r|(B_{2R})}{R^{n-1}}\right]^{\frac{1}{1+\epsilon}} + c_n \sum_r \left[\frac{|\mu_r|(B_{2R})}{R^{n-1}}\right] \left(\fint_{B_{2R}} (|Du_r| + 1 + \epsilon) dx_r\right)^{-\epsilon}$$
$$+ c_n \omega(R) \sum_r \fint_{B_{2R}} (|Du_r| + 1 + \epsilon) dx_r. \tag{4.29}$$

Proof. We start proving that the inequality

$$\sum_r \fint_{B_R} |Dw_r - Dv_r| dx_r \le c_n \omega(R) \sum_r \fint_{B_{2R}} (|Dw_r| + 1 + \epsilon) dx_r \tag{4.30}$$

holds for a constant c_n depending on n, ϵ. Indeed by [13] and assumptions (1.2) we have that v_r is a Q_n-minimizer of the functional

$$z_r \in W^{1,2+\epsilon}(B_R) \mapsto \int_{B_R} (|Dz_r| + 1 + \epsilon)^{2+\epsilon} dx_r \tag{4.31}$$

for some $Q_n \equiv Q_n(n, \epsilon) \ge 1$, and therefore

$$\sum_r \int_{B_R} |Dv_r|^{2+\epsilon} dx_r \le Q_n \sum_r \int_{B_R} (|Dw_r| + 1 + \epsilon)^{2+\epsilon} dx_r. \tag{4.32}$$

Moreover, a well-known version of Gehring's lemma applies to w_r here – see for instance the version presented in [13] – and leads to find a constant $\chi_0 \equiv \chi_0(n, \epsilon) > 1$ such that the reverse Hölder type inequality

$$\left(\sum_r \fint_{B_{\rho/2}} (|Dw_r| + 1 + \epsilon)^{\chi_0(2+\epsilon)} dx_r\right)^{1/\chi_0} \le c_n \sum_r \fint_{B_\rho} (|Dw_r| + 1 + \epsilon)^{2+\epsilon} dx_r$$

holds whenever $B_\rho \subseteq B_{2R}$ (this time not necessarily concentric to B_R) for a constant c_n depending on n, ϵ. In turn, applying Lemma 3.2 with $g^r \equiv (|Dw_r| + 1 + \epsilon)^{2+\epsilon}$, leads to establish that also

$$\left(\sum_r \int_{B_R} (|Dw_r| + 1 + \epsilon)^{2+\epsilon} dx_r\right)^{1/(2+\epsilon)} \le c_n \sum_r \int_{B_{2R}} (|Dw_r| + 1 + \epsilon) dx_r \tag{4.33}$$

holds. Now using (2.2) and eventually (2.3), the fact that both v_r and w_r are solutions, $(1.2)_3$ and again Young's inequality, we have

$$\sum_r \int_{B_R} (|Dv_r|^2 + |Dw_r|^2 + (1+\epsilon)^2)^{\epsilon/2} |Dw_r - Dv_r|^2 dx_r$$

$$\leq c_n \sum_r \int_{B_R} |V(Dw_r) - V(Dv_r)|^2 dx_r$$

$$\leq c_n \sum_r \int_{B_R} \langle a(x_0, Dw_r) - a(x_0, Dv_r), Dw_r - Dv_r \rangle dx_r$$

$$= c_n \sum_r \int_{B_R} \langle a(x_0, Dw_r) - a(x_r, Dv_r), Dw_r - Dv_r \rangle dx_r$$

$$\leq c_n (1+\epsilon)\omega(R) \sum_r \int_{B_R} (|Dw_r|^2 + (1+\epsilon)^2)^{(1+\epsilon)/2} |Dw_r - Dv_r| dx_r$$

$$\leq c_n (1+\epsilon)\omega(R) \sum_r \int_{B_R} (|Dv_r|^2 + |Dw_r|^2 + (1+\epsilon)^2)^{(1+\epsilon)/2} |Dw_r - Dv_r| dx_r$$

$$\leq \frac{1}{2} \sum_r \int_{B_R} (|Dv_r|^2 + |Dw_r|^2 + (1+\epsilon)^2)^{\frac{\epsilon}{2}} |Dw_r - Dv_r|^2 dx_r$$

$$+ c_n [(1+\epsilon)\omega(R)]^2 \sum_r \int_{B_R} (|Dv_r|^2 + |Dw_r|^2 + (1+\epsilon)^2)^{(2+\epsilon)/2} dx_r$$

Therefore, using again (2.2) we obtain

$$\sum_r \int_{B_R} |V(Dw_r) - V(Dv_r)|^2 dx_r \leq c_n [(\omega(R)]^2 \sum_r \int_{B_R} (|Dv_r|^2 + |Dw_r|^2 + (1+\epsilon)^2)^{(2+\epsilon)/2} dx_r$$

and by (4.32) also

$$\sum_r \int_{B_R} |V(Dw_r) - V(Dv_r)|^2 dx_r \leq c_n [(\omega(R)]^2 \sum_r \int_{B_R} (|Dw_r| + 1 + \epsilon)^{2+\epsilon} dx_r \tag{4.34}$$

for $c_n \equiv c_n(n, \epsilon)$. Similarly to (4.22) we now have

$$\sum_r |Dw_r - Dv_r|^{2+\epsilon} \leq c_n \sum_r |V(Dw_r) - V(Dv_r)|^{2+\epsilon} (|Dv_r|^2 + |Dw_r|^2 + (1+\epsilon)^2)^{-\epsilon(2+\epsilon)/4}$$

and therefore using the last estimate, (4.32) and Hölder's inequality in (4.34) yields

$$\sum_r \fint_{B_R} |Dw_r - Dv_r|^{2+\epsilon} dx_r$$

$$\leq c_n \sum_r \left(\fint_{B_R} |V(Dw_r) - V(Dv_r)|^2 dx_r \right)^{\frac{2+\epsilon}{2}} \left(\fint_{B_R} (|Dv_r|^2 + |Dw_r|^2 + (1+\epsilon)^2)^{(2+\epsilon)/2} dx_r \right)^{\frac{-\epsilon}{2}}$$

$$\leq c_n [(\omega(R)]^{2+\epsilon} \sum_r \fint_{B_R} (|Dw_r| + 1 + \epsilon)^{2+\epsilon} dx_r. \tag{4.35}$$

In turn, using first Hölder's inequality, (4.35) and finally (4.33) we have

$$\sum_r \fint_{B_R} |Dw_r - Dv_r| dx_r \leq c_n \sum_r \left(\fint_{B_R} |Dw_r - Dv_r|^{2+\epsilon} dx_r \right)^{1/(2+\epsilon)}$$

$$\leq c_n\omega(R)\sum_r\left(\fint_{B_R}(|Dw_r|+1+\epsilon)^{2+\epsilon}dx_r\right)^{1/(2+\epsilon)}$$

$$\leq c_n\omega(R)\sum_r\fint_{B_{2R}}(|Dw_r|+1+\epsilon)dx_r,$$

so that the proof of (4.30) follows. Using (4.30) together with (4.15) we have

$$\sum_r\fint_{B_R}|Du_r-Dv_r|dx_r\ \leq c_n\sum_r\left[\frac{|\mu_r|(B_{2R})}{R^{n-1}}\right]^{\frac{1}{1+\epsilon}}+c_n\sum_r\left[\frac{|\mu_r|(B_{2R})}{R^{n-1}}\right]\left(\fint_{B_{2R}}(|Du_r|+1+\epsilon)dx_r\right)^{-\epsilon}$$

$$+c_n\omega(R)\sum_r\fint_{B_{2R}}(|Dw_r|+1+\epsilon)dx_r$$

and using again (4.15) to estimate the last integral in the previous inequality (and recalling that $\omega(R)\leq 1$) we finally conclude with (4.29).

We proceed with the case $\epsilon\leq 0$.

Lemma 4.4. *Let $u_r\in C^1(\Omega_r)$ be as in Theorem 1.1, then there exist constants $\beta\in(0,1]$, $c_{n+1}\geq 1$ depending on n,ϵ such that the following estimate holds whenever $B_\rho\subseteq B_R\subseteq B_{2R}\subseteq\Omega_r$ are concentric balls:*

$$\sum_r\fint_{B_\rho}\left|Du_r-(Du_r)_{B_\rho}\right|dx_r\ \leq\ c_{n+1}\left(\frac{\rho}{R}\right)^\beta\sum_r\fint_{B_{2R}}|Du_r-(Du_r)_{B_{2R}}|dx_r+c_n\sum_r\left(\frac{R}{\rho}\right)^n\left[\frac{|\mu_r|(B_{2R})}{R^{n-1}}\right]^{\frac{1}{1+\epsilon}}$$

$$+c_n\left(\frac{R}{\rho}\right)^n\sum_r\left[\frac{|\mu_r|(B_{2R})}{R^{n-1}}\right]\left(\fint_{B_{2R}}(|Du_r|+1+\epsilon)dx_r\right)^{-\epsilon}$$

$$+c_n\left(\frac{R}{\rho}\right)^n\sum_r\omega(R)\fint_{B_{2R}}(|Du_r|+1+\epsilon)dx_r. \tag{4.36}$$

Proof. We report the simple proof for the sake of completeness. Starting by B_{2R} we define the comparison functions v_r and w_r as in (4.28) and (4.1), respectively. Then we compare Du_r and Dv_r by mean of (4.29), using (3.1) as basic reference estimate for v_r, that we eventually transfer to u_r:

$$\sum_r\fint_{B_\rho}\left|Du_r-(Du_r)_{B_\rho}\right|dx_r\ \leq 2\sum_r\fint_{B_\rho}\left|Du_r-(Dv_r)_{B_\rho}\right|dx_r$$

$$\leq 2\sum_r\fint_{B_\rho}\left|Dv_r-(Dv_r)_{B_\rho}\right|dx_r+2\sum_r\fint_{B_\rho}|Du_r-Dv_r|dx_r$$

$$\leq c_n\left(\frac{\rho}{R}\right)^\beta\sum_r\fint_{B_R}|Dv_r-(Dv_r)_{B_R}|dx_r+c_n\left(\frac{R}{\rho}\right)^n\sum_r\fint_{B_R}|Du_r-Dv_r|dx_r$$

$$\leq c_n\left(\frac{\rho}{R}\right)^\beta\sum_r\fint_{B_{2R}}|Du_r-(Du_r)_{B_{2R}}|dx_r+c_n\left(\frac{R}{\rho}\right)^n\sum_r\fint_{B_R}|Du_r-Dv_r|dx_r$$

In order to get (4.36) it is now sufficient to estimate the last integral in the previous inequality by means of (4.29).

We now give the suitable version of the last two lemmata in the case $\epsilon\geq 0$; this involves the use of the $V(\cdot)$- map.

Lemma 4.5. *Under the assumptions of the Theorem* 1.2, *with* v_r *as in* (4.28) *and* w_r *as in* (4.1), *there exists a constant* $c_n \equiv c_n(n,\epsilon)$ *such that the following inequality holds:*

$$\sum_r \fint_{B_R} |V(Du_r) - V(Dv_r)| dx_r$$

$$\leq c_n \sum_r \left[\frac{|\mu_r|(B_{2R})}{R^{n-1}}\right]^{\frac{2+\epsilon}{2(1+\epsilon)}} + c_n\omega(R) \sum_r \fint_{B_{2R}} \left(|V(Du_r)| + (1+\epsilon)^{(2+\epsilon)/2}\right) dx_r.$$

Proof. The proof is a modification of that of Lemma 4.3. We restart from (4.34) – that holds for $\epsilon \geq 0$ as well – then, using Hölder's inequality we have

$$\sum_r \fint_{B_R} |V(Du_r) - V(Dv_r)| dx_r \leq c_n \sum_r \left(\fint_{B_R} |V(Du_r) - V(Dv_r)|^2 dx_r\right)^{1/2}$$

$$\leq c_n\omega(R) \sum_r \left(\fint_{B_R} (|Dw_r| + 1 + \epsilon)^{2+\epsilon} dx_r\right)^{1/2}$$

Applying lemma 3.2 with $g^r \equiv (|Dw_r| + 1 + \epsilon)^{2+\epsilon}$,, leads to

$$\sum_r \left(\fint_{B_R} (|Dw_r| + 1 + \epsilon)^{2+\epsilon} dx_r\right)^{1/(2+\epsilon)} \leq c_n \sum_r \left(\fint_{B_{2R}} (|Dw_r| + 1 + \epsilon)^{(2+\epsilon)/2} dx_r\right)^{2/(2+\epsilon)}$$

holds. Combining the two last inequalities we obtain

$$\sum_r \fint_{B_R} |V(Dw_r) - V(Dv_r)| dx_r \leq c_n\omega(R) \sum_r \fint_{B_{2R}} (|Dw_r| + 1 + \epsilon)^{(2+\epsilon)/2} dx_r. \tag{4.37}$$

In turn, using Young's inequality we observe that when $\epsilon \geq 0$ it holds that

$$|z_r|^{(2+\epsilon)/2} \leq |V(z_r)| \tag{4.38}$$

and therefore (4.3) yields

$$\sum_r \fint_{B_{2R}} |Dw_r|^{(2+\epsilon)/2} dx_r \leq \sum_r \fint_{B_{2R}} |V(Dw_r) - V(Du_r)| dx_r + \sum_r \fint_{B_{2R}} |V(Du_r)| dx_r$$

$$\leq c_n \sum_r \left[\frac{|\mu_r|(B_{2R})}{R^{n-1}}\right]^{\frac{2+\epsilon}{2(1+\epsilon)}} + \sum_r \fint_{B_{2R}} |V(Du_r)| dx_r.$$

Combining the last estimate with (4.37) yields

$$\sum_r \fint_{B_{2R}} |V(Dw_r) - V(Dv_r)| dx_r$$

$$\leq c_n \sum_r \left[\frac{|\mu_r|(B_{2R})}{R^{n-1}}\right]^{\frac{2+\epsilon}{2(1+\epsilon)}} + c_n\omega(R) \sum_r \fint_{B_{2R}} \left(|V(Du_r)| + (1+\epsilon)^{(2+\epsilon)/2}\right) dx_r,$$

and the proof is complete.

The next lemma can be now obtained as Lemma 4.4 using Lemma 4.5 in place of Lemma 4.3, and the decay estimate (3.17) in place of (3.19).

Lemma 4.6. *Let* $u_r \in C^1(\Omega_r)$ *be as in Theorem* 1.2, *then there exist constants* $\beta \in (0,1]$, $c_{n+1} \geq 1$ *depending on depending on* $n\epsilon$ *such that the following estimate holds whenever* $B_\rho \subseteq B_R \subseteq B_{2R} \subseteq \Omega_r$ *are concentric balls:*

$$\sum_r \fint_{B_\rho} \left|V(Du_r) - (V(Du_r))_{B_\rho}\right| dx_r \leq c_{n+1} \left(\frac{\rho}{R}\right)^\beta \sum_r \fint_{B_{2R}} \left|V(Du_r) - (V(Du_r))_{B_{2R}}\right| dx_r$$

$$+c_n \left(\frac{R}{\rho}\right)^n \sum_r \left[\frac{|\mu_r|(B_{2R})}{R^{n-1}}\right]^{\frac{1}{1+\epsilon}} + c_n \left(\frac{R}{\rho}\right)^n \omega(R) \sum_r \fint_{B_{2R}} \left(|V(Du_r)| + (1+\epsilon)^{(2+\epsilon)/2}\right) dx_r. \qquad (4.39)$$

5. Proof of Theorem 1.1

In the rest of the proof all the balls will be concentric and centered at the point $x_r \in \Omega_r$ identified by the statement of the theorem; all of them will be contained in Ω_r. In particular we start with a ball $B(x_r, 2R) \equiv B_{2R} \subset \Omega_r$ as in the statement of the Theorem. All the radii R will be such that $R \leq \tilde{R}$ where the quantity $\tilde{R} > 0$ will be chosen along the proof in dependence of the data $n, \epsilon, \omega(\cdot)$. The main point of the proof is to show how the peculiar quantity (1.20) appearing in the right-hand side of (4.15) can be *reabsorbed in a way that make the Riesz potential appear*, along the iteration/summation procedure.

Step 1: *A preliminary estimate*. Referring to estimate (4.36), we select an integer $H^r \equiv H^r(n, \epsilon) \geq 1$ large enough to have

$$c_n \left(\frac{1}{H^r}\right)^\beta \leq \frac{1}{4}. \qquad (5.1)$$

Applying (4.36) on arbitrary balls $B_\rho \equiv B_{R/2H^r} \subseteq B_{R/2} \subset B_R$ and using the fact that $\omega(\cdot)$ in non-decreasing we gain

$$\sum_r \fint_{B_{R/2H^r}} \left|Du_r - (Du_r)_{B_{R/2H^r}}\right| dy_r \leq \frac{1}{4} \sum_r \fint_{B_R} \left|Du_r - (Du_r)_{B_R}\right| dy_r + c_{n+2} \sum_r \left[\frac{|\mu_r|(B_R)}{R^{n-1}}\right]^{\frac{1}{1+\epsilon}}$$

$$+ c_{n+2} \sum_r \left[\frac{|\mu_r|(B_R)}{R^{n-1}}\right] \left(\fint_{B_R} (|Du_r| + 1 + \epsilon) dy_r\right)^{-\epsilon} + c_{n+2}\omega(R) \sum_r \fint_{B_R} (|Du_r| + 1 + \epsilon) dy_r, \qquad (5.2)$$

where c_{n+2} depends on n, ϵ, H^r and therefore ultimately on n, ϵ. We reduce the value of $\tilde{R}$- in a way that makes it depending on n, ϵ and $\omega(\cdot)$ - to get

$$c_{n+2}\omega(\tilde{R}) \leq \frac{1}{4}, \qquad (5.3)$$

and using some further elementary estimates we gain

$$\sum_r \fint_{B_{R/2H^r}} \left|Du_r - (Du_r)_{B_{R/2H^r}}\right| dy_r \leq \frac{1}{2} \sum_r \fint_{B_R} \left|Du_r - (Du_r)_{B_R}\right| dy_r + c_{n+2} \sum_r \left[\frac{|\mu_r|(B_R)}{R^{n-1}}\right]^{\frac{1}{1+\epsilon}}$$

$$+c_{n+2} \sum_r \left[\frac{|\mu_r|(B_R)}{R^{n-1}}\right] \left(\fint_{B_R} (|Du_r| + 1 + \epsilon) dy_r\right)^{-\epsilon} + c_{n+2}\omega(R) \sum_r \left(\left|(Du_r)_{B_R}\right| + 1 + \epsilon\right). \qquad (5.4)$$

With the ball $B(x_r, 2R) \subseteq \Omega_r$ being fixed at the beginning of statement of Theorem 1.2, for $i \in \{0,1,2,\cdots\}$, let us define

$$B_i := B\left(x_r, R/(2H^r)^i\right) =: B(x_r, R_i) \quad \text{and} \quad k_i := \left|(Du_r)_{B_i}\right|, \qquad (5.5)$$

and

$$A_i := \fint_{B_i} \left|Du_r - (Du_r)_{B_i}\right| dy_r. \qquad (5.6)$$

For every integer $m \in \mathbb{N}$ we define and estimate

$$k_{m+1} = \sum_{i=0}^{m} (k_{i+1} - k_i) + k_0 \leq \sum_{i=0}^{m} \fint_{B_{i+1}} |Du_r - (Du_r)_{B_i}| dy_r + k_0 \leq (2H^r)^n \sum_{i=0}^{m} A_i + k_0. \tag{5.7}$$

To estimate the right-hand side in (5.7) we observe that (5.4) used with $R \equiv R_{i-1}$ yields, when ever $i \geq 1$

$$A_i \leq \frac{1}{2} A_{i-1} + c_{n+2} \sum_r \left[\frac{|\mu_r|(B_{i-1})}{R_{i-1}^{n-1}}\right]^{\frac{1}{1+\epsilon}} + c_{n+2} \sum_r \left[\frac{|\mu_r|((B_{i-1}))}{R_{i-1}^{n-1}}\right] \left(\fint_{B_{i-1}} (|Du_r| + 1 + \epsilon) dy_r\right)^{-\epsilon}$$
$$+ c_{n+2} \omega(R_{i-1})(k_{i-1} + 1 + \epsilon).$$

Summing up over $i \in \{1, \cdots, m\}$ the previous inequality yields

$$\sum_{i=1}^{m} A_i \leq \frac{1}{2} \sum_{i=0}^{m-1} A_i + c_{n+2} \sum_r \sum_{i=0}^{m-1} \left[\frac{|\mu_r|(B_i)}{R_i^{n-1}}\right]^{\frac{1}{1+\epsilon}} + c_{n+2} \sum_r \sum_{i=0}^{m-1} \left[\frac{|\mu_r|((B_i))}{R_i^{n-1}}\right] \left(\fint_{B_i} (|Du_r| + 1 + \epsilon) dy_r\right)^{-\epsilon}$$
$$+ c_{n+2} \sum_{i=0}^{m-1} \omega(R_i)(k_i + 1 + \epsilon),$$

and therefore

$$\sum_{i=1}^{m} A_i \leq A_0 + 2c_{n+2} \sum_r \sum_{i=0}^{m-1} \left[\frac{|\mu_r|(B_i)}{R_i^{n-1}}\right]^{\frac{1}{1+\epsilon}} + 2c_{n+2} \sum_r \sum_{i=0}^{m-1} \left[\frac{|\mu_r|((B_i))}{R_i^{n-1}}\right] \left(\fint_{B_i} (|Du_r| + 1 + \epsilon) dy_r\right)^{-\epsilon}$$
$$+2c_{n+2} \sum_{i=0}^{m-1} \omega(R_i)(k_i + 1 + \epsilon). \tag{5.8}$$

Using the last inequality in (5.7) yields, for every integer $m \geq 1$

$$k_{m+1} \leq c_n A_0 + c_n k_0 + c_n \sum_r \sum_{i=0}^{m-1} \left[\frac{|\mu_r|(B_i)}{R_i^{n-1}}\right]^{\frac{1}{1+\epsilon}} + c_n \sum_r \sum_{i=0}^{m-1} \left[\frac{|\mu_r|((B_i))}{R_i^{n-1}}\right] \left(\fint_{B_i} (|Du_r| + 1 + \epsilon) dy_r\right)^{-\epsilon}$$
$$+c_n \sum_{i=0}^{m-1} \omega(R_i)(k_i + 1 + \epsilon). \tag{5.9}$$

and the constant c_n depends on n, ϵ - keep in mind the dependence of H^r.

Step 2: A conditional estimate. This step is dedicated to the proof of an estimate that holds provided in turn a certain pointwise bound holds as well, and the radius $\tilde{R}$ is further reduced. This is in the following:

Lemma 5.1. *Assume that there exists an integer $\tilde{m} \in \mathbb{N} \cup \{\infty\}$ such that $\tilde{m} \geq 1$ and*

$$\fint_{B_i} |Du_r| dy_r \leq |Du_r(x_r)| \qquad \textit{holds whenever} \qquad 0 \leq i \leq \tilde{m} - 1. \tag{5.10}$$

Then for every $\epsilon \in (0,1)$ there exists a constant $\tilde{c}_n \equiv \tilde{c}_n(\epsilon) \geq 1$ such that

$$k_m \leq 2c_{n+4} M + 2c_{n+3} \epsilon |Du_r(x_r)| \tag{5.11}$$

holds whenever $m \leq \tilde{m} + 1$ and provided $R \leq \tilde{R}$. Here $c_{n+3}, c_{n+4} \geq 1$ and $\tilde{R} > 0$ are constants depending on n, ϵ, while

$$M \equiv M(\epsilon) := \sum_r \fint_{B_R} (|Du_r| + 1 + \epsilon)\, dy_r + (1 + c_{n+3} \tilde{c}_n(\epsilon)) \sum_r \left[\mathbf{I}_1^{|\mu_r|}(x_r, 2R)\right]^{\frac{1}{1+\epsilon}}. \tag{1.12}$$

Proof. By (5.10), whenever $1 \le m \le \tilde{m}$ it holds that

$$k_{m+1} \le c_n \left(A_0 + k_0 + \sum_r \sum_{i=0}^{m-1} \left[\frac{|\mu_r|(B_i)}{R_i^{n-1}} \right]^{\frac{1}{1+\epsilon}} \right) + c_n \sum_r (|Du_r(x_r)|^{-\epsilon} + (1+\epsilon)^{-\epsilon}) \sum_{i=0}^{m-1} \left[\frac{|\mu_r|((B_i))}{R_i^{n-1}} \right]$$

$$+ c_n \sum_{i=0}^{m-1} \omega(R_i)(k_i + 1 + \epsilon). \qquad (5.13)$$

We now notice that

$$\sum_r \sum_{i=0}^{m-1} \frac{|\mu_r|(B_i)}{R_i^{n-1}} \le \sum_r \sum_{i=0}^{\infty} \frac{|\mu_r|(B_i)}{R_i^{n-1}}$$

$$\le \frac{2^{n-1}}{\log 2} \sum_r \int_R^{2R} \frac{|\mu_r|\big(B(x_0,\rho)\big)}{\rho^{n-1}} \frac{d\rho}{\rho} + \sum_r \frac{(2H^r)^{n-1}}{\log 2H^r} \sum_{i=0}^{\infty} \int_{R_{i+1}}^{R_i} \frac{|\mu_r|\big(B(x_0,\rho)\big)}{\rho^{n-1}} \frac{d\rho}{\rho}$$

$$\le \sum_r c_n(H^r) \mathbf{I}_1^{|\mu_r|}(x_r, 2R). \qquad (5.14)$$

Moreover, using the elementary inequality (1.17) with $\epsilon = 0$ - notice that $\epsilon \ge 0$ as we are here assuming that $\epsilon \le 0$- together with (5.14), we also have that

$$\sum_r \sum_{i=0}^{m-1} \left[\frac{|\mu_r|(B_i)}{R_i^{n-1}} \right]^{\frac{1}{1+\epsilon}} \le \sum_r \left[\sum_{i=0}^{\infty} \frac{|\mu_r|(B_i)}{R_i^{n-1}} \right]^{\frac{1}{1+\epsilon}} \le \sum_r c_n(H^r) \left[\mathbf{I}_1^{|\mu_r|}(x_r, 2R) \right]^{\frac{1}{1+\epsilon}}, \qquad (5.15)$$

holds. Finally, as $\omega(\cdot)$ is non-decreasing, we have

$$\sum_{i=0}^{m-1} \omega(R_i) \le \sum_{i=0}^{\infty} \omega(R_i) \le \sum_r c_n(H^r) \int_0^{2R} \omega(\rho) \frac{d\rho}{\rho} = \sum_r c_n(H^r) d(2R), \qquad (5.16)$$

where the quantity $d(\cdot)$ has been defined in (1.3). We now further reduce $\tilde{R}$ in order to have that $d(2\tilde{R}) \le 1$ so that

$$d(2R) \le 1. \qquad (5.17)$$

Moreover, we record the elementary estimates

$$A_0 + k_0 + d(2R)(1+\epsilon) \le 3 \sum_r \fint_{B_R} (|Du_r| + 1 + \epsilon)\, dy_r, \qquad (5.18)$$

$$k_1 \le 2^n \sum_r (H^r)^n \fint_{B_R} |Du_r| dy_r \le c_n M \qquad (5.19)$$

where the constant c_n again depends on n, ϵ since H^r depends on such quantities. Notice that although M in (5.12) has not been fully defined, (5.19) holds for any M having the structure in (5.12).

Connecting (5.14) – (5.19) to (5.13) now yields

$$k_{m+1} \le c_n \sum_r \fint_{B_R} (|Du_r| + 1 + \epsilon)\, dy_r + c_n \sum_r \left[\mathbf{I}_1^{|\mu_r|}(x_r, 2R) \right]^{\frac{1}{1+\epsilon}}$$

$$+c_{n+3}\sum_r \mathbf{I}_1^{|\mu_r|}(x_r,2R)(|Du_r(x_r)|^{-\epsilon}+(1+\epsilon)^{-\epsilon})$$

$$+c_n\sum_{i=0}^{m-1}\omega(R_i)(k_i+1+\epsilon), \qquad (5.20)$$

that holds whenever $1 \le m \le \tilde{m}$ and for constants c_n, c_{n+3} depending on n,ϵ. In order to estimate the terms appearing in the right-hand side of (5.20), when $\epsilon < 0$ we apply Young's inequality with conjugate exponents $-1/\epsilon$ and $1/(1+\epsilon)$, and with $\epsilon \in (0,1)$ (to be chosen towards the end of the proof) we again

$$\sum_r \mathbf{I}_1^{|\mu_r|}(x_r,2R)|Du_r(x_r)|^{-\epsilon} \le \tilde{c}_n(\epsilon)\sum_r\left[\mathbf{I}_1^{|\mu_r|}(x_r,2R)\right]^{\frac{1}{1+\epsilon}} - \epsilon^2\sum_r|Du_r(x_r)| \qquad (5.21)$$

where

$$\tilde{c}_n(\epsilon) \coloneqq (1+\epsilon)\epsilon^{\frac{\epsilon}{1+\epsilon}} \qquad (5.22)$$

and similarly

$$\sum_r \mathbf{I}_1^{|\mu_r|}(x_r,2R)(1+\epsilon)^{-\epsilon} \le c_n\sum_r\left[\mathbf{I}_1^{|\mu_r|}(x_r,2R)\right]^{\frac{1}{1+\epsilon}} + 1+\epsilon. \qquad (5.23)$$

At this stage ϵ is still a free parameter to be chosen later and affecting the constant $\tilde{c}_n(\epsilon)$ in (5.22).

Now, with M defined as in (5.12), incorporating $\tilde{c}(\epsilon)$ introduced in (5.22), using (5.18), we have that (5.20) gives that

$$k_{m+1} \le c_{n+4}M + c_{n+5}\sum_{i=0}^{m-1}\omega(R_i)k_i + c_{n+3}\epsilon\sum_r|Du_r(x_r)| \qquad (5.24)$$

holds whenever $1 \le m \le \tilde{m}$, where the new constants $c_{n+4}, c_{n+5} \ge 1$ also depend on n,ϵ.

Now we come to the induction argument and we determine the value of $\tilde{R}$ by further reducing it; indeed we take $\tilde{R}$ such that

$$d(2\tilde{R}) \le \min\{1/(8c_{n+5}), 1/(8c_{n+4}), 1/(8c_{n+3})\}. \qquad (5.25)$$

Notice that the previous choice determines a smaller value of the radius $\tilde{R}$, that nevertheless can be chosen in a way that makes it depending on n,ϵ and $\omega(\cdot)$ since $c_{n+3}, c_{n+4}, c_{n+5}$ depends on n,ϵ.

In order to complete the proof of the lemma, we recall (5.18) and then prove that the following inequality holds whenever $0 \le i \le \tilde{m}+1$:

$$k_i \le 2c_{n+4}M + 2c_{n+3}\epsilon\sum_r|Du_r(x_r)|, \qquad (5.26)$$

where c_{n+3}, c_{n+4} are again the constants appearing in (5.24) and depending on n,ϵ. The proof is of course by induction. The cases $i = 0,1$ simply follow from (5.18) – (5.19). Next, we assume the validity of (5.26) for every $i \le m$ with $m \le \tilde{m}$, and prove it for $i = m+1$. By using estimate (5.24) and the induction assumption (5.26) for $i \le m-1$, and finally using also (5.25) we have

$$k_{m+1} \le c_{n+4}M + 2c_{n+4}c_{n+5}M\sum_{i=0}^{m-1}\omega(R_i)k_i + \sum_r\left[c_{n+3}\epsilon + 2c_{n+4}c_{n+3}\epsilon\sum_{i=0}^{m-1}\omega(R_i)\right]|Du_r(x_r)|$$

$$\leq c_{n+4}M + 2c_{n+4}c_{n+5}d(2R)M + c_{n+3}\epsilon \sum_r [1 + 2c_{n+4}c_{n+5}d(2R)]|Du_r(x_r)|$$

$$\leq 2c_{n+4}M + 2c_{n+3}\epsilon \sum_r |Du_r(x_r)|. \qquad (5.27)$$

This is (5.26) for $i = m + 1$ so that by induction (5.26) holds whenever $i \leq \tilde{m} + 1$. The proof of Lemma 5.1 is now complete.

Step 3: *Alternative.* We define the set

$$\mathbb{S} := \left\{ i \in \mathbb{N}: \sum_r |Du_r(x_r)| \geq \sum_r \fint_{B_i} |Du_r|\, dy_r \right\},$$

and distinguish two cases.

Case 1: $\mathbb{S} = \mathbb{N}$. In this case we have that

$$\sum_r \fint_{B_i} |Du_r|\, dy_r \leq \sum_r |Du_r(x_r)| \qquad \text{for every} \qquad i \in \mathbb{N}$$

and therefore we may apply Lemma 5.1 with $\tilde{m} = \infty$. In particular, this gives that

$$k_m \leq 2c_{n+4}M + 2c_{n+3}\epsilon \sum_r |Du_r(x_r)|,$$

holds whenever $m \in \mathbb{N}$, where M is defined in (5.12) and $\epsilon \in (0,1)$ is still a free parameter affecting M via the constant $\tilde{c}_n(\epsilon)$ defined in (5.22). Now, letting $m \to \infty$ in the previous inequality, and recalling that Du_r is here assumed to be continuous, yields

$$\sum_r |Du_r(x_r)| = \lim_{m\to\infty} k_m \leq 2c_{n+4}M + 2c_{n+3}\epsilon \sum_r |Du_r(x_r)|. \qquad (5.28)$$

We now choose $\epsilon = 1/(4c_{n+3})$ so the previous inequality gives

$$\sum_r |Du_r(x_r)| \leq 2c_{n+4}M. \qquad (5.29)$$

We now notice that since c_{n+3} depends n, ϵ we have that so is the dependence of ϵ and therefore of the (large) constant $\tilde{c}_n(\epsilon)$ appearing in the definition of the quantity Min (5.12) and in (5.22). All in all, using (5.12) in (5.29) we have proved that

$$\sum_r |Du_r(x_r)| \leq c_n \sum_r \fint_{B(x_r,R)} (|Du_r| + 1 + \epsilon)\, dy_r + c_n \sum_r \left[\mathbf{I}_1^{|\mu_r|}(x_r, 2R)\right]^{\frac{1}{1+\epsilon}} \qquad (5.30)$$

holds for a constant c_n depending on n, ϵ, whenever $R \leq \tilde{R}$, where $\tilde{R}$ in turn depends on $n, \epsilon, \omega(\cdot)$. By obviously changing the radius – see Step 4 below – estimate (5.30) implies (1.10) when $R \leq \tilde{R}$. We shall remove this restriction later, finally obtaining the validity of (1.10) for every ball $B(x_r, 2R) \subset \Omega_r$, but with a new constant that depends on n, ϵ and $\omega(\cdot)$, as prescribed in the statement of the theorem. This will be done in step 4, at the end of the proof.

We now proceed with the proof; our aim is to prove (5.30) in the case $\mathbb{S} \neq \mathbb{N}$.

Case 2: $\mathbb{S} \neq \mathbb{N}$.Then we let $\tilde{m} := \min(\mathbb{N}\backslash\mathbb{S}) \geq 0$; this means that

$$\sum_r |Du_r(x_r)| < \sum_r \fint_{B_{\tilde{m}}} |Du_r|\, dy_r \qquad (5.31)$$

and

$$\sum_r \fint_{B_i} |Du_r|\, dy_r \leq \sum_r |Du_r(x_r)| \quad , \text{ whenever } 0 \leq i \leq \tilde{m}-1 \ , \tag{5.32}$$

with the last estimate that holds whenever $\tilde{m} > 0$.We further distinguish two cases; the first is when $\tilde{m} = 0$; this means that $\sum_r |Du_r(x_r)| < \sum_r (|Du_r|)_{B_0}$ and therefore (5.30) trivially follows. The other case is when $\tilde{m} \geq 1$; we then use (5.31) as follows:

$$\sum_r |Du_r(x_r)| < \sum_r \fint_{B_{\tilde{m}}} |Du_r|\, dy_r$$

$$\leq \sum_r \fint_{B_{\tilde{m}}} \left|Du_r - (Du_r)_{B_{\tilde{m}}}\right| dy_r + \sum_r \left|(Du_r)_{B_{\tilde{m}}}\right| = A_{\tilde{m}} + k_{\tilde{m}}. \tag{5.33}$$

Next, we use Lemma 5.1 that gives (5.11) and in particular

$$k_{\tilde{m}} \leq 2c_{n+4}M + 2c_{n+3}\epsilon \sum_r |Du_r(x_r)|, \tag{5.34}$$

with $\epsilon \in (0,1)$ free to be chosen, affecting M in the way described in (5.12). On the other hand, combining (5.8) and (5.32) and again using (5.11) gives

$$A_{\tilde{m}} \leq A_0 + c_n \sum_r \sum_{i=0}^{\tilde{m}-1} \left[\frac{|\mu_r|(B_i)}{R_i^{n-1}}\right]^{\frac{1}{1+\epsilon}} + c_n \sum_r \left(|Du_r(x_r)|^{-\epsilon} + (1+\epsilon)^{-\epsilon}\right) \sum_{i=0}^{\tilde{m}-1} \frac{|\mu_r|(B_i)}{R_i^{n-1}}$$

$$+ c_n \sum_r \left[2c_{n+4}M + 2c_{n+3}\epsilon |Du_r(x_r)| + 1 + \epsilon\right] \sum_{i=0}^{\tilde{m}-1} \omega(R_i).$$

Again using (5.14) – (5.16) and (5.18) in the previous estimate yields

$$A_{\tilde{m}} \leq c_n \sum_r \fint_{B_R} (|Du_r| + 1 + \epsilon)\, dy_r + c_n \sum_r \left[\mathbf{I}_1^{|\mu_r|}(x_r, 2R)\right]^{\frac{1}{1+\epsilon}}$$

$$+ c_n \sum_r \mathbf{I}_1^{|\mu_r|}(x_r, 2R)\left(|Du_r(x_r)|^{-\epsilon} + (1+\epsilon)^{-\epsilon}\right)$$

$$+ c_n d(2R)\left[2c_{n+4}M + 2c_{n+3}\epsilon \sum_r |Du_r(x_r)| + 1 + \epsilon\right], \tag{5.35}$$

where $c_n \equiv c_n(n,\epsilon)$. Estimating as in (5.21) – (5.23) in (5.35), and using (5.25), we have

$$A_{\tilde{m}} \leq c_n M + c_n \epsilon \sum_r |Du_r(x_r)| \tag{5.36}$$

for yet a new constant c_n depending on n, ϵ and where M is defined accordingly to (5.29). Using (5.36) and (5.34) in (5.33) finally gives

$$\sum_r |Du_r(x_r)| \leq c_n M + c_n \epsilon \sum_r |Du_r(x_r)|$$

for $c_n \equiv c_n(n,\epsilon)$. Choosing $\epsilon = 1/(2c_n)$ again gives $|Du_r(x_r)| \leq c_n M$ and recalling the definition of M in (5.12) we once again obtain (5.30), which is valid under the same conditions of the Case 1, that is, provided $R \leq \tilde{R}$.

Step 4: Getting rid of the condition $R \leq \tilde{R}$. We finally prove estimate (1.10) also in the case $R > \tilde{R}$. Take a ball $B_R \equiv B(x_r, R) \subset \Omega_r$, with $R > \tilde{R}$; (5.30) gives

$$\sum_r |Du_r(x_r)| \leq c_n \sum_r \fint_{B(x_r,\tilde{R}/2)} (|Du_r| + 1 + \epsilon)\, dy_r + c_n \sum_r \left[\mathbf{I}_1^{|\mu_r|}(x_r, \tilde{R})\right]^{\frac{1}{1+\epsilon}}$$

and then we estimate

$$c_n \sum_r \fint_{B(x_r,\tilde{R}/2)} (|Du_r| + 1 + \epsilon)\, dy_r \leq c_n 2^n \left(\frac{R}{\tilde{R}}\right)^n \sum_r \fint_{B(x_r,R)} (|Du_r| + 1 + \epsilon)\, dy_r$$
$$\leq \sum_r \left(\frac{\dim(\Omega_r)}{\tilde{R}}\right)^n \fint_{B(x_r,R)} (|Du_r| + 1 + \epsilon)\, dy_r$$

and trivially

$$\sum_r \mathbf{I}_1^{|\mu_r|}(x_r, \tilde{R}) \leq \sum_r \mathbf{I}_1^{|\mu_r|}(x_r, R),$$

so that (1.10) follows with a new constant – $c_n/\tilde{R}^n$ instead of c_n – which depends on n, ϵ, and additionally on $\omega(\cdot)$ due to the presence of $\tilde{R}$, which has been previously by determined by choosing $\omega(\tilde{R})$ suitably small. The proof is complete.

6. Proof of Theorem 1.2

The proof is similar to that of Theorem 1.1, being actually much simpler as no mixed quantity of the type (1.20) shows up in the right-hand side of (4.39) and consequently no alternative – i.e. Case 1 and Case 2 in the proof of Theorem 1.2 – is needed. A minor difference will occur in that the constants involved will exhibit an additional dependence on the number $1 + \epsilon$ introduced in (1.13). We shall therefore confine ourselves to give just a sketch of the proof. After choosing H^r exactly as in (5.1) – but this time referring to Lemma 4.6 for the constants c_{n+1} and β – we everywhere consider the quantity

$$\sum_r \fint_{B_{R/2H^r}} \left|V(Du_r) - (V(Du_r))_{B_{R/2H^r}}\right| dy_r$$

in place of the "linear excess"

$$\sum_r \fint_{B_{R/2H^r}} \left|Du_r - (Du_r)_{B_{R/2H^r}}\right| dy_r.$$

We choose the balls as in (5.5) while now we define

$$k_i := \left|(V(Du_r))_{B_i}\right| \quad \text{and} \quad A_i := \fint_{B_i} \left|V(Du_r) - (V(Du_r))_{B_i}\right| dy_r. \tag{6.1}$$

In other words, instead of dealing with averages of Du_r as in the proof of Theorem 1.1, we deal with averages of $V(Du_r)$. Proceeding as in the proof of Theorem 1.1 we arrive at the following analog of (5.9):

$$k_{m+1} \leq c_n A_0 + c_n k_0 + c_n \sum_{i=0}^{m-1} \left[\frac{|\mu_r|(B_i)}{R_i^{n-1}}\right]^{\frac{2+\epsilon}{2(1+\epsilon)}} + c_n \sum_{i=0}^{m-1} \omega(R_i)(k_i + 1 + \epsilon). \tag{6.2}$$

Exactly as in (5.14) we observe that

$$\sum_r \sum_{i=0}^{m-1} \left[\frac{|\mu_r|(B_i)}{R_i^{n-1}}\right]^{\frac{2+\epsilon}{2(1+\epsilon)}} \leq \frac{2^{\frac{(2+\epsilon)(n-1)}{2(1+\epsilon)}}}{\log 2} \sum_r \int_R^{2R} \left[\frac{|\mu_r|(B(x_0,\rho))}{\rho^{n-1}}\right]^{\frac{2+\epsilon}{2(1+\epsilon)}} \frac{d\rho}{\rho}$$

$$+ \sum_r \frac{(2H^r)^{\frac{(2+\epsilon)(n-1)}{2(1+\epsilon)}}}{\log 2H^r} \sum_{i=0}^{\infty} \int_{R_{i+1}}^{R_i} \left[\frac{|\mu_r|(B(x_0,\rho))}{\rho^{n-1}}\right]^{\frac{2+\epsilon}{2(1+\epsilon)}} \frac{d\rho}{\rho}$$

$$\leq \sum_r c_n(H^r) \int_0^{2R} \left(\frac{|\mu_r|(B(x_0,\rho))}{\rho^{n-1}}\right)^{\frac{2+\epsilon}{2(1+\epsilon)}} \frac{d\rho}{\rho}. \tag{6.3}$$

Defining this time

$$M := \sum_r \fint_{B_R} (|V(Du_r)| + (1+\epsilon)^{(2+\epsilon)/2}) dy_r + \sum_r \int_0^{2R} \left(\frac{|\mu_r|(B(x_r,\rho))}{\rho^{n-1}}\right)^{\frac{2+\epsilon}{2(1+\epsilon)}} \frac{d\rho}{\rho}$$

$$\leq \sum_r \fint_{B_R} (|Du_r| + 1 + \epsilon)^{(2+\epsilon)/2} dy_r + \sum_r \int_0^{2R} \left(\frac{|\mu_r|(B(x_r,\rho))}{\rho^{n-1}}\right)^{\frac{2+\epsilon}{2(1+\epsilon)}} \frac{d\rho}{\rho},$$

and making (6.3) into account, we have that (6.2) implies

$$k_{m+1} \leq c_{n+4} M + c_{n+5} \sum_{i=0}^{m-1} \omega(R_i) k_i \tag{6.4}$$

for constants c_{n+4}, c_{n+5} depending on n, ϵ. Arguing as in the proof of Theorem 1.1 we then prove by induction that $k_m \leq 2c_{n+4}M$ holds for every $m \in \mathbb{N}$; indeed, observe that this time no alternative as in Step 3 of Theorem 1.1 occurs and the induction of Lemma 5.1 can be performed without assuming (5.10). Therefore, by (4.38), we conclude observing that

$$\sum_r |Du_r(x_r)|^{(2+\epsilon)/2} \leq \sum_r |V(Du_r(x_r))| \leq \lim_{m\to\infty} k_m \leq 2c_{n+4}M.$$

The previous relation proves (1.15) whenever $R \leq \tilde{R}$ and $\tilde{R}$ is a fixed radius depending on n, ϵ, and found as in the proof of Theorem 1.1. The general case $R > 0$ follows as in Step 4 of Theorem 1.1.

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